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# Theory for Some Asynchronous Time-Division Switches

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*A two-wire active asynchronous time-division switch has recently been proposed by J. O. Dimmick, T. G. Lewis, and J. F. O'Neill. The interrupted energy transfer from one filter to another is accomplished asynchronously in order that more efficient use may be made of processing time of talker pairs on the switching bus. After first formulating, in a concise mathematical fashion, the effect of passing a signal through such a randomly time-varying circuit, we focus attention on optimizing an important filter response function. Typically, two percent rms jitter in the transfer times yields an output S/N of 30 dB independent of signal spectrum. The timing stabilization required to obtain small jitter is also discussed and an exact solution for exponential processing time is obtained. This latter result may be put to good use in studying the efficiency of the switch. Conservatively, asynchronous operation should increase traffic capacity threefold.*

*Finally, a speech wave which passes through a sample-and-hold circuit with random sampling times is considered upon being reconstructed with a fixed filter. This is a model for a four-wire active asynchronous switch, and results are compared with the two-wire situation.*

## I. INTRODUCTION

Voice switching systems have most commonly been based on controlling electromechanical switches which select and hold a spatially distinct path for each conversation. The technology used to implement such a space-division network (crossbar or ferreed switches) usually results, in practice, of an individual path having much larger bandwidth than is required for faithful transmission of the signal. The space and cost of these switches makes other solutions desirable for many applications. One line of attack has been to keep the space-division concept, but replace the electromechanical relays with semiconductor switches. These techniques, however, still suffer from the hard-to-grow nature of multistage space division networks. A more promising solution seems to be to place all the conversations on a wideband bus using time-division techniques. In fact, the 101 Electronic Switching System (ESS) is such a time-division switch. This system uses resonant energy transfer to "move" periodically measured samples of a speech wave from an incoming line to an outgoing one. In the same vein, an asynchronous time-division switch has recently been proposed by J. O. Dimmick, T. G. Lewis, and J. F. O'Neill.[1] This switching arrangement makes use of active energy transfer between filters rather than resonant energy transfer and allows a variable time slot for transferring each speech sample through the switch. The asynchronous nature of this switch allows a more efficient use of processing time than is possible to achieve synchronously. However, a consequence of this virtue is a periodic sampling of the input waves. While the synchronous switch can make use of the usual sampling theory to guarantee faithful reproduction, the asynchronous switch cannot. Further, the modifications of the theory which are required to discuss the proposed asynchronous switch are not simple, for the random sampling causes some feedback energy which is later retransmitted, further adding to the output noise. Our immediate purpose will be, then, to present theoretical work relevant to this problem. We discuss the quality of transmission [measured by the output signal-to-noise ratio (S/N)] as a function of jitter in the sample values and as a functional of a certain filter response function upon which the feedback energy depends. The optimum function is found. Also a technique suggested in Ref. 1 for keeping the jitter small is discussed theoretically, and the combined question of how jitter, quality of transmission, and increased efficiency are related is answered.

Section X summarizes our conclusions and, barring some terminology introduced in the text, may be read next.

## II. MATHEMATICAL MODEL

Consider the diagram in Fig. 1 which represents a talker and listener on a switching bus. There will be many such pairs on a particular bus, but we need now concentrate on only one. The way the switch works is that, at approximately periodic instants of time, the two identical

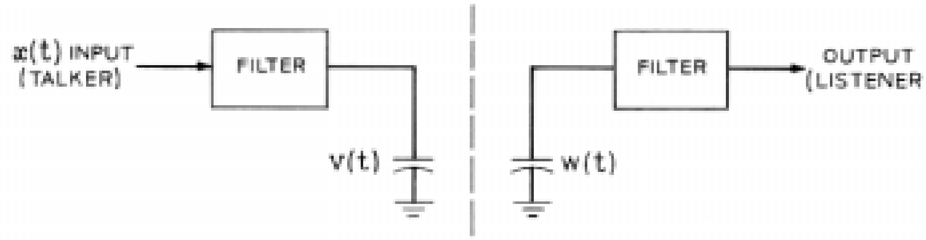


Fig. 1—Model for a "talker"—"listener" pair on an asynchronous switching bus.

capacitors which are shown in Fig. 1 are interchanged* and energy transfer is effected. The following assumptions are made:

(*i*) The input signal $x(t)$ is bandlimited to $W$ Hz and the switching occurs at times $t_n = nT + \epsilon_n$ where $T \leqq 2W$ and $\epsilon_n$ is small compared to $T$.

(*ii*) The filtering aspect of the filter is neglected in the sense that if *no* switching is performed, then $v(t) = x(t)$. In particular this means that if current impulses of area $x(nT)$ are applied at times $nT$ to the listeners capacitor, $x(t)$ will occur at the output.

(*iii*) Suppose that one volt is placed on the listening capacitor when no energy is stored in the filter. The voltage across the capacitor ($t \geqq 0$) under these conditions will be called $z(t)$, and we assume $z(0+) = 1$.

Our goal will be to determine, for given spectrum of the input, and for given second-order statistics of the $\epsilon_n$ sequence, the output S/N. The proper design of $z(t)$ will be of major concern in the analysis.

We shall call the problem just described two-wire switching.

## III. FUNDAMENTAL EQUATIONS

In this section an exact equation which describes the two-wire talker-listener switching situation will be derived. As in Fig. 1, let $v(t)$ be the actual voltage on the talker's capacitor and $w(t)$ the actual voltage on the listener's capacitor. In the absence of switching, $v(t) = x(t)$. In addition, at times $nT + \epsilon_n$ voltages $[w(nT + \epsilon_n -) - v(nT + \epsilon_n -)]$ are placed on the capacitor, where $w(t-)$ means the limiting

* Of course in reality they are not interchanged. What happens is that at the given instant of time at which the "switch" is to occur, the instantaneous voltages are measured, certain current sources (not shown) are activated and a fixed value of current flows for a duration just sufficient to effect the interchange of charges, and hence voltages, of the equal capacitors. All this occurs in a negligible period of time compared with the response time of the filters.

value of $w(t')$ as $t'$ approaches $t$ from below. Since switching occurs at times $nT + \epsilon_n$, continuity cannot be assumed at these instants and left and right limits must be distinguished. Using the definition of $z(t)$ we have by the superposition principle

$$v(t) = x(t) + \sum_{n=0}^{l(t)} [w(nT + \epsilon_n -) - v(nT + \epsilon_n -)]z(t - nT - \epsilon_n) \tag{1}$$

where $l(t)$ is the integer which satisfies

$$l(t)T + \epsilon_l \leqq t < (l(t) + 1)T + \epsilon_{l+1}.$$

Likewise for $w(t)$ we may write

$$w(t) = \sum_{n=0}^{l(t)} [v(nT + \epsilon_n -) - w(nT + \epsilon_n -)]z(t - nT - \epsilon_n). \tag{2}$$

If we let $t \rightarrow (kT + \epsilon_k -)$ in equations (1) and (2), we have the pair of equations

$$v(kT + \epsilon_k -) = x(kT + \epsilon_k) - \sum_{n=0}^{l(t)} [v(nT + \epsilon_n -) - w(nT + \epsilon_n -)] \cdot z[(k - n)T + \epsilon_k - \epsilon_n -] \tag{3}$$

and

$$w(kT + \epsilon_k -) = \sum_{n=0}^{l(t)} [v(nT + \epsilon_n -) - w(nT + \epsilon_n -)] \cdot z[(k - n)T + \epsilon_k - \epsilon_n -]. \tag{4}$$

It is very useful to now introduce the vectors $Y$, $V$, $W$ and a matrix $Z$ defined by

$$\begin{aligned} y_k &= x(kT + \epsilon_k), \\ v_k &= v(kT + \epsilon_k -), \\ w_k &= w(kT + \epsilon_k -), \end{aligned} \tag{5}$$

and

$$Z_{kn} = z[(k - n)T + \epsilon_k - \epsilon_n -]. \tag{6}$$

Note $Z_{kn} = 0$ if $k \leqq n$ since $z(t) = 0$ for $t < 0$. Using equations (5) and (6), (3) and (4) become

$$V = Y - Z(V - W), \tag{7}$$

$$W = Z(V - W). \tag{8}$$

Solving the above pair for $(V - W)$ gives*

$$(V - W) = [I + 2Z]^{-1}Y. \tag{9}$$

Equation (9) determines exactly the behavior of the switch, for the output is determined by applying the sequence $\{V_k - W_k\}$ at times $kT + \epsilon_k$ to the filter (here assumed ideal). We emphasize that the jitter enters not only through the time at which the δ-functions are applied, but also in the quantities $Z$ and $Y$.

Consider now a $z(t)$ such as that shown in Fig. 2 which passes through zero at all times $kT$, $k > 0$. If further there is no jitter, then $Z = 0$, and equations (5) and (9) show that impulses of area $x(nT)$ are applied at times $nT$, thus giving $x(t)$ at the output. In short, in the absence of jitter, any $z(t)$ which passes through zero at all positive integer multiples of the sampling interval will be optimum. One might now conclude that a $z(t)$ which is zero in a sufficiently large neighborhood of each such crossing will not see the jitter and would therefore be optimum in the presence of jitter. However the following argument (by J. F. O'Neill) suggests that such is not the case. Consider sampling a dc signal at random times. We must get enough power through the switch to reproduce the signal. Surely if we sample late, we are lagging in power and we would like to increase the area of the impulse; likewise if we sample early, we seem to be supplying extra power and so should decrease the impulse area. There should be a design of $z(t)$ which would conspire with the jitter to reduce jitter noise even below that noise obtained for the class of $z(t)$ which do not see the jitter.

