THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 50, No. 4, April, 1971
Printed in U.S.A.

Maximum Power Transmission Between Two Reflector Antennas in the Fresnel Zone

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(Manuscript received August 11, 1970)

Power transfer between two ellipsoidal reflector antennas with common focal points and dual-mode feeds has been investigated. Assuming a circularly symmetric feed pattern, the H-plane pattern of an open-end circular waveguide excited by the TE_{11} mode, the maximum transmission coefficient between reflector apertures is found to be within 0.5 percent of the value computed for transmission between two circular apertures with optimum illumination described by the generalized prolate spheroidal function. Transmission loss between two reflector antennas versus illumination taper is computed for various values of the parameter $[p = (ka_1a_2)/R]$. The minimum transmission loss is obtained as a compromise between feed spill-over and aperture transmission efficiency.

In view of the inconvenience of building different ellipsoidal reflectors for different antenna spacings in order to achieve maximum power transfer, we examine the feasibility of using a defocused ellipsoid to simulate ellipsoids of different focal lengths. The deviation of the aperture phase distribution of a defocused ellipsoid from a required spherical phase front is approximately given by an explicit expression. A simple upper bound of the transmission loss due to small phase deviation is obtained for a given maximum phase deviation.

I. INTRODUCTION

Millimeter waves have never been used in long-haul radio transmission systems because of rain attenuation. However, wideband transmission over short span (say less than 1 km) can be accomplished by millimeter wave systems, for example, the *Picturephone*® distribution in large cities proposed by R. Kompfner. The very high transmission efficiency may leave a large margin for rain attenuation to insure high reliability. Here the transmitting and receiving antennas may be within the Fresnel zones of each other.

Maximum power transfer between two apertures in the Fresnel region takes place when the field distribution appears as the lowest order mode of a confocal open resonator.[1, 2] Two ellipsoidal reflectors with common focal points may provide the required aperture phase distribution which is a spherical wave front with the center of curvature located at the center of the other aperture. The optimum amplitude distribution is a prolate spheroidal function for a rectangular aperture or a generalized prolate spheroidal function for a circular aperture. S. Takeshita[3] has shown that truncated gaussian illumination, which is asymptotically identical to the generalized prolate spheroidal function, may give a transmission efficiency between two circular apertures almost as good as that using generalized prolate spheroidal illumination. However, the truncated gaussian distribution only looks simpler in terms of mathematical manipulation while its practical realization appears not any easier than that of prolate speroidal functions. Furthermore, if a lens or reflector is used to produce the spherical phase front, the feed spill-over must be taken into account. The maximum power transfer between the feeds of two reflector antennas will be obtained as a compromise between feed spill-over loss and aperture transmission efficiency. This procedure is similar to optimizing the gain of a paraboloidal antenna. The aperture blocking effect can be made very small by using the periscope type structure[4] which virtually eliminates the feed supports. This paper will present the calculated results of Fresnel zone transmission between two ellipsoidal reflector antennas with dual-mode feeds. In view of the inconvenience of building different ellipsoidal reflectors for different antenna spacings, we will also examine the feasibility of using a defocused ellipsoid to simulate ellipsoids of different focal lengths.

II. MAXIMUM POWER TRANSFER

Neglecting the interaction between the antennas and assuming that the tangential components of the electric and magnetic fields are related by the free space impedance at each point of the two apertures A_1 and A_2, the ratio of the received to transmitted power between two apertures* at any separation can be shown[5] to be

* Not to be confused with transmission between two antennas which includes spill-over of the feeds.

$$\frac{P_R}{P_T} = \frac{\left| \int_{A_1} \int_{A_2} E_1^t \frac{\exp(-jkL)}{L} E_2^t \, ds_1 \, ds_2 \right|^2}{\lambda^2 \left\{ \int_{A_1} | E_1^t |^2 \, ds_1 \right\} \left\{ \int_{A_2} | E_2^t |^2 \, ds_2 \right\}} \tag{1}$$

where E_1^t and E_2^t are tangential components of the aperture field distributions when A_1 and A_2 are transmitting respectively. Using the small angle Fresnel approximation, the distance L may be approximated by

$$\begin{aligned} L &= [R^2 + (x - \xi)^2 + (y - \eta)^2]^{\frac{1}{2}}, \\ &\approx R + \frac{(x - \xi)^2 + (y - \eta)^2}{2R}, \end{aligned} \tag{2}$$

