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Digital Phase Demodulator

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Injection-locked oscillators are shown to act as narrow-band tunable filters for FM signals if the modulation rate is much larger than the locking bandwidth. The filtering action of the injection-locked oscillator for FM signals is found analogous to that of a high Q passive cavity. The effective Q of the injection-locked oscillator can be as high as 10^5 if the stability of the injected signal carrier and oscillator frequencies is better than 10^6.

These filtering properties can be applied to a digital demodulator for coherent phase detection of a coded FM signal. The local source which is required for coherent phase detection is provided by using a fraction of the received signal to lock an oscillator. Sideband suppression and carrier amplification of the injected signal are achieved simultaneously by using the filtering action of the injection-locked oscillator.

The simplicity of this digital demodulator makes it appear useful for repeaters in microwave radio relays.

I. INTRODUCTION

Injection-locked oscillators can perform a wide variety of functions required in microwave radio relays, such as amplification, amplitude limitation,[1] frequency modulation[2] and demodulation[3] to mention only the most important applications.

It is shown in this paper that injection-locked oscillators can also be used as narrow band tunable filters for angle modulated signals. These filtering properties can be used in a digital demodulator for coherent phase detection and such a demodulator is described here. Its configuration, shown in Fig. 1, is similar to the injection-locked oscillator FM receiver proposed by C. L. Ruthroff.[3] The principles of operation, however, are different. For proper operation of the injection-locked oscillator FM receiver the output signal of the oscillator contains all of the frequency modulation on the input signal. This signal is multiplied by a fraction of the input signal to

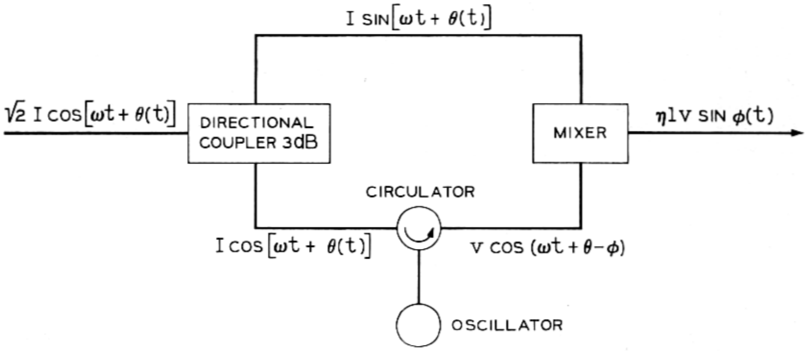

Fig. 1—Scheme of the digital demodulator for coherent phase detection.

produce the demodulated output. In the digital demodulator, the filtering properties of an injection-locked oscillator are used to remove the frequency modulation from the input signal and to deliver a sinusoidal output at the frequency and phase of the unmodulated carrier; this carrier is used as the local reference clock signal in a synchronous detector. The mixing of the received signal with the local source gives a current dependent on the phase of the received signal provided the time average of the phase modulation stays small over periods shorter than the time constant of the oscillator.

The first part of this paper includes an analysis of the filtering properties of the injection-locked oscillator for phase modulated signals. An approximate analytical solution for the filtering action is derived from the locking equation. This solution is then compared with exact numerical calculations and with experimental results.

The second part of this paper describes the properties and the limitations of a digital demodulator which uses the filtering action of the injection-locked oscillator analyzed in the first section.

II. ANALYSIS OF THE INJECTION-LOCKED OSCILLATOR FILTER

2.1 *Locking Equation Analysis*

The injection-locked oscillator performs two functions in the digital demodulator: it removes the phase modulation of the input signal and it amplifies the carrier signal up to the free-running oscillator output level. The oscillator output signal which is obtained can be used as a reference signal for synchronous detection.

The filtering properties can be derived from the locking equation. These properties can also be obtained from a frequency domain analysis which provides a clearer physical picture of the phenomena.

Let us consider first the locking equation analysis. Assume that the injection-locked oscillator is driven by an injected signal phase modulated by a function $\theta(t)$, thus $i(t) = I \cos [\omega t + \theta(t)]$. Locking occurs if ω is sufficiently close to the natural oscillator frequency ω_0. Within the locking range, the oscillator output voltage is given by $v(t) = V(t) \cos [\omega t + \theta(t) - \varphi(t)]$.[4] $\varphi(t)$, called the tracking angle, is the difference in phase between the input and output signals of the oscillator. The phases of the input and output signals are related by the well-known locking equation[5,6]

$$\frac{d\varphi}{dt} = \frac{d\theta}{dt} + \omega - \omega_0 - \Delta\omega_L \sin \varphi(t), \tag{1}$$

where

$$\Delta\omega_L = I/(2V(t)G_L) \times \omega_0/Q$$

is one-half the locking bandwidth for an unmodulated injected signal of amplitude I, G_L is the load and Q is the external loaded circuit Q. The output signal amplitude $V(t)$ is usually nearly constant[1] and therefore $\Delta\omega_L$ may be assumed to be time independent.

Removing the phase modulation from the injected signal requires that the phase of the oscillator output signal, $\theta(t) - \varphi(t)$, becomes time independent. It will be shown that this condition is approximately fulfilled if the rate of phase modulation is much larger than the locking bandwidth.