## IV. WHAT IS THE OUTPUT NOISE?

Define the vector

$$\gamma \equiv V - W \tag{10}$$

and the functions

$$\psi_n(t) = \frac{\sin \frac{\pi}{T}(t - nT)}{\frac{\pi}{T}(t - nT)} \tag{11}$$

for all integer $n$. The properties of $\psi_n(t)$ that we use are

$$\psi_n(t) = \psi_{-n}(-t), \tag{12}$$

* The inverse of $(I + 2Z)$ always exists since it is triangular and all its diagonal elements are unity.

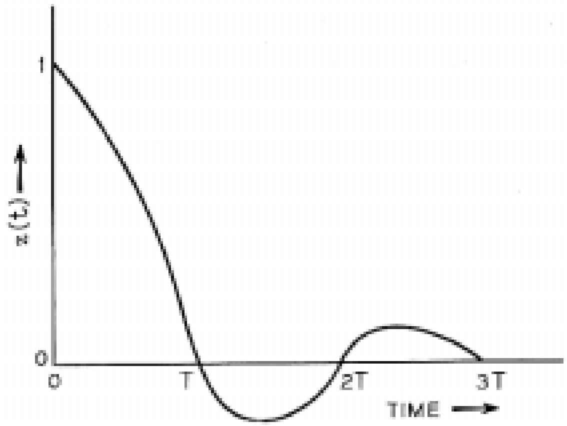


Fig. 2—An illustrative curve for $z(t)$.

$$\int_{-\infty}^{\infty} \psi_n(t)\psi_m(t)\, dt = T\delta_{nm} , \tag{13}$$

and

$$\psi_n(t_1 - t_2) = \sum_{m=-\infty}^{\infty} \psi_{n+m}(t_1)\psi_m(t_2). \tag{14}$$

We assume that the reconstructing filter impulse response is given by $\psi_0(t)$ and thus the output error signal is

$$e(t) = \sum_{n=0}^{\infty} \gamma_n \psi_n(t - \epsilon_n) - \sum_{n=0}^{\infty} x_n \psi_n(t) \tag{15}$$

and has average power* $N$ where

$$N = \lim_{k\to\infty} \frac{1}{KT} \int_{-KT}^{KT} e^2(t)\, dt = \lim_{k\to\infty} \frac{1}{KT} \int_{-\infty}^{\infty} e_K^2(t)\, dt. \tag{16}$$

In the right side of equation (16), we have introduced $e_k(t)$ which is the error signal truncated to $K$ pulses; that is, the upper limit of the sums in equation (15) is $K$ instead of infinity. If we use equation (14) to expand $\psi_n(t - \epsilon_n)$ which occurs in equation (15), find $e^2(t)$ and do the time integral, we obtain

$$\int_{-\infty}^{\infty} e_K^2(t) = T \sum_{n,k=0}^{K} \gamma_n\gamma_k \sum_{m=-\infty}^{\infty} \psi_m(\epsilon_n)\psi_{n+m-k}(\epsilon_k) + T \sum_{0}^{K} x_n^2 - 2T \sum_{n,k=0}^{K} x_n\gamma_k\psi_{n-k}(\epsilon_k). \tag{17}$$

* This corresponds to the noise in a band up to $1/2T$ Hz. If $T$ is less than the Nyquist rate, then some out-of-band noise is included here. In practice $T$ will be about half the Nyquist interval in order to make filter design problems easier, but then the inclusion of the out-of-band noise seems fair at this stage. Figure 3 shows a picture of the relevant bandwidths.

Equation (14) may be used again to recombine the first terms of the right side of equation (17) and so

$$\frac{1}{KT}\int_{-\infty}^{\infty} e_K^2(t) = \frac{1}{K}\sum_{n,k=0}^{K}\gamma_n\gamma_k\psi_{n-k}(\epsilon_k-\epsilon_n)+\frac{1}{K}\sum_{0}^{K}x_n^2 - \frac{2}{K}\sum_{n,k=0}^{K}x_n\gamma_k\psi_{n-k}(\epsilon_k). \tag{18}$$

Recall again that

$$\begin{aligned} Z_{kn} &= z[(k-n)T+\epsilon_k-\epsilon_n],\\ \gamma_k &= (1+2Z)_{kl}^{-1}y_l\ , \\ y_l &= x(lT+\epsilon_l). \end{aligned} \tag{19}$$

It is useful to define a new matrix $\theta$ by

$$\theta = \frac{2Z}{1+2Z} \tag{20}$$

so that

$$\gamma = Y - \theta Y. \tag{21}$$

Substitution of equation (21) in equation (18) splits the expressions into two types of terms. The first type, called collectively $A_K$, do not involve the filter $z(t)$ while the remaining ones, called $B_K$, do. We thus have

$$N = \lim_{K\to\infty} E[A_K] + \lim_{K\to\infty} E[B_K] \equiv A + B, \tag{22}$$

where in equation (22), the expectation is taken over both the signal and jitter statistics. The quantities $A_k$ and $B_k$ are given by

$$A_K = \frac{1}{K}\sum_{n,k=0}^{K} y_n y_k\psi_{n-k}(\epsilon_k-\epsilon_n)+\frac{1}{K}\sum_{0}^{K}x_n^2-\frac{2}{K}\sum_{n,k=0}^{K}x_n y_k\psi_{n-k}(\epsilon_k), \tag{23}$$

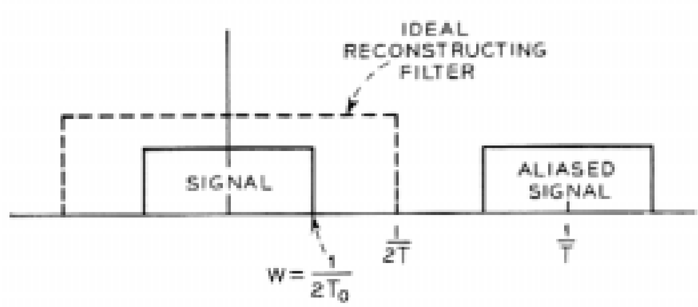


Fig. 3—Relevant bandwidths in terms of the sampling interval $T$.

$$B_K = -\frac{2}{K}\sum_{n,k=0}^{K} y_n(\theta y)_k[\psi_{n-k}(\epsilon_k - \epsilon_n) - \psi_{n-k}(\epsilon_k)]$$

$$- \frac{2}{K}\sum_{n,k=0}^{K} (\theta y)_k[y_n - x_n]\psi_{n-k}(\epsilon_k)$$

$$+ \frac{1}{K}\sum_{n,k=0}^{K} (\theta y)_n(\theta y)_k\psi_{n-k}(\epsilon_k - \epsilon_n). \qquad (24)$$

One may note that if $z(t)$ is such that when the filter is designed so as not to feel the jitter (as described in Section III), then $B = 0$ and the noise would be given by the filter independent term $A$ alone. The function of the filter design is to make $B$ as negative as possible.

To proceed further we make essential use of the fact that the $\epsilon_k$ are small. In addition, since $z(kT) = 0$ is an optimum solution in the absence of jitter we shall keep the requirement

$$z(kT) = 0, \qquad k = 1, 2, \cdots \qquad (25)$$

and see what further optimization can now be made. This will amount to designing the slopes of the function $z(t)$ when it goes through zero. The evaluation of $A$ and $B$ for small $\epsilon$ is carried out in Appendix B. Introducing the correlation function of the signal

$$R(\tau) = E[x(t)x(t + \tau)] \qquad (26)$$

and the correlation function of the jitter

$$J(k - n) = E[\epsilon_k\epsilon_n] \qquad (27)$$

we find that when the jitter is independent from sample to sample,

$$A = \lim E[A_K] = \sigma_\epsilon^2\left[-\ddot{R}(0) + \frac{1}{3}\frac{\pi^2}{T^2}R(0)\right]. \qquad (28)$$

In equation (28), $\sigma_\epsilon^2$ is the variance of the jitter and is given, for independent jitter, by

$$\sigma_\epsilon^2 = \overline{\epsilon^2} - \bar{\epsilon}^2 = J(0) - J(n \neq 0). \qquad (29)$$

To write the corresponding expression for $B_k$ we introduce, as is done in Appendix B, the derivatives $\dot{z}_s$ of the function $z(t)$ at the zeros, that is

$$\dot{z}_s \equiv \frac{d}{dt}z(t)\bigg|_{t=sT} \qquad s = 1, 2, 3, \cdots \qquad (30)$$

and also the constants*

$$\Gamma_s = \dot{\psi}_s(0) = \dot{\psi}_0(-s) = \frac{(-1)^{s+1}}{Ts}, \qquad s \neq 0. \tag{31}$$

Then, for jitter independent from sample to sample,

$$B = \lim E[B_K]$$
$$= 4R(0)\sigma_\epsilon^2\left[\sum_{s=1}^{\infty} \Gamma_s \dot{z}_s + 2\sum_{s=1}^{\infty} \dot{z}_s^2 - \sum_{s=1}^{\infty} \frac{\dot{R}(s)}{R(0)} \dot{z}_s + 2 \sum_{\substack{s,t=1\\ s>t}}^{\infty} \dot{z}_s \dot{z}_t \frac{R(s-t)}{R(0)}\right]. \tag{32}$$

Expressions for $A_k$ and $B_k$ which include the effects of correlation in the jitter are also derived in Appendix B and will be discussed in Section V. At the moment we merely state that positive correlation between jitter values will tend to reduce the noise power given by the sum of equations (28) and (32). Equation (32) gives, incidentally, more of a physical interpretation as to the breakup of the noise into $A$ and $B$ terms. The filter mentioned at the end of Section III which does not see the jitter certainly has $\dot{z}(t) = 0$ at each crossing, and thus from equation (32), $B = 0$ for that filter. Thus the $A$ term is the noise power for the "blind" filter; it will be improved upon whenever $B$ is negative.