where the coordinate system is illustrated in Fig. 1. Then the near field power transmission formula becomes

$$T = \frac{P_R}{P_T} = \frac{\left| \int_{A_1} \int_{A_2} E_1 \exp\left[j \frac{k(x\xi + y\eta)}{R} \right] E_2 \, ds_1 \, ds_2 \right|^2}{R^2 \lambda^2 \left\{ \int_{A_1} | E_1 |^2 \, ds_1 \right\} \left\{ \int_{A_2} | E_2 |^2 \, ds_2 \right\}} \tag{3}$$

where

$$E_1 = E_1^t \exp\left[-j \frac{k(x^2 + y^2)}{2R} \right], \tag{4a}$$

$$E_2 = E_2^t \exp\left[-j \frac{k(\xi^2 + \eta^2)}{2R} \right]. \tag{4b}$$

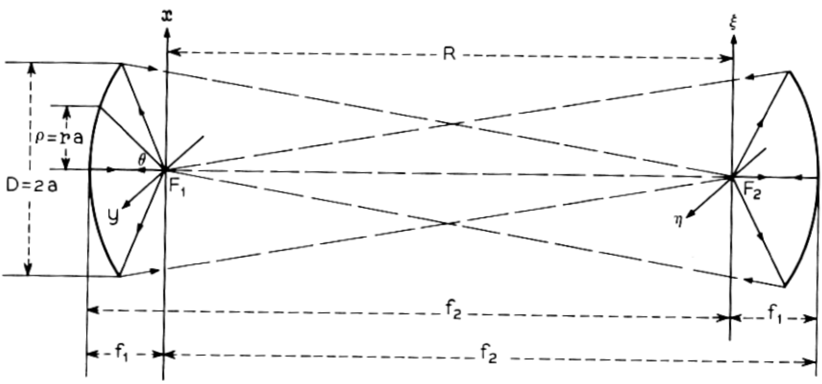

Fig. 1—Two ellipsoidal reflectors with common focal points.

If the aperture distributions are circularly symmetric, equation (3) can be reduced to

$$\frac{T}{p} = \frac{\left| \int_0^1 \int_0^1 E_1(r)\sqrt{r}\, E_2(r')\sqrt{r'}\, J_0(prr')\sqrt{prr'}\, dr\, dr' \right|^2}{\left\{\int_0^1 | E_1(r)\sqrt{r} |^2\, dr\right\}\left\{\int_0^1 | E_2(r')\sqrt{r'} |^2\, dr'\right\}} \tag{5}$$

where $p = (ka_1a_2)/R$ for two circular apertures of radii a_1 and a_2. One notes that the transmitting and receiving apertures may have different radii, while the optimum illumination function is identical for the two apertures. When both E_1 and E_2 are real, the Schwartz principle gives the following condition for the maximization of equation (5)

$$\sqrt{\frac{T}{p}}\, E(r)\sqrt{r} = \int_0^1 E(r')\sqrt{r'}\, J_0(prr')\sqrt{prr'}\, dr'. \tag{6}$$

The subscript of E has been dropped because equation (6) is satisfied by both E_1 and E_2. The above integral equation has been thoroughly investigated by D. Slepian[6] and $\phi(r) = E(r)\sqrt{r}$ is designated as a generalized prolate spheroidal function. The optimum transmission coefficient T taken from Slepian's work is given in Table I for reference.

Now a simple way of approximately realizing the optimum aperture distribution appears to be the illumination of an ellipsoidal reflector by a dual-mode feel. In the vicinity of the reflector, the reflected field will follow the geometrical optics rays which are pointed toward the remote focal point as shown in Fig. 1, and thus the required spherical phase front will be created in the aperture. An experiment on dual-mode apertures[7], one to two wavelengths in diameter, showed the measured patterns to be circularly symmetric and essentially in agreement with the H-plane pattern of an open-end circular waveguide excited by TE_{11} mode, i.e.,

$$F(\theta) = \left[\sqrt{1 - \left(\frac{1.841}{u}\right)^2} + \cos\theta\right] \frac{J_1'(u \sin\theta)}{1 - \left(\frac{u \sin\theta}{1.841}\right)^2} \tag{7}$$

where u is the circumference of the waveguide in wavelengths. Since the ellipsoidal reflector is very closely a defocused paraboloid, as will be shown in the next section, the space attenuation factor needed for the variable distance from the feed to the reflector surface is essentially the same as that of a paraboloid. Then the aperture distribution is related to the feed pattern by