Let us consider an input signal with a sinusoidal phase modulation given by

$$\theta(t) = \theta_0 \sin \Omega t. \tag{2}$$

Substitution of equation (2) into equation (1) gives

$$\frac{1}{\Omega}\frac{d\varphi}{dt} - \theta_0 \cos \Omega t = \frac{\Delta\omega_L}{\Omega}\left[\frac{\omega - \omega_0}{\Delta\omega_L} - \sin \varphi(t)\right]. \tag{3}$$

With a rate of phase modulation $\Omega \gg \Delta\omega_L$, the right side of equation (3) remains much smaller than unity as long as $|\omega - \omega_0| \leqq \Delta\omega_L$. Equation (3) can therefore be approximated by

$$\frac{1}{\Omega}\frac{d}{dt}[\varphi(t) - \theta(t)] \approx 0 \tag{4}$$

which gives for the output signal a phase approximately time independent, thus

$$\varphi(t) - \theta(t) \approx \varphi_0 . \tag{5}$$

The constant of integration, φ_0 , can be obtained from the locking condition which is obtained by taking the time average of equation (1) yielding

$$\left\langle \frac{d\varphi}{dt} \right\rangle = \left\langle \frac{d\theta}{dt} \right\rangle + \omega - \omega_0 - \Delta\omega_L \langle \sin \varphi(t) \rangle. \tag{6}$$

From equations (2) and (4),

$$\left\langle \frac{d\theta}{dt} \right\rangle = 0, \qquad \left\langle \frac{d\varphi}{dt} \right\rangle = 0, \tag{6a}$$

and

$$\begin{aligned} \langle \sin \varphi(t) \rangle &= \langle \sin (\varphi_0 + \theta_0 \sin \Omega t) \rangle \\ &= J_0(\theta_0) \sin \varphi_0 , \end{aligned}$$

where J_0 is the Bessel function of order zero. Substitution of equation (6a) into equation (6) gives the locking condition

$$\omega - \omega_0 = \Delta\omega_L J_0(\theta_0) \sin \varphi_0 . \tag{7}$$

The frequency range of locking is determined by the condition $| \sin \varphi_0 | \leqq 1$. Therefore the locking bandwidth is, from equation (7),

$$2(\omega - \omega_0)_{\max} = 2\, \Delta\omega_L J_0(\theta_0). \tag{8}$$

The maximum locking bandwidth is reduced by the factor $J_0(\theta_0)$ compared with the unmodulated case; in particular, it becomes equal to zero for $\theta_0 = 2.405$ radians.

These results have been obtained from a first-order solution of equation (3). The magnitude of the filtering effect, resulting from the residual phase modulation $\theta(t) - \varphi(t)$ of the oscillator output signal, can be calculated by solving equation (3) to the second order.

Before calculating the magnitude of the filtering effect, it is interesting to give a physical picture of this phenomenon through a frequency domain analysis. Experimental results observed with a spectrum analyzer will be compared with the results of this analysis.

2.2 *Frequency Domain Discussion*

Let us consider as before an oscillator locked by an injected signal with a sinusoidal phase modulation, thus

$$i(t) = I \cos [\omega t + \theta_0 \sin \Omega t]. \tag{9}$$

The current expression, expanded in Bessel series, can be written as[7]

$$i(t) = I\left\{J_0(\theta_0) \cos \omega t + \sum_{\substack{-\infty \\ n\neq 0}}^{+\infty} J_n(\theta_0) \cos (\omega + n\Omega)t\right\}. \tag{10}$$

If one assumes that the carrier frequency ω is within the locking bandwidth $\Delta\omega_L$ and that $\Omega \gg \Delta\omega_L$, the spectral components at $\omega + n\Omega$ ($n = \pm 1, \pm 2, \cdots$ etc) have a small effect on the oscillator. It can be expected in a first approximation that the oscillator is locked by the injected current component $IJ_0(\theta_0) \cos \omega t$ corresponding to the carrier frequency. The locking bandwidth, which is proportional to the effective driving current amplitude $IJ_0(\theta_0)$, is reduced by the factor $J_0(\theta_0)$ as found previously in Section 2.1. The locking bandwidth decreases with increasing index of modulation and becomes equal to zero for $\theta_0 = 2.405$. This particular case corresponds to an injected FM signal with a suppressed carrier.

The sideband suppression effect, of the injection-locked oscillator, can be seen clearly in this analysis as an increasing function of the rate of phase modulation. Furthermore it can be expected from this analysis that locking can also occur for any of the spectral components at $\omega + n\Omega$ ($n = \pm 1, \pm 2, \cdots$). Locking can be obtained by tuning the oscillator natural frequency to $\omega + n\Omega$. The locking range for each frequency $\omega + n\Omega$ is proportional to the spectral line amplitude, $IJ_n(\theta_0)$.

2.3 *Experimental Verification*

The validity of the assumptions made in this analysis has been checked by locking a 35-MHz oscillator with an FM injected signal. The index of modulation, the rate of modulation and the locking bandwidth were adjusted to be about $\pi/2$, 100 kHz, and 10 kHz, respectively.