If we now define the functional $F[z]$ of $z(t)$ by

$$F[z] = \sum_{s=1}^{\infty} \Gamma_s \dot{z}_s + 2\sum_{s=1}^{\infty} \dot{z}_s^2 - \sum_{s=1}^{\infty} \frac{\dot{R}(s)}{R(0)} \dot{z}_s + 2 \sum_{\substack{s,t=1\\ s>t}}^{\infty} \dot{z}_s \dot{z}_t \frac{R(s-t)}{R(0)}, \tag{33}$$

then optimum choice of any $\dot{z}_k$ is found by simple differentiation of equation (33):

$$\frac{\partial F[z]}{\partial \dot{z}_k} = 0 = \Gamma_k + 2\dot{z}_k - \frac{\dot{R}(k)}{R(0)} + 2\sum_{t=1}^{\infty} \dot{z}_t \frac{R(k-t)}{R(0)}, \qquad k = 1, 2, \cdots. \tag{34}$$

Thus for given signal statistics, equation (34) must be solved for the optimum set of $\{\dot{z}_k\}$. In reality, equation (34) does not have to be taken seriously for all positive integer $k$, since the response of a realistic filter will die off rapidly with time. Thus in equation (33), most of the $\dot{z}_k$ can be set to zero and only a few retained. If the first $Z$ are thus retained we shall refer to this as designing the first $Z$ zero

* Recall the definition of $\psi_s(t)$ given in equation (11). Also in equation (31), dots denote differentiation with respect to the time variable.

crossings. Of course equation (34) will hold only for the $k$ such that $\dot{z}_k \neq 0$ *a priori*.

As our first example we consider the case where the signal spectrum is flat and extends to $\Omega_{\max} = \pi/T$ rad/sec. We refer to this as the full spectrum case. The interval $T$ in this case is also the Nyquist interval, and no oversampling is done as one would expect to do in practice (the latter case will be treated shortly). We have

Full Spectrum Case:

$$\begin{aligned} R(\tau) &= R(0)\psi_0(\tau), \\ R(s) &= 0, \qquad s \neq 0, \\ \dot{R}(0) &= 0, \\ \dot{R}(s) &= R(0)\frac{(-1)^s}{Ts}, \qquad s \neq 0, \\ -\ddot{R}(0) &= R(0)\frac{1}{3}\frac{\pi^2}{T^2}. \end{aligned} \tag{35}$$

Thus equation (34) yields as the optimum solution for designing all zeros in the full spectrum case

$$\dot{z}_k = \frac{(-1)^k}{2Tk} = -\frac{\Gamma_k}{2}. \tag{36}$$

Proceeding to evaluate $A$ and $B$, we have

$$\begin{aligned} A &= R(0)\cdot\frac{2}{3}\frac{\pi^2}{T^2}\sigma_\epsilon^2, \\ B &= -R(0)\cdot\frac{1}{3}\frac{\pi^2}{T^2}\sigma_\epsilon^2. \end{aligned} \tag{37}$$

Since $R(0)$ is the signal power, we have

$$\frac{S}{N} = \frac{1}{3}\frac{\pi^2}{T^2}\sigma_\epsilon^2. \tag{38}$$

An important fact can be gleaned from equation (37): one should not generally expect the optimum filter to be very much better than the "blind" filter.* In the reasonable example which we have just worked, only a gain of 3 dB is achieved.

* Of course the blind filter is itself a highly designed filter.

TABLE I—FULL SPECTRUM

| Output S/N | Jitter Standard Deviation $\sigma_s$ |
|---|---|
| 15 dB | $0.1\,T$ |
| 21 dB | $0.05\,T$ |
| 29 dB | $0.02\,T$ |
| 35 dB | $0.01\,T$ |
| 55 dB | $0.001\,T$ |

Table I calculates equation (38) for several values of jitter. For an output S/N of 30 dB, two percent jitter is required.

## V. EFFECTS OF SIGNAL CORRELATION

Equation (34), the basic design equation for the filter response $z(t)$, can be conveniently rewritten in matrix notation

$$(I + M)\dot{z} = A \tag{39}$$

where the vectors $\dot{z}$ and $A$ have components

$$(\dot{z})_k = \dot{z}_k\,, \tag{40a}$$

$$A_k = \frac{1}{2}\left[\frac{\dot{R}(k)}{R(0)} - \Gamma_k\right] \qquad k = 1, 2, \cdots,$$

and the matrix $M$ is defined by

$$M_{kl} = \frac{R(k-l)}{R(0)}. \tag{40b}$$

The matrix $I$ is of course the identity. The dimension of all the above quantities is $Z$, where $Z$ is the number of zero crossings that one wishes to design for.

To solve equation (39) one must in general invert the matrix $(I + M)$. This was possible in the full spectrum case because $M$ was a diagonal matrix; in general it is hard to do exactly, unless the dimension $Z$ is small. A case of special interest for applications is a qualitative understanding of the situation for a flat signal bandwidth up to $\Omega_{\max} = \pi/2T$. This corresponds to sampling at twice the Nyquist rate and better approximates the situation to be encountered in practice. We call it the half spectrum case. We have

Half Spectrum Case:

$$R(sT) = R(0)\frac{2}{\pi s}\sin\frac{\pi}{2}s, \qquad s \neq 0;$$

$$\dot{R}(sT) = R(0)\left[\frac{\cos\frac{\pi}{2}s}{sT} - \frac{2}{\pi}\frac{\sin\frac{\pi}{2}s}{s^2T}\right]; \tag{41}$$

$$-\ddot{R}(0) = \frac{1}{3}\left(\frac{\pi}{2T}\right)^2 \cdot R(0).$$

The $A$ term yields, from equation (28)

$$\frac{A}{R(0)} = \frac{5}{12}\frac{\pi^2}{T^2}\sigma_\epsilon^2 = \frac{\sigma_\epsilon^2}{T^2}(4.16). \tag{42}$$

In the $B$ term we must solve equation (39). We do so by inverting the matrix by hand for the cases of designing either the first zero, the first two zeros, or the first three zeros ($Z = 1, 2, 3$ respectively). We obtain

$$\dot{z}_{\text{opt}} \underset{Z=1}{=} \frac{1}{T}[-0.409],$$

$$\dot{z}_{\text{opt}} \underset{Z=2}{=} \frac{1}{T}\begin{bmatrix}-0.454\\+0.145\end{bmatrix}, \tag{43}$$

$$\dot{z}_{\text{opt}} \underset{Z=3}{=} \frac{1}{T}\begin{bmatrix}-0.442\\+0.179\\-0.118\end{bmatrix}.$$

These numbers give for $B$

$$\frac{B}{R(0)} \underset{Z=1}{=} \frac{\sigma_\epsilon^2}{T^2}(-1.34),$$

$$\frac{B}{R(0)} \underset{Z=2}{=} \frac{\sigma_\epsilon^2}{T^2}(-1.49), \tag{44}$$

$$\frac{B}{R(0)} \underset{Z=3}{=} \frac{\sigma_\epsilon^2}{T^2}(-1.50).$$

From equations (44) and (42) we note that $Z = 1$ design improves the "blind" filter by only 1.7 dB and $Z = 2, 3$ by 1.8 dB. Output S/N are given in Table II. For a given fraction of jitter essentially the same output S/N is obtained as for the full spectrum case shown in Fig. 1. Let

TABLE II—HALF SPECTRUM*

| Output S/N | | | $\sigma_s/T$ |
|---|---|---|---|
| Z | | | |
| 1 | 2 | 3 | |
| 15.5 dB | | | 0.1 |
| 21.5 dB | S | S | 0.05 |
| 29.5 dB | A | A | 0.02 |
| 35.5 dB | M | M | 0.01 |
| 55.5 dB | E | E | 0.001 |

* The noise is here measured in a bandwidth twice the signal bandwidth.

us emphasize here that the jitter is listed as a fraction of the nominal sampling time and not in absolute units. Thus for a *fixed* signal spectrum and *fixed* jitter in *seconds*, one percent jitter for the full spectrum case would correspond to two percent for half spectrum situation.

The effect of positive correlation can be seen by comparing equation (43) with equation (36). Positive correlation tends to flatten the slopes at the zero crossings somewhat. Pursuing positive signal correlation to the utmost, we consider one more case, the case of a dc signal. In this case $\dot{R}(\tau) = 0$, and we want to solve

$$(I + M)\dot{z} = -\tfrac{1}{2}\Gamma \tag{45}$$

where

$$I + M = \begin{bmatrix} 2 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 1 & 1 & \cdots \\ 1 & 1 & 2 & 1 & \cdots \\ 1 & 1 & 1 & 2 & \cdots \end{bmatrix}. \tag{46}$$

Let us first consider the $(Z \times Z)$ version of equation (45). One may verify that for dc

$$(I + M)^{-1} = I - \frac{1}{Z + 1} U \tag{47}$$

where $U$ is the $Z \times Z$ matrix that has all its elements equal to unity. If we let $V$ equal the $Z$ dimensional vector which has all its components equal to unity, then we get

$$\dot{z} = -\frac{\Gamma}{2} + \frac{1}{2(Z + 1)T}\left[\sum_{n=1}^{Z} \frac{(-1)^{n+1}}{n}\right] V. \tag{48}$$

Finally for dc signal we have

$$\frac{A}{R(0)} = \frac{\sigma_\epsilon^2}{T^2} \cdot \frac{\pi^2}{3} \tag{49}$$

and

$$\frac{B}{R(0)} = \sigma_\epsilon^2(-4)\dot{z}'\left(-\frac{\Gamma}{2}\right)$$

which has the limiting value $(Z \rightarrow \infty)$

$$\frac{B}{R(0)} = \frac{-\sigma_\epsilon^2}{T^2} \sum_{s=1}^{\infty} \frac{1}{s^2} = \frac{-\sigma_\epsilon^2}{T^2} \frac{\pi^2}{6}. \tag{50}$$

Again we see for an exact solution that only 3 dB is gained by pursuing optimum design past the "blind" filter design. The output (S/N) is 3 dB better than equation (38); one would expect dc to be influenced less by jitter.