$$E(r) = F(\theta) \cos^2 \frac{\theta}{2} \tag{8}$$

where $ar/2f = \tan \theta/2$ and f is the focal length of the reflector. The combination of equations (7) and (8) can be used to determine the value of u for any corresponding illumination taper as plotted in Fig. 2. Substituting equation (8) into equation (5), numerical integration will give the transmission coefficient between two ellipsoidal reflector apertures excited by dual-mode feeds. The computed data have been plotted in Fig. 3 for $f/D = 0.5$ and various values of the parameter p. The maximum efficiency using dual-mode feed and excluding spill-over has been tabulated in Table I for comparison with the optimum efficiency using amplitude illumination of generalized prolate spheroidal functions.

The agreement between the second and third columns in Table I is indeed excellent. The insensitivity of the transmission coefficient to small differences in illumination is not surprising in view of the stationary property of equation (5). Table I does not show the maximum efficiency with dual-mode feed for $p = 2$ and 10, because the mathematical model in equation (7) for the dual-mode pattern has not been experimentally verified for the circumference of the waveguide outside the range $3 \leqq u \leqq 6$. However, this model covers the most interesting range and demonstrates that efforts to synthesize a truncated gaussian distribution are unwarranted. As far as maximum power transfer between two apertures is concerned, any aperture distribution which resembles the generalized prolate spheroidal function may achieve practically the optimum transmission efficiency. As an example of tolerable discrepancy, the dual-mode feed illumination function of a taper strength which maximizes the transmission coefficient for $p = 5$ is compared with the corresponding generalized prolate spheroidal function

Table I—Power Transfer Efficiency of Two Circular Apertures

$p = ka_1a_2/R$	Optimum Illumination Efficiency	Dual Mode Illumination Maximum Efficiency
2.0	0.630	
3.0	0.887	0.886
4.0	0.975	0.972
5.0	0.995	0.992
10.0	1.000	

in Fig. 4. One notes that the over-all similarity is sufficient to achieve transmission coefficients differing by only 0.3 percent while the edge illuminations differ by more than 3 dB.

Next we turn our attention to the spill-over loss of a feed. If the reflector is illuminated by a circularly symmetric feed pattern $F(\theta)$, then the fraction of the energy intercepted by the reflector will be

$$\alpha = \int_0^{\theta_0} [F(\theta)]^2 \sin\theta \, d\theta \bigg/ \int_0^{\pi/2} [F(\theta)]^2 \sin\theta \, d\theta \tag{9}$$

where θ_0 is the half angle subtended by the reflector at the focus and the back lobes have been neglected. Substituting equation (7) into equation (9), we compute the total spill-over loss in decibels from $2[10 \log_{10} (1/\alpha)]$ as shown in Fig. 3. The presence of two feeds in the system accounts for the factor two. It is seen that the spill-over loss is as important as the power transfer loss between two reflector apertures in determining the total transmission loss between two reflector antennas. The minimum total transmission loss between feeds will always be greater than the power transfer loss between the reflector apertures, and always occurs at an illumination taper stronger than that for the maximum transmission between two apertures. Any slight decrease in transmission loss due to deviation of the aperture distribution from the generalized prolate spheroidal function will certainly be swamped by the feed spill-over loss. Figure 3 indicates that the illumination taper corresponding to the maximum power transfer decreases as the parameter p decreases. As the distance between two reflectors increases toward the far zone condition $p = 0$, the optimum illumination taper

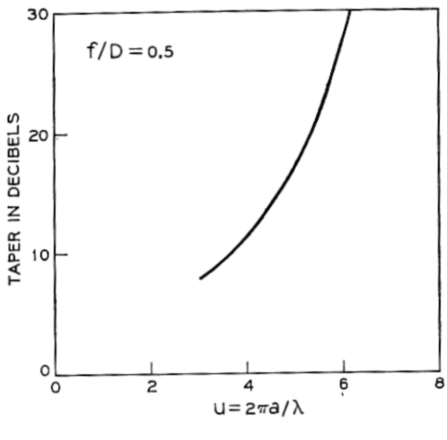

Fig. 2—Relation between feed pattern parameter and taper.