Stable locking was obtained by tuning the injected signal frequency such that the main spectral lines, $J_0(\pi/2) \cos \omega t$, $J_{\pm 1}(\pi/2) \cos (\omega \pm \Omega)t$ and $J_{\pm 2}(\pi/2 \cos (\omega \pm 2\Omega)t$ lie consecutively in the locking range. Figure 2 shows the locking obtained with the three first spectral components and also shows the characteristic beat modulation which occurs just before locking. The largest locking ranges correspond as expected to the spectral components $J_{\pm 1}(\pi/2) \cos (\omega \pm \Omega)t$ which have the largest amplitudes.

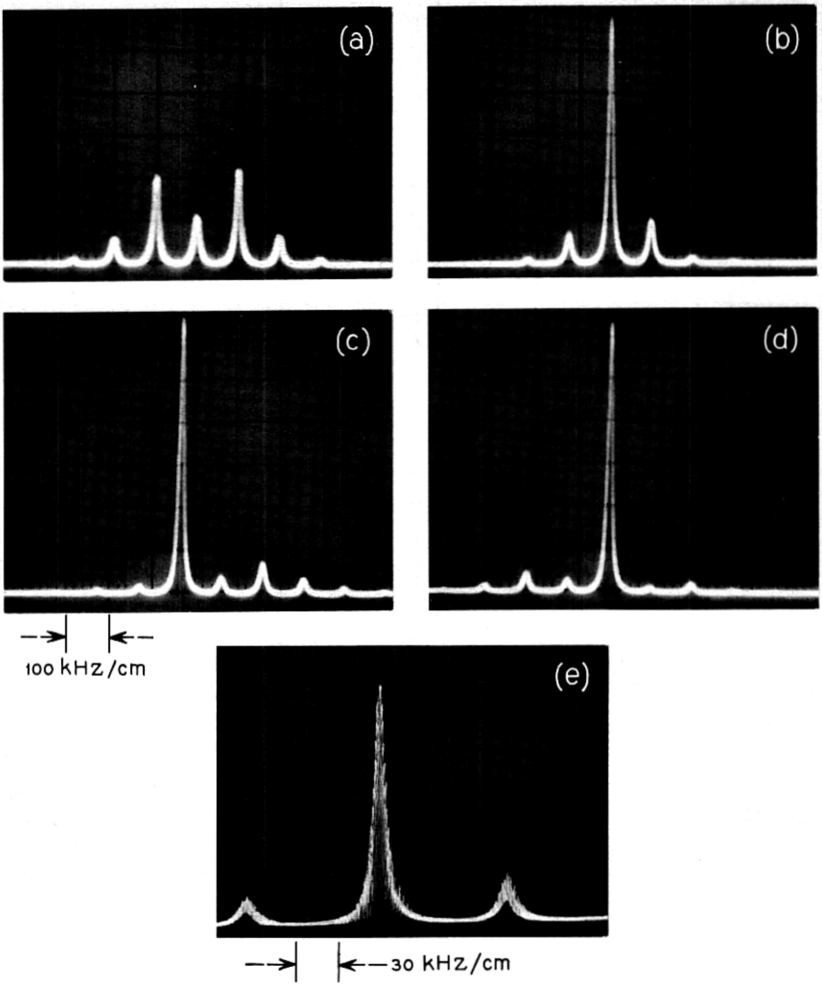

Fig. 2—Spectra of the injected signal and the oscillator output signal: (a) injected signal spectrum, (b) output signal with locking at ω, (c) output signal with locking at $\omega - \Omega$, (d) output signal with locking at $\omega + \Omega$, and (e) output signal just before locking. ($\theta \approx \pi/2 \sin \Omega t$, $\Omega = 2\pi \times 100$ kHz, $\Delta\omega_L \approx 2\pi \times 10$ kHz, $F_0 = 35$ MHz)

2.4 *Sideband Attenuation*

The filtering effect of the injection-locked oscillator suppresses the sidebands of the output signal. This effect is shown in Fig. 2 where the spectra of the input and output signals of the oscillator can be compared.

Sideband attenuation can be defined by

$$\left(\frac{\text{Sideband Power}}{\text{Carrier Power}}\right)_{\text{output}} \Big/ \left(\frac{\text{Sideband Power}}{\text{Carrier Power}}\right)_{\text{input}}.$$

This ratio is related to the residual phase modulation $\theta(t) - \varphi(t)$ which can be calculated by solving equation (3) to the second order. The calculations are given in Appendix A for $\theta_0 = \pi/2$. The steady-state solution is

$$\varphi(t) \simeq \frac{\pi}{2} \sin \Omega t + \varphi_0 + 2 \frac{\Delta\omega_L}{\Omega} \left\{ J_1\left(\frac{\pi}{2}\right) \cos \varphi_0 \cos \Omega t - \frac{J_2\left(\frac{\pi}{2}\right)}{2} \sin \varphi_0 \sin 2\Omega t \right\}, \tag{11}$$

where from equation (7)

$$\sin \varphi_0 = \frac{\omega - \omega_0}{J_0\left(\frac{\pi}{2}\right)\Delta\omega_L}. \tag{12}$$

For simplicity let us assume that $\omega = \omega_0$, thus the injected current is

$$i(t) = I \cos \left[\omega_0 t + \frac{\pi}{2} \sin \Omega t \right]. \tag{13}$$