We note the remarkable fact that as $Z \rightarrow \infty$, equation (48) yields exactly the same solution $\dot{z}_k = -\Gamma_k/2$, as one gets for the full spectrum case. This is not true for a fixed finite Z.

Judging from our two exact and one approximate solution, *designing*

$$\dot{z}_k = -\frac{\Gamma_k}{2} \tag{51}$$

*is an optimum design independent of signal spectrum. Also the "blind" filter is a very good design independent of signal spectrum, being about only 3 dB worse than optimum.* This all assumes that the jitter is independent from sample to sample.

## VI. EFFECTS OF JITTER CORRELATION

Until now, any correlation between jitter samples has been ignored. In fact, some positive correlation is to be expected in the jitter statistics. We do not feel it is large for the way we have described the jitter in the previous sections* (call it $\epsilon$-jitter), so that our previous design is not affected. Nevertheless, we should note that any positive correlation which exists between the jitter values will improve the output S/N above that obtained by assuming that the jitter from sample to sample is independent. The physical argument as to why this should be true is quite simple. Assume $\epsilon_i$ is a nonzero constant, the same con-

* See Section VII.

stant for all $i$. Then we are sampling at time $t_n = nT + \epsilon$. From equation (19), $Z = 0$ and we are supplying current impulses of area $x(n\tau + \epsilon)$ at times $n\tau + \epsilon$ to the output filter and the signal is reconstructed perfectly. Positive correlation helps.

An important check on our work thus far can be made by assuming $\epsilon =$ const., and checking that the $A$ and $B$ terms, which for jitter are given in Appendix B, vanish. Since for this case

$$J(0) = J(s), \qquad \text{for all } s, \tag{52}$$

the $B$ term given in equation (112) obviously vanishes. The explicit demonstration of the vanishing of the $A$ term in equation (107) is not at all so obvious. Term (107), the $A$ term, for constant jitter, is seen to be proportional to [note $\dot{\psi}_0(0) = 0$]

$$-\ddot{R}(0) - \sum_s R(s)\ddot{\psi}_s(0) + 2 \sum_s R(s)\dot{\psi}_s(0)$$
$$= -\ddot{R}(0) - \sum_s R(s)\ddot{\psi}_0(sT) - 2 \sum_s \dot{R}(s)\dot{\psi}_0(sT). \tag{53}$$

Define a function of time $g(t)$ by

$$g(t) \equiv \frac{d^2}{dt^2}\,[R(t)\psi_0(t)].$$

Then the right side of equation (53) is

$$\sum_{s=-\infty}^{\infty} g(sT)$$

and by Poisson's sum formula

$$\sum_{s=-\infty}^{\infty} g(sT) = \frac{1}{T} \sum_{m=-\infty}^{\infty} G\left(\frac{2\pi}{T}\, m\right) \tag{54}$$

where $G(\omega)$ is the Fourier Transform of $g(t)$. Now $\psi_0(t)$ is band-limited to $|\omega| < \pi/T$ and so is $R(t)$. The spectrum of $R(t)\psi_0(t)$ extends only to $2\pi/T$ and in fact vanishes at $\omega = 2\pi/T$ since it is a convolution. Thus only the $m = 0$ term of equation (54) could possibly contribute, but this contribution also vanishes because the second time derivative in the definition of $g(t)$ introduces a double zero in the spectrum of $G(\omega)$ at $\omega = 0$.

Further use will be made of the expression for output noise with correlated jitter when the $\epsilon$-model for jitter is compared with another model discussed later in Section IX.

VII. TIMING STABILIZATION

At this point we return to consideration of many talker-listener pairs. Let $L$ be the number of possible such pairs. This number depends on how rapidly each talker can be processed (switched). For example, if each talker is sampled at a nominal rate of $1/T$ and $t_i$ is the processing time for the $i$th speaker, a constraint which reads something like

$$\sum_{i=1}^{L} t_i \approx T \tag{55}$$

must be imposed. Further, the timing errors $\epsilon_i$ are hopefully small. A scheme for stabilizing the sampling rate has, in fact, been proposed but T. G. Lewis.[1,2] An interpretation of this scheme due to Saltzberg and Pasternak is shown in Fig. 4. For immediate convenience let us not worry about normalization and let the dashed line have unity slope. The large dots in Fig. 4 represent a talker starting to be sampled, and let the length of time required for the $i$th sample be $t_i$. The time $t_i$ represents the length of time that the current source pumps current to effect the interchange of the talker-listener capacitors and, in accordance with our earlier assumptions about measurement times being short compared to filter bandwidths, we have $t_i \ll T$. After many counts (measurements), say $L$, we return to resample a given talker. We particularly note that a talker is never sampled early. We see that in Fig. 4 after the time $t_3$, our path hits the 45° line early, and we

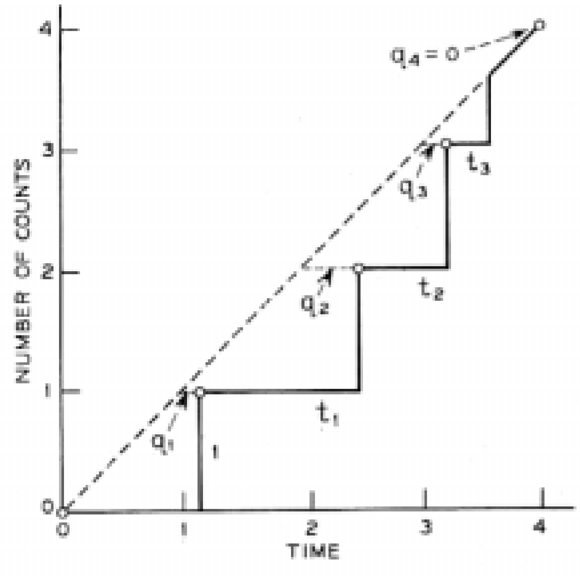


Fig. 4—Timing stabilization.

"pad" time until the fourth count occurs. Also, as shown, the variables $q_n$ represent the horizontal distance from the 45° line to the dot representing the count. They are positive random variables and are given by the equation*

$$q_n = \max \begin{cases} 0 \\ q_{n-1} + t_{n-1} - 1. \end{cases} \tag{56}$$

The random length of times $t_{n-1}$ are assumed to be identically distributed and independent of $q_{n-1}$. We shall proceed to derive an integral equation for the probability distribution function for $q_n$.† We note the possibility of a $\delta$-function at $q_n = 0$ and thus write for the density function $q_n(x)$

$$q_n(x) = \alpha_n\, \delta(x) + (1 - \alpha_n)p_n(x). \tag{57}$$

The number $\alpha_n$ is the probability of having $q_n = 0$, and $p_n(x)$ describes the continuous portion of the density. If we let $u(t)$ denote the density of $t_n$ and $*$ denote convolution, we have, using equation (56),

$$\alpha_n = \alpha_{n-1} \int_0^1 u(t)\, dt + (1 - \alpha_{n-1}) \int_0^1 [p_{n-1} * u]\, dt', \tag{58a}$$

$$d_n(x) = \alpha_{n-1} \frac{u(t + 1)}{\int_1^\infty u(t')\, dt'} + (1 - \alpha_{n-1}) \frac{\int_0^\infty p_{n-1}(t')u(x + 1 - t')\, dt'}{\int_1^\infty dt \int_0^\infty p_{n-1}(t')u(t - t')\, dt'}. \tag{58b}$$

A steady-state $\alpha$ and $p(x)$ would obey equation (58) with all indices removed. We shall write the steady-state equation. Let $K$ denote the known constant,

$$K \equiv \int_1^\infty u(t')\, dt', \tag{59}$$

let, for $x > 0$, $v(t)$ denote the known density function

$$v(t) \equiv \frac{u(t + 1)}{K}, \tag{60}$$

* Except for a time scale normalization, the sequence $\{q_{nL}\}$ $n = 1, 2 \cdots$ corresponds to the sequence $\{\epsilon_n\}$ of previous sections when there are $L$ talkers on the switch.

† B R. Saltzberg and G. P. Pasternak[3] had also derived an integral equation for $q_n$, and no doubt our equation is substantially equivalent to their equation. An entirely different approach (i.e., an equation for a different set of variables to describe the same problem) has also been discussed by J. Salz and R. D. Gitlin.[4]

and introduce the *unknown* constant

$$\rho = \int_0^1 [p * u(t')] \, dt'. \tag{61}$$

Then the steady-state solution ($\lim n \to \infty$) satisfies, if it exists,

$$\alpha = (1 - K)\alpha + (1 - \alpha)\rho, \tag{62a}$$

$$p(x) = \alpha v(x) + (1 - \alpha) \int_0^\infty p(t')v(x - t') \, dt'. \tag{62b}$$

If we let $C(\omega)$ denote the characteristic function of $p(x)$ then equation (62) yields

$$C(\omega) = \frac{\alpha V(\omega)}{1 - (1 - \alpha) V(\omega)} \tag{63}$$

where

$$\alpha = \frac{\rho}{K + \rho}. \tag{64}$$

One may now imagine determining the unknown constant $\rho$ in the following manner. Equation (63) determines $C(\omega)$ and therefore the density $p(x)$ *in terms of* $\rho$. Form the convolution of $p$ with $u$ and set the finite integral of this convolution equal to $\rho$ in accordance with equation (61). This then is an equation which can be used for the numerical determination of $\rho$. For the case

$$u(t) = \beta \exp(-\beta t) \tag{65}$$

the procedure may be carried through exactly. We find that the density $p(x)$ of the $q$ variable is also exponential and is

$$p(x) = \alpha\beta \exp(-\alpha\beta x), \tag{66}$$

where the probability $\alpha$ of having $q$ exactly zero is related to $\beta$ by

$$1 - \alpha = \exp(-\beta\alpha). \tag{67}$$

Using the inequality

$$1 - x < e^{-x}, \qquad x \neq 0,$$

we see that equation (67) has a nonvanishing solution for $\alpha$ if and only if $\beta > 1$. Realizing that the average of $t_i$ is $1/\beta$, this says the system will be stable if the average duration of $t_i$ is less than one.