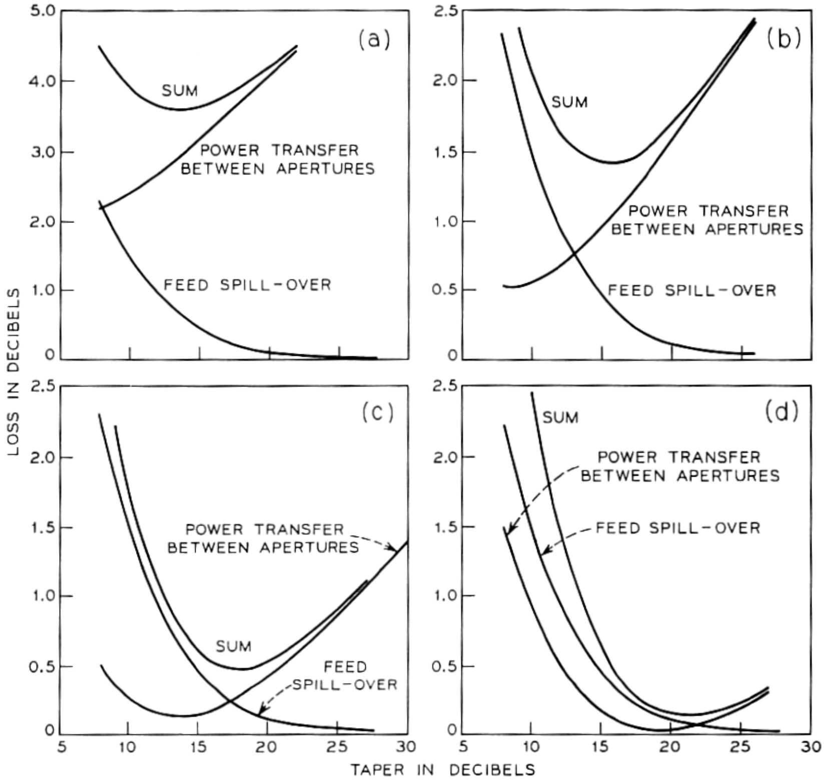

Fig. 3—Power transfer between ellipsoidal reflector antennas with dual mode feeds. (a) $ka_1a_2/R = 2$, (b) $ka_1a_2/R = 3$, (c) $ka_1a_2/R = 4$, (d) $ka_1a_2/R = 5$.

will approach 12 dB which maximizes the Fraunhofer gain of a paraboloidal antenna.[8] One notes that 12-dB aperture taper corresponds to 10-dB feed pattern taper, allowing a 2-dB space attenuation factor, for 0.5 f/D ratio. When f/D decreases to smaller values, the spill-over loss curve will be shifted to the right because of increasing space attenuation factor. Then the minimum of the sum will be also moved to the right and upward. Large f/D ratio which requires narrow feed pattern may cause significant loss due to aperture blocking which has been neglected in the above calculations.

III. OPTIMUM DEFOCUSING OF AN ELLIPSOID

In the preceding section we have shown that ellipsoidal reflectors are needed for maximum power transfer between two reflector antennas in the Fresnel region. The required focal lengths of the ellipsoidal re-

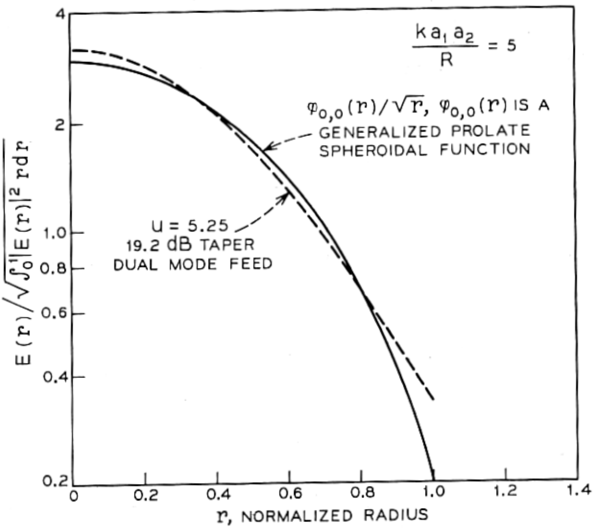

Fig. 4—Comparison between optimum aperture distributions.

flectors are determined by the distance between the reflectors and the desired f/D ratio. It is obviously inconvenient to build different ellipsoidal reflectors for different antenna spacings; however, approximating ellipsoids of different focal lengths by a defocused ellipsoid can be an attractive possibility in some practical situations. The necessary displacement of the feed along the axis will be first determined. Then we will calculate the approximate phase deviation between the wave front reflected from a defocused ellipsoid and the spherical wave front in the aperture of a required ellipsoidal reflector. An upper bound of the transmission loss will be obtained for a given maximum phase deviation.