The tracking angle becomes

$$\varphi(t) \simeq \frac{\pi}{2} \sin \Omega t + 2 \frac{\Delta\omega_L}{\Omega} J_1\left(\frac{\pi}{2}\right) \cos \Omega t, \tag{14}$$

and the oscillator output voltage is

$$v(t) = V \cos \left[\omega_0 t - 2 \frac{\Delta\omega_L}{\Omega} J_1\left(\frac{\pi}{2}\right) \cos \Omega t \right]. \tag{15}$$

The expressions of the input and output power, expanded in Bessel series, yield for the ratio of sideband power to carrier power:

a) *Input signal*

$$\left(\frac{\text{Sideband Power}}{\text{Carrier Power}}\right)_{\text{input}} = \frac{1 - J_0^2\left(\frac{\pi}{2}\right)}{J_0^2\left(\frac{\pi}{2}\right)} = 3.484. \tag{16}$$

b) *Output signal*

$$\left(\frac{\text{Sideband Power}}{\text{Carrier Power}}\right)_{\text{output}} = \frac{1 - J_0^2\left[2\,\frac{\Delta\omega_L}{\Omega}\,J_1\left(\frac{\pi}{2}\right)\right]}{J_0^2\left[2\,\frac{\Delta\omega_L}{\Omega}\,J_1\left(\frac{\pi}{2}\right)\right]} = 0.0064. \qquad (17)$$

The calculated sideband attenuation, in the present case, is about 27.4 dB.

In order to verify these results, equation (3) has been solved numerically for the following parameter values: $\theta_0 = \pi/2$, $(\omega - \omega_0)/\Omega = -0.01$ and $(\Delta\omega_L)/\Omega = 0.1$. Results of this computation are shown in Fig. 3 and are compared in Table I with the second-order approximation given by equation (11).

2.5 *Comparison Between the Active and Passive Resonators*

In the case of an injected current phase-modulated by a sinusoidal phase excursion, the filtering properties of the injection-locked oscillator characterized by a single-pole resonator with a negative resistance can be compared to the filtering effect of the same passive circuit.

Assuming an injected current equal to $I \cos\,(\omega_0 t + \pi/2 \sin \Omega t)$ the power ratio between the spectral component at $\omega_0 + \Omega$ and the carrier at ω_0 is, for the passive resonator,

$$\frac{1}{1 + 4Q^2\left(\frac{\Omega}{\omega_0}\right)^2}\left[\frac{J_1\left(\frac{\pi}{2}\right)}{J_0\left(\frac{\pi}{2}\right)}\right]^2 \approx \frac{1}{4Q^2}\left(\frac{\omega_0}{\Omega}\right)^2\left[\frac{J_1\left(\frac{\pi}{2}\right)}{J_0\left(\frac{\pi}{2}\right)}\right]^2. \qquad (18)$$

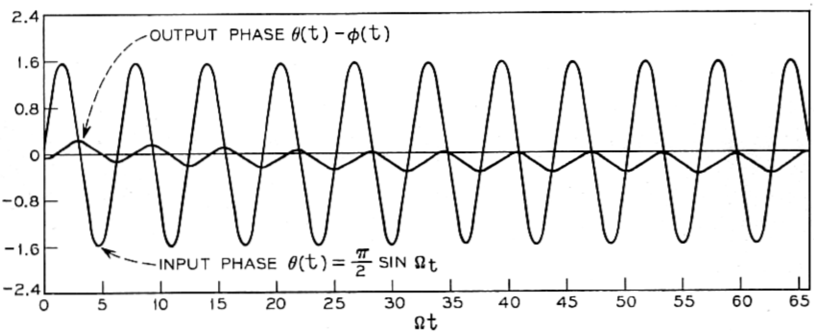

Fig. 3—Phase of the oscillator output signal for an injected signal phase modulated by $\theta(t) = \pi/2 \sin \Omega t$. $[(\Delta\omega_L)/\Omega = 0.1,\ (\omega - \omega_0)/\Omega = -0.01,\ \theta_0 = \pi/2.]$

TABLE I—COMPARISON BETWEEN THE ANALYTIC APPROXIMATION SOLUTION AND THE COMPUTED SOLUTION

Analytic Second-Order Approximation Solution	Numerically Computed Solution
$\sin\varphi_0 \simeq \dfrac{\omega-\omega_0}{J_0(\pi/2)\Delta\omega_L} = -0.212$	$\sin\varphi_0 = -0.199$
$\lvert\varphi-\varphi_0-\theta\rvert_{\max} = 0.112$ Sideband attenuation = 27.4 dB	$\lvert\varphi-\varphi_0-\theta\rvert_{\max} = 0.160$ Sideband attenuation = 23.7 dB

Taking into account that $\Delta\omega_L = \omega_0/Q(P_i/P_0)^{1/2}$, the same ratio for the locked oscillator gives, for $\Omega \gg \Delta\omega_L$,

$$\frac{1}{Q^2}\left(\frac{\omega_0}{\Omega}\right)^2 J_1^2\left(\frac{\pi}{2}\right)\left[\frac{P_i}{P_0}\right] \tag{19}$$

where P_0/P_i is the locking gain. For an FM injected signal having a sinusoidal phase excursion of amplitude $\pi/2$, the locked oscillator acts as a singly resonant filter with an effective Q given by

$$Q_e = Q\left(\frac{P_0}{P_i}\right)^{\frac{1}{2}} \frac{1}{2J_0\left(\dfrac{\pi}{2}\right)} \simeq Q\left(\frac{P_0}{P_i}\right)^{\frac{1}{2}} = \left[\frac{\Delta\omega_L}{\omega_0}\right]^{-1}. \tag{20}$$

The maximum effective Q which can be obtained depends on the minimum-locking bandwidth achievable. The minimum-locking bandwidth is limited by

(*i*) minimum frequency offset $\omega - \omega_0$,
(*ii*) oscillator free running frequency stability $\delta\omega_0/\omega_0$,
(*iii*) injected signal frequency stability $\delta\omega/\omega$.