From the above considerations we calculate that

$$\sigma_q^2 = \frac{1 - \alpha^2}{\alpha^2 \beta^2}. \tag{68}$$

## VIII. EFFICIENCY OF SWITCH

We propose here to give an idea of how much is gained by asynchronous (but stabilized) versus synchronous sampling. We have shown in the previous section that (introducing unnormalized quantities now) for a speech processing duration distributed exponentially, i.e.,

$$u(t) = \beta e^{-\beta t}, \tag{69}$$

we have

$$\sigma_s^2 = \frac{1 - \alpha^2}{\alpha^2 \beta^2} \tag{70}$$

where $\alpha$ and $\beta$ are related by

$$1 - \alpha = \exp\left[-\frac{\alpha\beta T}{L}\right], \tag{71}$$

which requires

$$\beta > \frac{L}{T}. \tag{72}$$

Again, $L$ is the number of talkers. To effect the comparison with the synchronous version of the above scheme a new parameter has to be introduced which represents the peak-to-average *voltage* (not power) ratio of the signal. Recall $1/\beta$ is the average processing time, and therefore represents the average voltage. In the synchronous case a maximum time, $t_{\max}$, is allowed for each talker. Clearly $t_{\max}$ is a representative of the peak voltage. The ratio $A$ is then

$$A = \beta t_{\max}. \tag{73}$$

If $L_s$ is the number of synchronous talkers, we also have

$$L_s t_{\max} = T. \tag{74}$$

Now use equation (70) to solve for $\alpha$, substitute the expression for $\alpha$ into equation (71), eliminate $\beta$ via equations (73) and (74) and obtain

$$1 - \frac{1}{\sqrt{1+Z^2}} = \exp\left[\frac{-A\left(\frac{L_S}{L}\right)}{\sqrt{1+Z^2}}\right] \tag{75}$$

where

$$Z^2 = A^2 L_S^2 \left(\frac{\sigma_s}{T}\right)^2. \tag{76}$$

An approximate solution to equation (75), which is accurate for large $L_S$, is that

$$\frac{L}{L_S} \approx A, \tag{77}$$

i.e., in the large $L_S$ limit, the ratio of possible asynchronous talkers to synchronous ones is approximately the peak-to-average voltage ratio of the speech signal. One would expect this number to be at least four. Actually for $L_S$ of interest (77) is not an accurate enough approximation. The solution of equations (75) and (76) is shown if Figs. 5 and 6 for $A = 4$ and 8 respectively. Taking $A = 4$ (which is conservative), we see from Fig. 5 that if the technology would permit 50 talkers to be put on the switch synchronously, the switch could accommodate 150 simultaneous speakers, sampled on a stabilized asynchronous manner, with an output S/N of 35 dB.* To repeat, factor of 3 increase in efficiency seems like a conservative estimate.

## IX. FOUR-WIRE CONSIDERATIONS

The last topic we consider is what we call a four-wire treatment of the problem. Here we treat the details of a model proposed by F. K. Becker[5] which should deal with active asynchronous energy transfer when four-wire facilities are available. The model is this. A speech waveform is sampled and held for a variable time $\Delta_i$ before the next sample is taken (see Fig. 7). The holding times $\Delta_i$ are independent, identically distributed, and have average value

$$\bar{\Delta}_i = T. \tag{78}$$

The initial speech waveform is approximately reconstructed by passing the jittered box car through a filter having an inband characteristic

$$h(\omega) = \frac{\frac{\omega T}{2}}{\sin\frac{\omega T}{2}} \exp\left(\frac{i\omega T}{2}\right). \tag{79}$$

* The S/N is read from Table II for $\sigma_s/T = 0.01$.

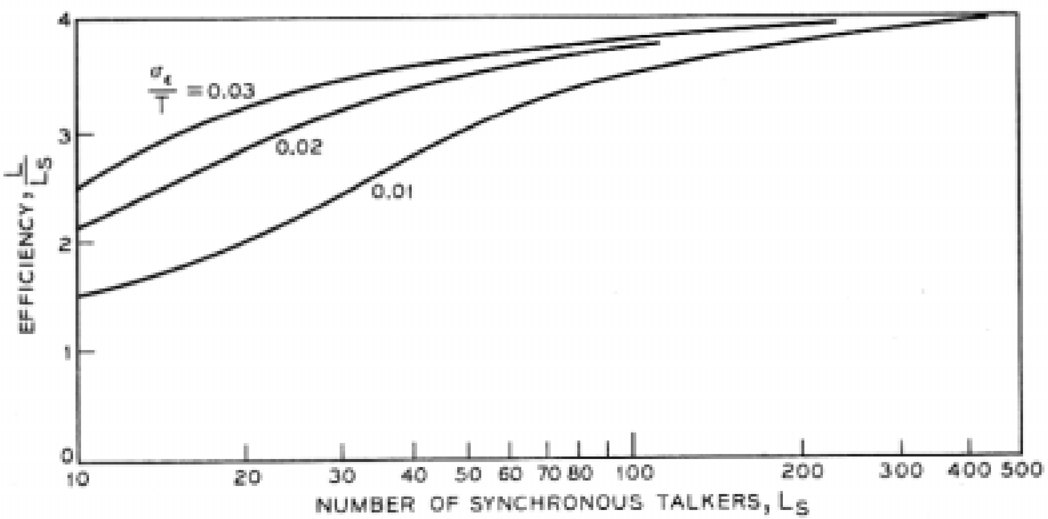


Fig. 5—Efficiency versus number of synchronous talkers for peak-to-average voltage ratio $A = 4$.

The $\Delta_i$ in the above model will not necessarily be assumed to be sharply distributed about $T$; later a stabilized version using an $\epsilon$-jitter model will be used.

Let $x(t)$ denote the speech wave and $x_J(t)$ denote the jittered box car version. Further denote the spectra of the two processes by $S(\omega)$ and $S_J(\omega)$ respectively. Our major interest shall focus on determining the spectrum $E(\omega)$ of the error

$$\text{error} = h * x_j(t) - x(t), \tag{80}$$

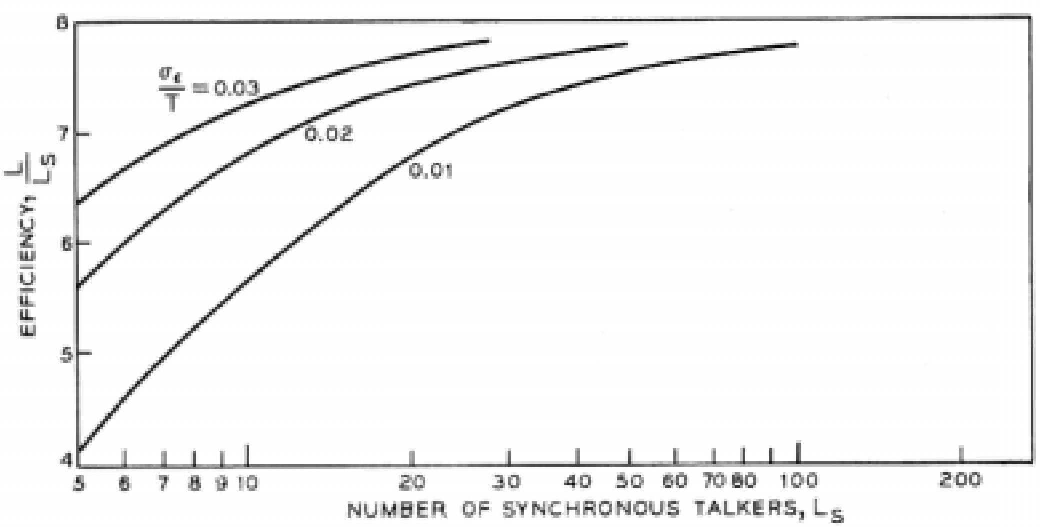


Fig. 6—Efficiency versus number of synchronous talkers for peak-to-average voltage ratio $A = 8$.

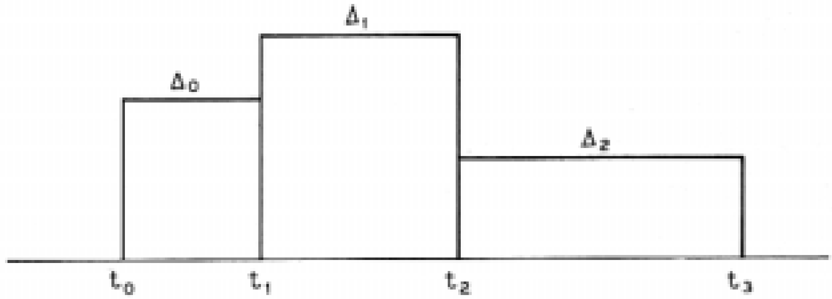


Fig. 7—"Jittered-boxcar" sampled speech wave.

assuming $h(t)$ is the reconstructing filter. The output noise power can then be obtained by integrating the spectrum. It may be shown that for the $\Delta$-model of jitter described above the *exact* error spectrum is

$$E(\omega) = S(\omega)\left|1 - \frac{i}{\omega T}h^*(\omega)[1 - C(\omega)]\right|^2 + \frac{2\,|h(\omega)|^2}{T\omega^2}\,\mathrm{Re}\left\{[1 - C^*(\omega)]P\int_{-\infty}^{\infty} S(\omega')\,\frac{1 - C(\omega')}{1 - C(\omega' - \omega)}\,\frac{d\omega'}{2\pi}\right\}. \qquad (81)$$

In equation (81), the function $C(\omega)$ is the characteristic function of the variable $\Delta$, that is,

$$C(\omega) = \int_{-\infty}^{\infty} \exp(i\omega\Delta)p(\Delta)\,d\Delta, \qquad (82)$$

$p(\Delta)$ being the probability density of $\Delta$. Also

$$h(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega t)h(t)\,dt.$$

The symbol $P$ in front of the integral in equation (81) denotes that the principal value is to be taken for the simple pole $1/[1 - C(\omega' - \omega)]$. If the term in the braces in equation (81) is rewritten as

$$\{\ \} = \int_{-\infty}^{\infty} S(\omega')\,\mathrm{Re}\left[\frac{(1 - C^*(\omega))(1 - C(\omega'))}{1 - C(\omega' - \omega)}\right]\frac{d\omega'}{2\pi}, \qquad (83)$$

the reader should be assured that now no singularity will arise in the integrand, and the principal value distinction need not be made.