The equation of an ellipsoid can be written as

$$Z = \frac{f + R}{2}\left[1 - \sqrt{1 - \frac{\rho^2}{fR}}\right] \tag{10}$$

where f and R* are the two focal lengths. Now let us consider another ellipsoid with corresponding focal lengths f' and R'

$$Z = \frac{f' + R'}{2}\left[1 - \sqrt{1 - \frac{\rho^2}{f'R'}}\right]. \tag{11}$$

* R may be taken as any distance between $f_2 - f_1$ and f_2, where f_1 and f_2 are the two focal lengths of the ellipsoid. This slight ambiguity results from the aperture approximation for reflector antenna, and is unimportant provided that f_2 is orders of magnitude greater than f_1.

These two ellipsoids coincide at the tip $\rho = 0$. We will impose the condition that the two ellipsoids also coincide at the edge $\rho = a$ as shown in Fig. 5. Taking the first three terms of the binomial expansion of the square root in equations (10) and (11), we obtain an expression for the required defocus distance:

$$\epsilon = f' - f = \frac{f^2(R - R')}{RR'} \left[1 + \left(\frac{a}{2f}\right)^2\right] \tag{12}$$

in which the approximations $f \ll R$ and $f' \ll R'$ have been used. Substituting equation (12) into equations (10) and (11), the approximate deviation between the two ellipsoids becomes

$$\Delta Z = \frac{1}{4}\left(\frac{a}{2f}\right)^2 r^2(1 - r^2)\, \frac{a^2(R - R')}{RR'} \tag{13}$$

where $r = \rho/a$ is the normalized radius. Multiplying equation (13) by the factor $(1 + \cos\theta)$ yields the phase deviation between the two wave fronts reflected from the two ellipsoids as shown in Fig. 5. Since these ellipsoids differ little from paraboloids, the relation $\tan\theta/2 = ar/2f$ is approximately valid. Then one arrives at the following expression

$$\frac{\delta}{\lambda} = \left[\tfrac{1}{2}r^2(1 - r^2)\, \frac{\left(\frac{a}{2f}\right)^2}{1 + \left(\frac{a}{2f}\right)^2 r^2}\right] \frac{a^2}{\lambda R'}\, \frac{R - R'}{R}. \tag{14}$$

The last factor in equation (14) is always less than unity when $R > R'$. This factor approaches unity when $R \to \infty$, i.e., approximating an ellipsoid by a defocused paraboloid. The quantity inside the bracket of equation (14) is the phase deviation[9] of a wave front produced by a

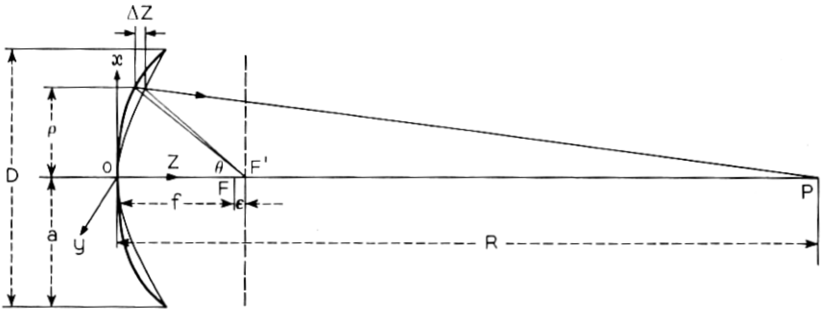

Fig. 5—Geometry of defocused ellipsoid.

defocused paraboloid from a desired spherical wave front for a Fresnel number ($a^2/\lambda R$) of unity, and has been plotted in Fig. 6 for various values of f/D ratio. The maximum phase deviation is found to be located at

$$\rho^2 = 4f^2\left[\sqrt{1 + \left(\frac{a}{2f}\right)^2} - 1\right]. \tag{15}$$

When $(a/2f)^2 \ll 1$, the above equation can be reduced to

$$\rho^2 = \frac{a^2}{2}\left[1 - \frac{1}{4}\left(\frac{a}{2f}\right)^2\right]. \tag{16}$$