The slow variation of the frequency offset $\omega - \omega_0$, between the injected signal frequency and the oscillator free-running frequency, can be made approximately equal to zero with a low-frequency feedback loop as shown in Ref. 3. In that scheme, the mixer output signal contains a current proportional to $\omega - \omega_0$; the oscillator natural frequency ω_0 can be kept tuned to ω by using this current to control a suitable oscillator parameter.

If one assumes a frequency stability of 10^{-6} for ω and ω_0 and takes a safety margin $\Delta\omega_L = 10\,\delta\omega$, one obtains a maximum effective Q given by

$$Q_{e\max} \simeq \frac{1}{10}\left[\frac{\Delta\omega_L}{\omega}\right]^{-1} = 10^5.$$

III. DIGITAL DEMODULATOR

3.1 *Coherent Phase Detection*

Coherent phase detection involves multiplication of the PM received signal by a local source which has a constant phase and the frequency of the carrier associated with the received signal. The local source, which is needed to achieve coherent phase detection, can be obtained by using the filtering properties of the injection-locked oscillator for phase-modulated signals.

A block diagram of the digital demodulator is shown in Fig. 1. The oscillator, in this configuration, is locked by a fraction of the received signal. The filtering properties of the injection-locked oscillator are used to remove the phase modulation from the input signal and to deliver a sinusoidal output at the frequency and phase of the unmodulated carrier; this carrier is used as the local reference signal for synchronous detection. The mixing of the received signal with the local source gives a current dependent on the phase of the received signal. Correct demodulation is obtained if the time average of the phase modulation stays small over periods shorter than $\Delta\omega_L^{-1}$. This restriction results from the impossibility of maintaining the phase of the reference signal constant if the phase of the injected signal has an average value different from zero. This problem arises for instance with a long pulse sequence of plus ones made by binary digital encoding using polar pulses.

3.2 *Demodulation of a Binary Polar Signal*

Let us consider an unmodulated carrier given by $\cos \omega t$. Starting at $t = 0$, the phase is modulated by a pulse train of plus ones made from raised cosines of maximum amplitude equal to $\pi/2$. The phase of the received signal can be written for $t \geqq 0$

$$\begin{aligned} \theta(t) &= \frac{\pi}{2} \sin^2 \Omega t \\ &= \frac{\pi}{4} [1 - \cos 2\Omega t] \end{aligned} \tag{21}$$

where 2Ω is the signaling frequency equal here to the rate of phase modulation. The phase modulation of the received signal has an average value equal to $\pi/4$. The phase of the local source $\theta(t) - \varphi(t)$ at $t > 0$ is obtained by the transient solution of the equation

$$\frac{d\varphi}{d(\Omega t)} = \frac{\pi}{2} \sin 2\Omega t - \frac{\Delta\omega_L}{\Omega} \sin \varphi(t), \tag{22}$$

where $\omega - \omega_0$ has been set equal to zero in order to simplify the analysis.

The filtering condition $\Delta\omega_L/\Omega \ll 1$ suggests that the time constant associated with the transient solution of equation (22) is large compared to $2\pi/\Omega$. An equation giving approximately the transient effect can be obtained by taking the time average of equation (22) over a large number of periods of sin $2\Omega t$. Equation (22) then becomes

$$\frac{d\langle\varphi\rangle}{dt} = -\Delta\omega_L \langle \sin \varphi \rangle. \tag{23}$$

Equation (23) can be solved approximately by replacing $\langle \sin \varphi \rangle$ by $\langle \varphi \rangle$ which yields

$$\langle \varphi \rangle \simeq \frac{\pi}{4} \exp(-\Delta\omega_L t). \tag{24}$$

The transient solution of equation (22) is therefore approximated by

$$\varphi(t) \simeq \frac{\pi}{4} [\exp(-\Delta\omega_L t) - \cos 2\Omega t], \tag{25}$$

where $-\pi/4 \cos 2\Omega t$ is the first order steady state solution of equation (22). The phase of the local source is given for $t \geqq 0$ by

$$\theta(t) - \varphi(t) \simeq \frac{\pi}{4} [1 - \exp(-\Delta\omega_L t)]. \tag{26}$$

An exact numerical solution, shown in Fig. 4, agrees well with the analytical solution given by equation (26). This result shows that the phase of the filtered signal used as a local source increases exponentially from zero to $\pi/4$ with a time constant equal to about $\Delta\omega_L^{-1}$.

Correct demodulation requires that the magnitude of the phase of the reference signal remains small compared to $\pi/4$. The maximum number of consecutive pulses of the same polarity, which can be decoded without error, increases with the ratio $\Omega/\Delta\omega_L$. It is important to note that the sideband suppression effect improves by the same factor.