For numerical purposes, we plot the error spectrum given in equations (81) and (83) for the case when $\Delta$ is gaussian distributed about mean value $T$, and for standard deviations $(\sigma_\Delta/T) = 0.316$, 0.1.*

* The random variable $\Delta$ is always positive, while the gaussian assumption allows it to become negative with some probability. We have verified that this effect is not significant here.

These larger variances are chosen here because we are mainly interested in the unstabilized version. Figures 8 and 9 show, respectively, these error spectra, assuming the input spectrum is flat up to a maximum frequency $\Omega = \pi$ rad/s, and assuming a sampling interval $T = 0.5$ s. This corresponds to sampling at twice the Nyquist rate. Also in calculating the out-of-band noise we have taken the reconstructing filter (79) to extend to *twice* the baseband spectrum. This emphasizes the higher frequencies more than desirable, perhaps. Thus in Fig. 9, the S/N for noise measured in twice the baseband interval is 24 dB while inband we have a S/N of 29 dB. Thus for independent $\Delta_i$ a 10 percent jitter about the mean value seems tolerable for sampling at about twice the Nyquist rate.*

In the event that the unstabilized four-wire version is unsatisfactory, we recalculate the error spectrum when timing is stabilized according to $\epsilon$-model discussed earlier. In this case the Fourier transform $e_L(\omega)$ of the truncated error signal is found to be given by

$$
\begin{aligned}
e_L(\omega) = \frac{i}{\omega} \sum_{n=0}^{L} x(nT + \epsilon_n) \exp\left(-i\omega nT - i\omega \frac{T}{2}\right) \\
\times \left[\exp\left(-\frac{i\omega T}{2} - i\omega\epsilon_{n+1}\right) - \exp\left(\frac{i\omega T}{2} - i\omega\epsilon_n\right)\right] \\
\times \left[\frac{\omega T}{2} \frac{1}{\sin \frac{\omega T}{2}} \exp\left(\frac{i\omega T}{2}\right)\right] - \binom{\text{same terms}}{\text{with } \epsilon = 0}. \qquad (84)
\end{aligned}
$$

Again, the reconstructing filter (79) has been assumed. Further details of the calculation will not be recorded, but we simply state that the error spectrum $E(\omega)$ is, retaining only second-order terms as in previous work involving $\epsilon$-jitter,

$$
\begin{aligned}
E(\omega) &= \lim_{L\to\infty} \frac{E\,|e^L(\omega)|^2}{L}, \\
&= \frac{T}{\sin^2 \frac{\omega T}{2}} \sum_{s=-\infty}^{\infty} \exp(-i\omega sT)\left\{\frac{\omega^2}{2} J(s)\left[R(s) - \frac{R(s+1)+R(s-1)}{2}\right]\right. \\
&\quad - \sin^2 \frac{\omega T}{2} \bar{R}(s) J(s)
\end{aligned}
$$

* Let us remark that the $\Delta$ model for jitter described here is significantly different from the $\epsilon$-model used previously. The $\Delta$ model implies positive correlation between adjacent $\epsilon$ variables and for a given variance produces less noise. We shall return to these questions shortly.

$$+ \frac{\omega}{2} \sin \frac{\omega T}{2} J(s) \left[ (-\dot{R}(s) + \dot{R}(s+1)) \exp\left(-\frac{i\omega T}{2}\right) \right.$$
$$\left. \left. - (-\dot{R}(s) + \dot{R}(s-1)) \exp\left(\frac{i\omega T}{2}\right) \right] \right\}. \tag{85}$$

One may verify that the above expression is indeed real, that is, it vanishes when $\epsilon_i = \epsilon$ and when the correlation function corresponds to a dc input signal. The noise power $N$ in any bandwidth $\Omega$ is gotten from equation (85) by

$$N = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} E(\omega)\, d\omega. \tag{86}$$

Choosing $\Omega = \pi/T$ we obtain from equation (86)

$$N = -\ddot{R}(0)J(0) + \frac{1}{T^2} \sum_{s=-\infty}^{\infty} \alpha_s\, J(s) \left[ R(s) - \frac{R(s+1) + R(s-1)}{2} \right]$$
$$+ \frac{1}{T} \sum_{s=-\infty}^{\infty} \beta_s\, J(s) \frac{\dot{R}(s+1) - \dot{R}(s-1)}{2}$$
$$- \frac{1}{T} \sum_{s=-\infty}^{\infty} \delta_s\, J(s) \left[ -\dot{R}(s) + \frac{\dot{R}(s+1) + \dot{R}(s-1)}{2} \right]. \tag{87}$$

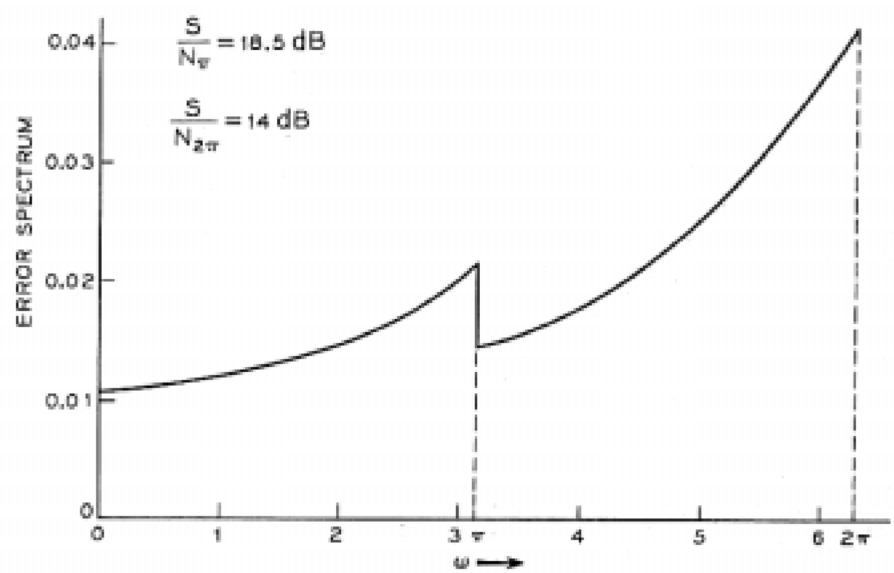


Fig. 8—Error spectrum for $T = 1/2$, $\sigma_\Delta/T = 0.316$. Undistorted spectrum is one for $-\pi \leqq \omega \leqq \pi$, and zero otherwise. $S/N_\pi = 18.5$ dB, $S/N_{2\pi} = 14$ dB.

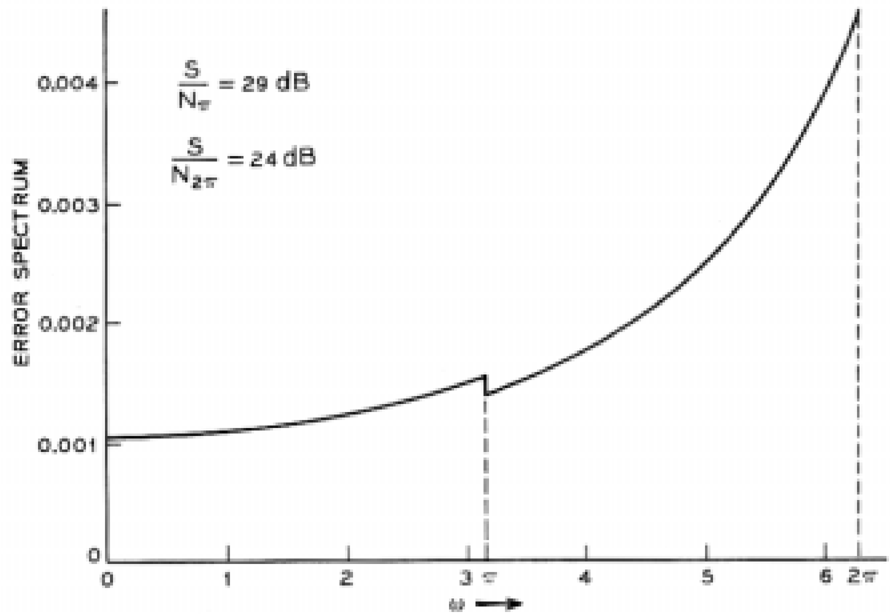


Fig. 9—Error spectrum for $T = 1/2$, $\sigma_b/T = 0.1$. Undistorted spectrum is one for $-\pi \leqq \omega \leqq \pi$, and zero otherwise. $S/N_\pi = 29$ dB, $S/N_{2\pi} = 24$ dB.

where

$$\alpha_s = \frac{1}{2\pi}\int_0^\pi \frac{\cos sx}{\sin^2 \dfrac{x}{2}}\, x^2\, dx, \tag{88a}$$

$$\beta_s = \frac{1}{\pi}\int_0^\pi x \cos sx \frac{\cos \dfrac{x}{2}}{\sin \dfrac{x}{2}}\, dx, \tag{88b}$$

$$\delta_s = \frac{1}{\pi}\int_0^\pi x \sin sx\, dx = \frac{(-1)^{s+1}}{s}, \qquad s \neq 0. \tag{88c}$$

Numerically

$$\begin{aligned} \alpha_0 &= 2\beta_0 \approx 2.78, \\ \alpha_1 &= -0.469, \\ \beta_0 &= 1.39, \\ \beta_1 &= 0.386. \end{aligned} \tag{89}$$

For the case of independent $\epsilon$-jitter, equation (87) reduces to

$$\begin{matrix}\text{noise power;} \\ \text{independent } \epsilon\end{matrix} = R(0)\sigma_\epsilon^2\left[-\frac{\ddot{R}(0)}{R(0)} + \frac{1}{T^2}\,\alpha_0\left(1 - \frac{R(1)}{R(0)}\right) + \frac{1}{T}\,\beta_0\,\frac{\dot{R}(1)}{R(0)}\right]. \tag{90}$$

The results for the stabilized full spectrum and half spectrum cases are obtained by using equations (35) and (41) in equation (90).