If the variation of antenna spacing covers a range from R_1 to R_2, the optimum ellipsoid for minimizing the phase deviation of equation (14) can be found by equating

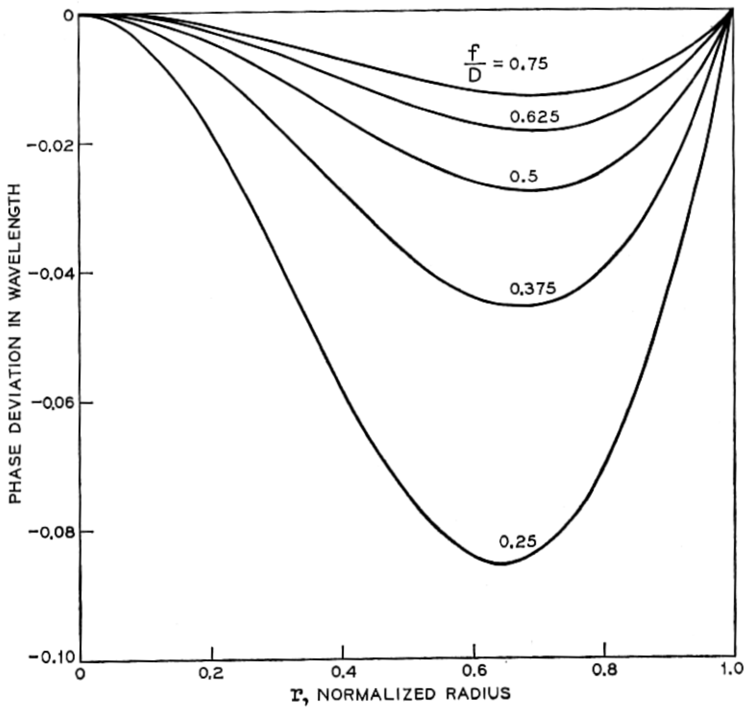

Fig. 6—Aperture phase deviation between a defocused paraboloid and an ellipsoid for unity Fresnel number.

$$\frac{R_2 - R'}{R_2 R'} = \frac{R' - R_1}{R_1 R'}. \tag{17}$$

Solving the above equation, we have

$$R' = \frac{2R_2 R_2}{R_1 + R_2}. \tag{18}$$

Substituting equation (18) into equation (14), the maximum phase deviation becomes

$$\frac{\delta}{\lambda} = \left[\tfrac{1}{2} r^2 (1 - r^2) \frac{\left(\frac{a}{2f}\right)^2}{1 + \left(\frac{a}{2f}\right)^2 r^2} \right] \frac{a^2}{\lambda R_1} \frac{R_2 - R_1}{2R_2}. \tag{19}$$

One notes that the above total phase deviation is twice the maximum absolute magnitude of the phase error for the defocused ellipsoid. In the Appendix an upper bound of the transmission loss due to small phase error has been approximately determined as $(m_1 + m_2)^2$ for optimum aperture distributions where m_1 and m_2 are the maximum absolute magnitudes of phase error of the two apertures respectively. As a numerical example, if the antenna spacing varies by a factor of 3, equation (19) and Fig. 6 will give $m_1 = m_2 = (\frac{1}{2}(\delta/\lambda)) \cdot 2\pi = 0.03$ for a Fresnel number of unity and an f/D ratio of 0.5. Then the fractional transmission loss due to phase error will be less than 0.4 percent.

IV. DISCUSSION

The above calculations have demonstrated the potential of optimum Fresnel zone transmission between two antennas with ellipsoidal reflectors illuminated by dual-mode feeds. In spite of the discrepancy between the illumination function of a dual-mode feed and the optimum illumination of a generalized prolate spheroidal function, the maximum power transmission between reflector apertures is practically the same for the two cases. However, the feed spill-over loss is as important as the transmission loss between reflector apertures in determining the total transmission loss between reflector antennas. More sophisticated feed design, such as the synthesis of more than two modes, may reduce the spill-over loss. Here the resulting increase in aperture blocking of the reflector is undesirable. The use of a lens in place of a reflector would avoid aperture blocking but would give rise to interface matching problems. Furthermore, the proper organization of many modes implies narrow bandwidth and stringent tolerances.