In general, the pulse polarity varies in a nearly random fashion from pulse to pulse. The phase of the reference signal is then a function of the random processes which give the pulse polarity distribution. Its maximum magnitude variation can be equal to $\pm\pi/4$. It will remain smaller than $| \pi/4 |$ if the probability of having positive or negative pulses is about the same over periods shorter than $\Delta\omega_L^{-1}$. This condition introduces some restriction on the coding.

3.3 *Demodulation of a Signal Symmetrically Phase Modulated*

The problem of the reference phase discussed in the previous section disappears in case the average phase deviation is equal to zero for each

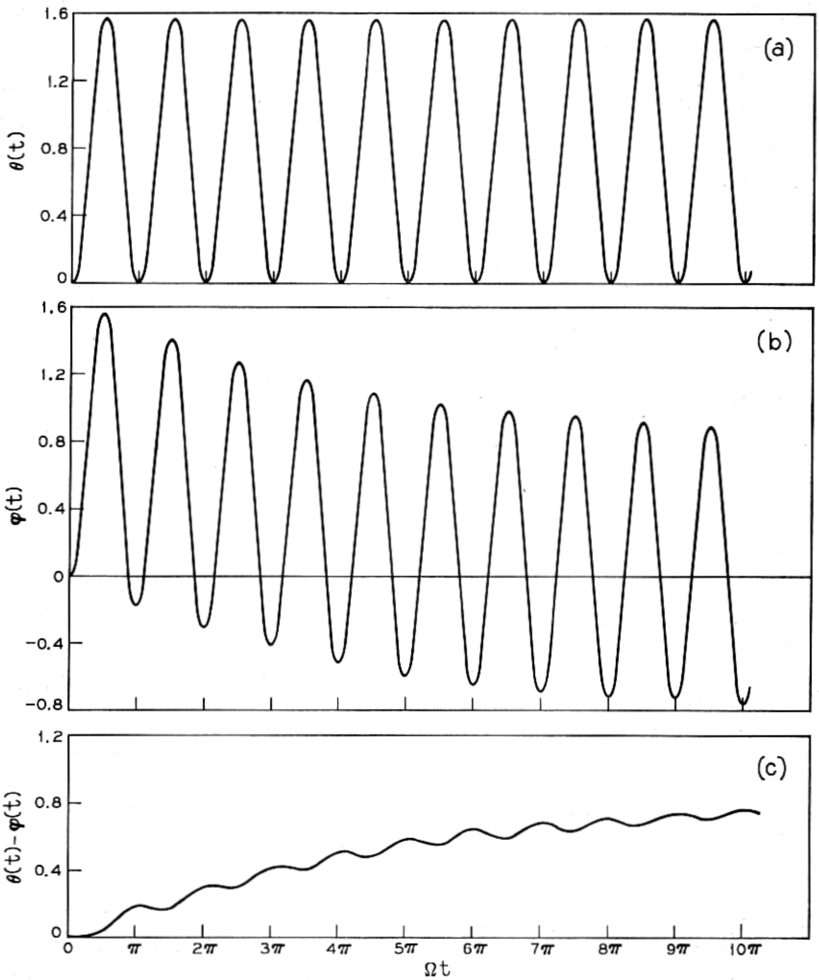

Fig. 4—Tracking angle and phase of the oscillator output signal for an injected signal phase modulated by $\theta(t) = (\pi/2)\sin^2 \Omega t$ with $(\Delta\omega_L)/\Omega = 0.1$. (a) Phase of the input signal, (b) tracking angle, (c) phase of the oscillator output signal.

pulse. A simple example of such a pulse shape is a sine wave starting from zero and limited to one period. The phase of the received signal can be written in this case[7]

$$\theta(t) = \sum_{k=-\infty}^{+\infty} a(k) \frac{\pi}{2} \sin \Omega(t - kT), \tag{27}$$

where $a(k)$ takes, for example, the value 0 or +1 according to a binary code. The phase of the received signal becomes, for a pulse train of +1s

$$\theta(t) = \frac{\pi}{2} \sin \Omega t. \tag{28}$$

This case has been solved in Section II, equation (11); it gives, if $(\omega - \omega_0)/\Delta\omega_L \ll 1$, a local source with a phase equal to

$$\theta(t) - \varphi(t) \approx \frac{\omega - \omega_0}{J_0\left(\frac{\pi}{2}\right)\Delta\omega_L} + 2\frac{\Delta\omega_L}{\Omega} J_1\left(\frac{\pi}{2}\right) \cos \Omega t. \tag{29}$$

This result is shown in Fig. 3.

The phase of the local source can be made nearly constant and adjusted close to zero by making $(\omega - \omega_0)/\Delta\omega_L$ and $(\Delta\omega_L)/\Omega$ very small.