Full Spectrum:

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right) = \frac{\sigma_\epsilon^2}{T^2}\left[\frac{\pi^2}{3} + \beta_0\right] \approx \frac{\sigma_\epsilon^2}{T^2}(4.67). \tag{91}$$

Half Spectrum:

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right) = \frac{\sigma_\epsilon^2}{T^2}\left[\frac{\pi^2}{12} + \alpha_0\left(1 - \frac{2}{\pi}\right) - \beta_0\frac{2}{\pi}\right] \approx \frac{\sigma_\epsilon^2}{T^2}(0.946). \tag{92}$$

By comparison with equation (38), the four-wire full spectrum answer (91) is about 1.5 dB worse than the optimum two-wire result, but the four-wire result has about a 4.5-dB advantage for the half-spectrum case [compare equation (92) with equations (42) and (44)]. The latter case corresponds more to the case of practical interest. We note further that in this comparison the four-wire might be penalized by two dB or so due to the $(x/\sin x)$ characteristic being used out of band also. Secondly, the more concentrated toward dc the signal spectrum is, the better the four-wire version will become since it has vanishing distortion for dc, while the optimum two-wire version does not.

Before leaving this topic, a comment on the relation between $\Delta$-jitter and $\epsilon$-jitter should be made. We begin by looking at two successive $\Delta$-variables $\Delta_n$ and $\Delta_{n+1}$ in terms of $\epsilon$-variables.

$$\begin{aligned} \Delta_n &= T + \epsilon_n - \epsilon_{n-1}, \\ \Delta_{n+1} &= T + \epsilon_{n+1} - \epsilon_n. \end{aligned} \tag{93}$$

Clearly

$$E(\Delta) = T, \tag{94a}$$

$$\sigma_\Delta^2 = 2\overline{\epsilon^2}(1 - \rho), \tag{94b}$$

$$E[(\Delta_{n+1} - T)(\Delta_n - T)] = 2\overline{\epsilon^2}\rho - \bar{\epsilon}^2 - \overline{\epsilon^2}. \tag{94c}$$

In writing equation (94 a through c), we define

$$E[\epsilon_n\epsilon_{n+1}] = \overline{\epsilon^2}\rho \tag{95}$$

and assume no $\epsilon$-correlation after one displacement, i.e.,

$$E(\epsilon_0\epsilon_2) = \bar{\epsilon}^2.$$

To make two successive $\Delta$ variables uncorrelated, we set equation

(94c) equal to zero and obtain

$$\rho = \frac{1}{2} + \frac{1}{2}\frac{\bar{\epsilon}^2}{\overline{\epsilon^2}} \tag{96}$$

and

$$\sigma_\Delta^2 = \sigma_\epsilon^2 . \tag{97}$$

Thus to compare independent Δ-jitter with ε-jitter, a good way to do it is to choose the same variance according to equation (97) and introduce a correlation between adjacent ε's by equation (96). This simple way of comparing the two schemes is not exact, but should be good if signal correlations do not persist for extended periods. In any event, it illustrates why for $\sigma_\Delta^2 = \sigma_\epsilon^2$ the independent Δ-model will yield appreciably better results than the independent ε model. The first implies positive correlation in the second.

## X. CONCLUSIONS

We began our study with consideration of the two-wire switching problem (Fig. 1). After obtaining a concise mathematical description of the operation of this switch, [equation (9)], attention focused on the optimum capacitor discharge $z(t)$ when one volt is placed on the capacitor (Fig. 2). We have seen that a good design for the slopes $\dot{z}_k$ of the function $z(t)$ as it passes through its zeros (see Fig. 2) are the values $\dot{z}_k = (-1)^k/(2kT)$. Exact solutions for full spectrum and for dc indicate that this result is not sensitive to the signal spectrum. This optimum design can be expected to yield only a 3 dB improvement over the "blind" filter which is defined to be flat at the zero crossings and consequently does not "see" the jitter. Typically two percent jitter yields an output S/N of 30 dB. Any positive correlation in the jitter will tend to improve this figure.

When the processing time of an individual speaker is exponentially distributed, an exact distribution of the timing jitter with a stabilized clock is obtained. This result is used to study the efficiency of the asynchronous switch. This increase in capacity over a switch using the same technology but employing a synchronous strategy is, roughly, the peak-to-average voltage ratio of the signal. More accurate descriptions are presented in Figs. 5 and 6. Conservatively, the asynchronous switch should handle three times the traffic.

Finally, a four-wire version modeled as a jittered boxcar reconstructed with a filter has been considered. Under the independent

$\Delta$-model for jitter, Fig. 7, 10 percent jitter can yield 30 dB S/N, largely because of the correlations this implies for $\epsilon$-jitter. Exact error spectra have been plotted for this case also (Figs. 8 and 9). With timing stabilization, and consequently the $\epsilon$-jitter model, the reconstructed boxcar should yield several dB improvement over the two-wire results for cases of practical interest (spectrum concentrated at dc).

## XI. ACKNOWLEDGMENT

We are happy to acknowledge several discussions with J. F. O'Neill and F. K. Becker during which several of the problems dealt with here were outlined.

## APPENDIX A

### *Some Lemmas on Limits of Sums*

We list here certain lemmas which will be needed in Appendix B.

*Lemma 1: If either* $\sum_{s=0}^{\infty} |f(s)|$ *or* $\sum_{s=0}^{\infty} sf(s)$ *converges,* then*

$$\lim_{K\to\infty} \frac{1}{K} \sum_{\substack{n,k=0 \\ n\geq k}}^{K} f(n-k) = \sum_{s=0}^{\infty} f(s). \tag{98}$$

*Lemma 2: If* $\sum_{s=0}^{\infty} f(s)$ *converges, then*

$$\lim_{K\to\infty} \frac{1}{K} \sum_{n=1}^{K} \sum_{s=1}^{n} f(s) = \sum_{s=1}^{\infty} f(s). \tag{99}$$

*Likewise*

$$\lim_{K\to\infty} \frac{1}{K} \sum_{n=1}^{K} \sum_{s,t=1}^{n} f(s,t) = \sum_{s,t=1}^{\infty} f(s,t). \tag{100}$$

*Lemma 3: If* $\sum_{\substack{s=1 \\ j=-\infty}}^{\infty} f(j,s)$ *converges, then*

$$\lim_{K\to\infty} \frac{1}{K} \sum_{k=1}^{K} \sum_{s=1}^{k} \sum_{j=-k}^{K-k} f(j,s) = \sum_{\substack{s=1 \\ j=-\infty}}^{\infty} f(j,s). \tag{101}$$

## APPENDIX B

### *Derivation of Output Noise Power*

Our first step is to derive the filter independent noise, i.e.

* Neither condition implies the other; each implies the convergence of $\Sigma f(s)$.

$$A = \lim_{K \to \infty} A_K$$

where $A_K$ is given by equation (23). Our expressions are valid to second order in the $\{\epsilon_l\}$. We note the following relations

$$y_n = x(nT + \epsilon_n) \approx x(nT) + \epsilon_n \dot{x}(nT) + \tfrac{1}{2}\epsilon_n^2 \ddot{x}(nT); \qquad (102)$$

$$\psi_m(\epsilon_l) \approx 1 + \ddot{\psi}_0 \frac{\epsilon_l^2}{2}, \qquad \text{if} \quad m = 0;$$

$$\approx \dot{\psi}_m(0)\epsilon_l + \tfrac{1}{2}\ddot{\psi}_m(0)\epsilon_l^2/2, \qquad \text{if} \quad m \neq 0; \qquad (103)$$

$$\dot{\psi}_k(0) = -\dot{\psi}_{-k}(0);$$

making use of this in equation (23), we have, to second order in $\{\epsilon_l\}$

$$A_K = \frac{1}{K} \sum_{n=0}^{K} (\epsilon_n^2 \dot{x}_n^2 - x_n^2 \ddot{\psi}_0(0)\epsilon_n^2)$$
$$- \frac{1}{K} \sum_{\substack{n,k=0 \\ n \neq k}}^{\infty} [x_n x_k \epsilon_n \epsilon_k - 2x_k \dot{x}_n \epsilon_n \epsilon_k \dot{\psi}_{n-k}(0)]. \qquad (104)$$

Averaging over the signal statistics by using equation (26) and the further relations

$$E[\dot{x}(t + \tau)x(t)] = \dot{R}(\tau) = -E[\dot{x}(t)x(t + \tau)],$$
$$E[\dot{x}(t + \tau)\dot{x}(t)] = -\ddot{R}(\tau), \qquad (105)$$

and also averaging over the jitter variables using equation (27) gives:

$$EA_K = \frac{1}{K} \sum_{n=0}^{K} [-\ddot{R}(0)J(0) - \ddot{\psi}_0(0)J(0)R(0)]$$
$$- \frac{1}{K} \sum_{\substack{n,k=0 \\ n \neq k}}^{K} [R(n-k)J(n-k)\ddot{\psi}_{n-k}(0) - 2\Gamma_{n-k}J(n-k)\dot{R}(n-k)]. \qquad (106)$$

Observing that the second sum is a symmetric function of $(n - k)$ and using Lemma 1 of Appendix A, the required limiting operation yields

$$A = -\ddot{R}(0)J(0) - \sum_{s=-\infty}^{\infty} R(s)J(s)\ddot{\psi}_s(0) + 2 \sum_{\substack{s=-\infty \\ s \neq 0}}^{\infty} \Gamma_s J(s)\dot{R}(s). \qquad (107)$$

We list for further possible use

$$\ddot{\psi}_s(0) = 2\,\frac{(-1)^{s+1}}{s^2T^2}\,, \qquad s \neq 0;$$
$$\ddot{\psi}_0(0) = -\frac{1}{3}\frac{\pi^2}{T^2}\,; \tag{108}$$

and also

$$\sum_{s=-\infty}^{\infty} \ddot{\psi}_s(0) = 0. \tag{109}$$

The latter relation may be derived by direct evaluation of the sum, of course, or by using the Poisson sum formula and the bandlimited nature of $\psi(t)$.