From a geometrical optics point of view, one is tempted to have the ellipsoidal reflectors focused at the midpoint between them. This scheme corresponds to a concentric resonator. The aperture distribution created by the two reflectors which are portions of the same ellipsoid, as shown in Fig. 1, is equivalent to a confocal resonator. The diffraction loss of a concentric resonator is much greater than than of a confocal resonator. The gaussian beam theory[10] predicts a beam waist, i.e., an effective focal region, in the middle of the confocal resonator. These observations indicate the failure of geometrical optics, although the reflectors are in the Fresnel zones of each other. Converting the reflected field into the aperture distribution employs geometrical optics ray tracing only in the immediate vicinity of the reflector. The validity of this procedure should be as good as that of calculating the diffraction pattern from the aperture distribution of a paraboloidal antenna.

The feasibility of using a defocused ellipsoid to simulate ellipsoids of different focal lengths has been investigated. The optimum defocusing of an ellipsoidal reflector for obtaining another spherical wavefront is similar to that of defocusing a spherical reflector for obtaining an approximate plane wave front. The explicit expression for phase deviation shows its simple dependence on the f/D ratio, the Fresnel number, and the variation of antenna spacing. Since a defocused paraboloid is a special case of a defocused ellipsoid, the criteria obtained here will also be useful for measuring the Fraunhofer radiation pattern of a paraboloidal antenna in the Fresnel region. In particular, it clarifies the inconsistency among various schemes for this latter problem, proposed by D. K. Cheng.[9,11]

V. ACKNOWLEDGMENT

The author wishes to thank A. B. Crawford for suggesting this problem and for helpful discussions.

APPENDIX

An upper bound of the transmission loss due to small phase error will be given below for nearly optimum aperture distributions. The phase error may be either deterministic or random. A small phase error simply introduces an additional factor $\exp(j\Delta\phi)$ into the numerator of equation (5), where $\Delta\phi = \Delta(r) + \Delta(r')$. For small values of $\Delta\phi$,

$$\exp\,(j\Delta\phi) \approx 1 - \tfrac{1}{2}(\Delta\phi)^2 + j\Delta\phi. \tag{20}$$

Then the modified equation (5) becomes

$$\frac{T}{p} = \left| \int_0^1 \int_0^1 M_1(r)M_2(r')J_0(prr')\sqrt{prr'}\left[1 - \frac{(\Delta\phi)^2}{2}\right] dr\,dr' \right|^2 + \left| \int_0^1 \int_0^1 M_1(r)M_2(r')J_0(prr')\sqrt{prr'}\,\Delta\phi\,dr\,dr' \right|^2 \tag{21}$$

where

$$M_i(r) = \frac{E_i(r)\sqrt{r}}{\int_0^1 |\,E_i(r)\sqrt{r}\,|^2\,dr}, \qquad i = 1, 2. \tag{22}$$

If E_1 and E_2 are optimum aperture distributions, then for any perturbation factors $A(r)$ and $B(r')$ with maximum magnitudes m_A and m_B, the following inequality holds,

$$\left| \int_0^1 \int_0^1 A(r)M_1(r)B(r')M_2(r')J_0(prr')\sqrt{prr'}\,dr\,dr' \right|^2 \leqq m_A^2 m_B^2 \left| \int_0^1 \int_0^1 M_1(r)M_2(r')J_0(prr')\sqrt{prr'}\,dr\,dr' \right|^2, \tag{23}$$

where A and B can be either of the following combinations

$$\begin{cases} A = \Delta(r) \\ B = \Delta(r') \end{cases}, \quad \begin{cases} A = [\Delta(r)]^2 \\ B = 1 \end{cases}, \quad \begin{cases} A = 1 \\ B = [\Delta(r')]^2 \end{cases}. \tag{24}$$

It follows that

$$T \geqq T_{\text{opt}} \left[1 - \frac{(m_1 + m_2)^2}{2}\right]^2, \tag{25}$$

where $m_1 = |\Delta(r)|_{\max}$ and $m_2 = |\Delta(r')|_{\max}$. The maximum fractional transmission loss due to small phase error is

$$1 - \frac{T}{T_{\text{opt}}} \leqq (m_1 + m_2)^2 \left[1 - \frac{(m_1 + m_2)^2}{4}\right]. \tag{26}$$

The above upper bound of the effect of phase error on the Fresnel zone transmission loss represents a conservative estimate.

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