This phase shift is also calculated for other types of pulse distributions. Its effect is shown in Fig. 5 which gives the tracking angle $\varphi(t)$ and the demodulated signal sin $[\varphi(t)]$ for an injected signal phase modulated by a pulse train of alternate ones and zero. These curves are calculated for the parameters $(\Delta\omega_L)/\Omega = 0.1$ and $\omega - \omega_0/\Delta\omega_L = \pm 0.1$. Figure 6 shows the same functions for a random phase modulation with the same parameter values. In these two cases, the distortion due to the phase shift of the local source is minimum for $(\omega - \omega_0)/\Delta\omega = -0.1$. Partial compensation is then obtained between the phase shift resulting from the frequency offset and the phase shift due to the residual of the filtering effect. In all cases the distortion can be minimized by setting $(\Delta\omega_L)/\Omega$ and $(\omega - \omega_0)/\Delta\omega_L$ small compared to unity. Correct demodulation can then be obtained without an encoding restriction.

IV. CONCLUSION

Injection-locked oscillators can be used as filters for sideband suppression of FM signals if the modulation rate is much larger than the locking range. These filtering properties can be summarized as

(*i*) high effective Q,
(*ii*) power amplification for the carrier, and
(*iii*) circuit simplicity.

The filtering properties of an injection-locked oscillator can be used in a digital demodulator to provide a local source for coherent phase detection of a particular class of digitally modulated signals. In particular, correct demodulation is obtained for pulse shapes which give an average phase deviation equal to zero for each pulse.

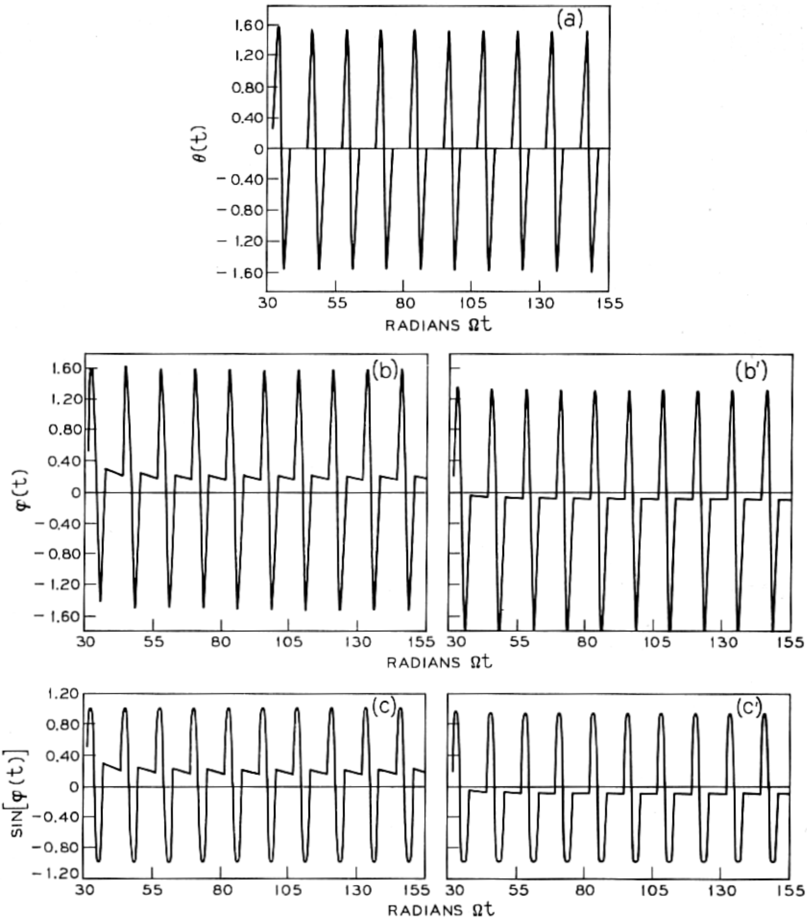

Fig. 5—Tracking angle and demodulated signal for an injected signal phase modulated by

$$\theta(t) = \frac{\pi}{2} \sin \Omega t \left[\frac{1}{2} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{\sin (2p + 1)\Omega t/2}{2p + 1} \right]$$

which corresponds to a pulse train of 1, 0, 1, 0, $\cdots$. (a) Input modulation, $\theta(t) = (\pi/2) \sin \Omega t\{1/2 + 2/\pi \sum_{0}^{\infty} [\sin (2p + 1)(\Omega t/2)]/(2p + 1)\}$. (b) Tracking angle $\phi(t)$ and (c) output mixer $\sin \phi(t)$, $(\Delta\omega_L/\Omega) = 0.1$ and $(\omega - \omega_0/\Omega) = 0.01$. (b′) Tracking angle $\phi(t)$ and (c′) output $\sin \phi(t)$, $(\Delta\omega_L/\Omega) = 0.1$ and $(\omega - \omega_0/\Omega) = -0.01$.