We now proceed to derive an expression for the filter dependent terms $B$, starting from equation (24). Gathering the second-order terms proceeds much as it did for the $A_k$ terms, with the following additional complication. Recall from equation (20) that $\theta$ is given by

$$\theta = \frac{2Z}{1 + 2Z} = 2Z - 4Z^2 + 8Z^3 - \cdots$$

From equation (6),

$$Z_{kn} = (\epsilon_k - \epsilon_n)\dot{z}_{k-n} + \begin{pmatrix}\text{higher}\\ \text{terms}\end{pmatrix}, \qquad \text{if} \quad k > n;$$
$$= 0, \qquad\qquad \text{if} \quad k \leq n.$$

Thus the matrix $\theta$ is at least linear in the $\{\epsilon_i\}$. Proceeding now to collect terms and averaging, we have to second order

$$\begin{aligned} E[B_K] = {} & \frac{4}{K}\sum_{n=1}^{K}\sum_{m,l=0}^{n-1} \dot{z}_{n-m}\dot{z}_{n-l}R(l - m) \\ & \cdot[J(0) + J(m - l) - J(n - m) - J(n - l)] \\ & - \frac{4}{K}\sum_{n=1}^{K}\sum_{m=0}^{n-1} \dot{z}_{n-m}\dot{R}(n - m)[J(0) - J(n - m)] \\ & + \frac{4}{K}\sum_{\substack{n,k=1\\ n\neq k}}^{K}\sum_{m=0}^{k-1} \Gamma_{n-k}\dot{z}_{k-m}R(n - m)[J(n - k) - J(n - m)]. \end{aligned} \tag{110}$$

In the second term of the above, we change summation variables by replacing the sum over $m$ with a sum over $s$, where

$$s = n - m, \qquad s = 1, 2, \cdots, n.$$

We make the same change in the first term, and also introduce

$$t = n - l, \qquad t = 1, 2, \cdots, n.$$

In the last term let

$$\begin{aligned} s &= k - m, \qquad s = 1, 2, \cdots, k, \\ j &= n - k, \qquad j = -(k-1), \cdots, K - k, \\ & \qquad\qquad\quad j \neq 0. \end{aligned}$$

Then

$$\begin{aligned} E[B_K] = \frac{4}{K} \sum_{n=1}^{K} \sum_{s,t=1}^{n} \dot{z}_s \dot{z}_t R(s-t)[J(0) + J(s-t) - J(s) - J(t)] \\ - \frac{4}{K} \sum_{n=1}^{K} \sum_{s=1}^{n} \dot{z}_s \dot{R}(s)[J(0) - J(s)] \\ + \frac{4}{K} \sum_{k=1}^{K} \sum_{s=1}^{k} \sum_{\substack{j=-k+1 \\ j \neq 0}}^{K-k} \Gamma_j \dot{z}_s R(j+s)[J(j) - J(j+s)]. \end{aligned} \tag{111}$$

Making use of Lemmas 2 and 3 of Appendix A yields

$$\begin{aligned} B = \lim_{K \to \infty} E[B_K] = -4 \sum_{s=1}^{\infty} \dot{z}_s R(s)[J(0) - J(s)] \\ + 4 \sum_{s,t=1}^{\infty} \dot{z}_s \dot{z}_t R(s-t)[J(0) + J(s-t) - J(s) - J(t)] \\ + 4 \sum_{s=1}^{\infty} \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \Gamma_j \dot{z}_s R(j+s)[J(j) - J(j+s)]. \end{aligned} \tag{112}$$

The simpler equations (28) and (32) which hold for independent jitter may be gotten from the above by using the result of Section VI. Namely for $J(s) = \text{const.}$, all $s$, the $A$ and $B$ terms vanish. When we have independent jitter $J(s) = \text{const.}$, $s \neq 0$. So we add and subtract a term with $J(0)$ replaced by $J(s \neq 0)$ and use equation (29) to obtain the independent jitter results.

## APPENDIX C

### *Exact Error Spectrum for Independent Δ-Jitter*

Consider a long time $L$, and denote by $[L]$ the best integer so that

$$L \cong [L]T. \tag{113}$$

The jittered box car (truncated to the long interval $L$) can then be represented as

$$x_J(t) = \sum_{n=0}^{[L]} x(t_n)B[t_{n+1}, t_n], \tag{114}$$

where $B$ is a unit box extending from $t_n$ to $t_{n+1}$. The Fourier transform of this truncated version is then

$$s_L(\omega) = \frac{i}{\omega} \sum_{n=0}^{[L]} x(t_n) \exp(-i\omega t_n)[\exp(-i\omega \Delta_n) - 1]. \tag{115}$$

The power spectra for a process $y(t)$ is generally calculated according to

$$G(\omega) = \lim_{L\to\infty} \frac{1}{L} E\, |y^L(\omega)|^2,$$

where $y^L(\omega)$ is the Fourier transform of $y(t)$ truncated to an interval $L$, and $E$ denotes an ensemble average with respect to all random parameters. If we apply the above to spectrum of the error signal (80), we obtain immediately, using notation of the text,

$$E(\omega) = |\, h(\omega)\, |^2\, S_J(\omega) + S(\omega) + D(\omega) \tag{116}$$

where

$$D(\omega) = -\lim_{[L]\to\infty} \frac{1}{[L]T} \cdot 2E \text{ Re } h^*(\omega) x_J^{L^*}(\omega) x^L(\omega). \tag{117}$$

Expressing $x^L(\omega)$ as a time integral and performing the signal averaging we get

$$D(\omega) = -\lim \frac{2}{[L]T} \text{ Re } h^*(\omega) \left\{ \frac{-i}{\omega} \sum_{n=0}^{[L]} \exp(i\omega t_n)[\exp(i\omega \Delta_n) - 1] \right. \\ \left. \times \int_{-\infty}^{\infty} \exp(-i\omega t') R(t' - t_n)\, dt \right\}. \tag{118}$$

Expressing the signal correlation function in terms of the spectrum, doing the integrals and remaining averages, gives

$$D(\omega) = \frac{-2}{T} \text{ Re } h^*(\omega) \left(\frac{i}{\omega}\right) S(\omega)[1 - C(\omega)]. \tag{119}$$

There remains the calculation of $S_J(\omega)$. Using equation (115), we have (performing averages over signals)

$$E\,|s^L(\omega)|^2 = \frac{1}{\omega^2} E \sum_{n,m=0}^{[L]} R\left[\sum_{k=0}^{n-1} \Delta_k - \sum_{k=0}^{m-1} \Delta_k\right]$$
$$\times [\exp(-i\omega\,\Delta_n) - 1](\exp(i\omega\,\Delta_m) - 1)\left[\exp - i\omega\left(\sum_{k=0}^{n-1} \Delta_k - \sum_{k=0}^{m-1} \Delta_k\right)\right] \tag{120}$$

where we note the fact

$$t_{n+1} = \sum_{k=0}^{n} \Delta_n. \tag{121}$$

Performing the $\Delta$-averaging yields

$$\frac{E\,|s^L(\omega)|^2}{[L]T} = \frac{R(0)}{[L]T\omega^2}\, 2[1 - \operatorname{Re} C(\omega)] \sum_{n=0}^{[L]} 1$$
$$+ \frac{2}{[L]T\omega^2} \operatorname{Re}\left\{[1 - C^*(\omega)] \sum_{n=m+1}^{[L]} \sum_{m=0}^{[L]}\right.$$
$$\left.\cdot \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, S(\omega')C(\omega' - \omega)\cdot[C(\omega' - \omega) - C(\omega')]\right\}. \tag{122}$$

To perform the limits of the sums we first replace $C^{n-m-1}(\omega' - \omega)$ by $[\exp(-\epsilon)C^{-n-m-1}(\omega' - \omega)]$ to insure convergence and then let $\epsilon \to 0$. We then have

$$S_J(\omega) = \lim_{[L]\to\infty} \frac{1}{[L]T} E\,|s^L(\omega)|^2$$
$$= \frac{2}{\omega^2 T} \operatorname{Re}\left\{[1 - C^*(\omega)] \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S(\omega') \frac{1 - C(\omega')}{1 - C(\omega' - \omega)e^{-\epsilon}}\right\}. \tag{123}$$

Making use of the general relation

$$\frac{1}{x \pm i\epsilon} = P\,\frac{1}{x} \mp i\pi\,\delta(x),$$

where $P$ denotes principal value, finally yields

$$S_J(\omega) = \frac{|1 - C(\omega)|^2}{\omega^2} \frac{S(\omega)}{T^2}$$
$$+ \frac{2}{T\omega^2} \operatorname{Re}\left\{[1 - C^*(\omega)]P \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, S(\omega') \frac{1 - C(\omega)}{1 - C(\omega' - \omega)}\right\}. \tag{124}$$

The interested reader may indeed check that for

$$C(\omega) = \exp(i\omega T),$$

the above expression reproduces the known result for a wave-form impulse modulated at precisely $T$-second intervals.

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