A binary digital coding using polar pulses requires a coding which gives a small average phase deviation over periods shorter than $\Delta\omega_L^{-1}$

V. ACKNOWLEDGMENT

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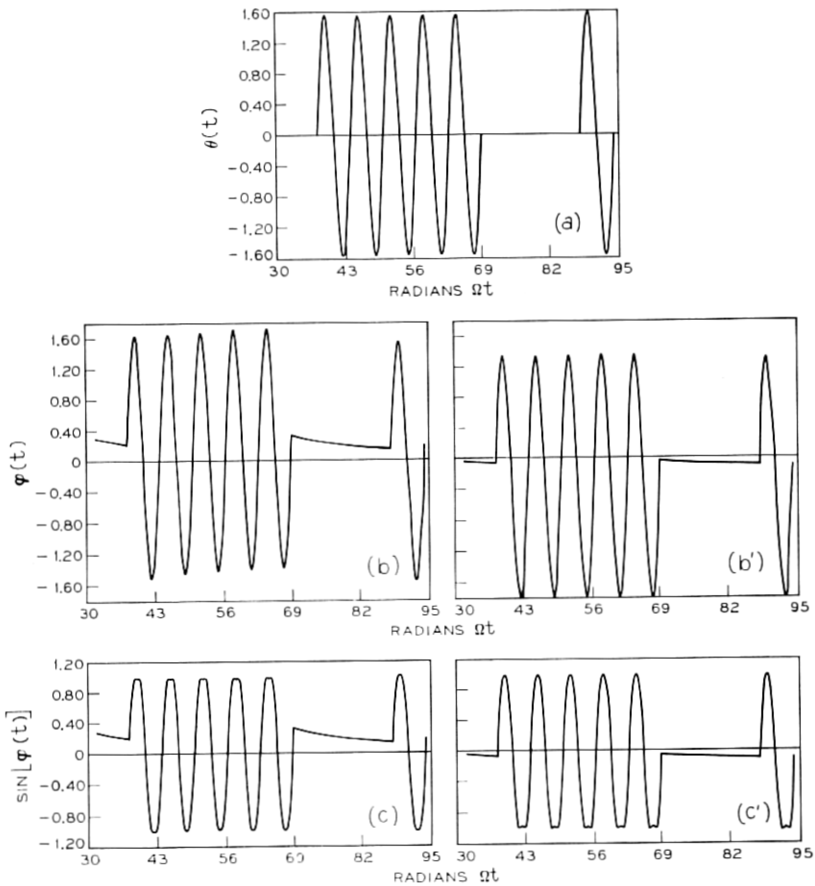

Fig. 6—Tracking angle and demodulated signal for an injected signal phase modulated by $\theta(t) = \Gamma(t)\pi/2 \sin \Omega t$ where $\Gamma(t)$ is a random telegraphic signal equal to +1 or zero with signaling frequency equal to Ω. (a) Random input modulation $\theta(t)$. (b) Tracking angle $\phi(t)$ and (c) mixer output $\sin \phi(t)$, $(\Delta\omega_L/\Omega) = 0.1$ and $(\omega - \omega_0/\Omega) = 0.01$. (b′) Tracking angle $\phi(t)$ and (c′) mixer output $\sin \phi(t)$, $(\Delta\omega_L/\Omega) = 0.1$ and $(\omega - \omega_0/\Omega) = -0.01$.

APPENDIX

Second-Order Solution of the Locking Equation

The locking equation (3), obtained for an injected signal phase modulated by $\theta(t) = \pi/2 \sin \Omega t$, can be written as:

$$\frac{d\varphi}{dx} = \frac{\pi}{2} \cos x + \beta - \alpha \sin \varphi \tag{30}$$

where $n = \Omega t$, $\beta = (\omega - \omega_0)/\Omega$ and $\alpha = (\Delta\omega_L)/\Omega$.

Equation (30) integrated for $\beta < \alpha \ll 1$ has a solution in the first approximation given by

$$\varphi_1 = \frac{\pi}{2} \sin x + \varphi_0 \tag{31}$$

with

$$\beta = \alpha J_0\left(\frac{\pi}{2}\right) \sin \varphi_0 . \tag{32}$$

The second-order approximation is calculated by putting $\varphi_2 = \varphi_1 + \eta$, with $\eta \sim \alpha \ll 1$. Substitution of these results into equation (30) gives, keeping the first-order terms in α,

$$\frac{d\eta}{dx} + \alpha\eta\left\{\left[J_0\left(\frac{\pi}{2}\right) + 2J_2\left(\frac{\pi}{2}\right) \cos 2x\right] \cos \varphi_0 - 2J_1\left(\frac{\pi}{2}\right) \sin x \sin \varphi_0\right\}$$
$$= -2\alpha\left\{J_1\left(\frac{\pi}{2}\right) \cos \varphi_0 \sin x + J_2\left(\frac{\pi}{2}\right) \sin \varphi_0 \cos 2x\right\}. \tag{33}$$

The approximate solution of equation (33) is

$$\eta = 2\alpha\left\{J_1\left(\frac{\pi}{2}\right) \cos \varphi_0 \cos x - \frac{J_2\left(\frac{\pi}{2}\right)}{2} \sin \varphi_0 \sin 2x\right\}$$
$$+ C \exp\left[-\alpha\left\{J_0\left(\frac{\pi}{2}\right)x + J_2\left(\frac{\pi}{2}\right) \cos \varphi_0 \right.\right.$$
$$\left.\left. \cdot \sin 2x + 2J_1\left(\frac{\pi}{2}\right) \sin \varphi_0 \sin x\right\}\right], \tag{34}$$

which gives for φ_2 after the initial transient

$$\varphi_2 = \frac{\pi}{2}\sin x + \varphi_0 + 2\alpha\left\{J_1\left(\frac{\pi}{2}\right)\cos\varphi_0\cos x - \frac{J_2\left(\frac{\pi}{2}\right)}{2}\sin\varphi_0\sin 2x\right\}. \tag{35}$$

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