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Linearized Dispersion Relation and Green's Function for Discrete-Charge-Transfer Devices with Incomplete Transfer

By W. B. JOYCE and W. J. BERTRAM

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A small-signal (linearized) theory of discrete-charge-transfer-device performance is presented for the case of incomplete charge transfer. Specifically, the dispersion relation is derived which relates the charge-transfer efficiencies presently characterizing these discrete (in space and time) devices to the usual measures of device or transmission-line performance based on the attenuation, dispersion, phase velocity, etc., of sine waves. In a more general sense this emphasizes the applicability of conventional signal theory to these new devices. The impulse solution or Green's function is then shown to be the equivalent of a bivariate distribution in probability theory. More generally the utility of (deterministically interpreted) probability theory is emphasized by showing the equivalence of a general small-signal theory to a random-walk process.

I. INTRODUCTION

In an important new class of discrete-charge-transfer devices including charge-coupled devices[1] (CCD's), bucket-brigade shift registers,[2-3] and other shift-register or image-detection or display devices, externally applied time-dependent voltages step captive charge along a chain of equivalent discrete storage stations. In some of these devices the charge transfer is imperfect with a fraction of the charge failing to advance and a fraction lost altogether during each step. Explicit expressions are constructed here for the dispersion relations and Green's functions which describe this imperfect performance under conditions when the fractions of charge that go astray can be described by constant parameters characteristic of the particular device.

The theory of analog signal processing is based on the properties of sine waves, and analog devices are conventionally characterized by the dispersion, attenuation, etc., which they cause. A second standard method of characterization is based on the distortion and attenuation of pulses. In contrast, discrete-charge-transfer devices are presently characterized by their charge-transfer efficiencies (fractions). The solutions given here were chosen to facilitate the application of conventional signal processing theory to these new discrete devices. That is, they show the equivalence of these three methods of characterization in the small-signal limit and provide the appropriate interrelating formulas. Specifically, Section III treats the sinusoidal representation, while Section IV considers pulses.

Although the physical interpretation of our equations is deterministic, it is also an objective in Section IV to show the direct applicability of many established results of probability theory. In the Appendix our basic equation is re-derived as the simplest nontrivial example of a discrete small-signal theory. In turn, the small-signal theory is seen to be the deterministic limit of a random-walk process.

Whether the mode of device operation be digital or analog, one is ordinarily interested in utilizing the full information capability; i.e., maximum bandwidth or wavelengths approaching twice the station separation in shortness. Thus, we at no time approximate the discrete equation by a (continuous) differential equation.

In summary our objective is to emphasize by way of a case of present interest that standard methods of device characterization, as well as established probability theory, can be applied to discrete charge-transfer devices.

II. BASIC EQUATION

Discrete-charge-transfer devices are often, to a good approximation if not exactly, discrete in both space and time. That is, the information-bearing charge is moved in discrete bursts in time from one spatially discrete storage station (e.g., capacitor or potential well) to the next along a line of stations. Consider $q_{x,t}$ the charge in station x at time t where x and t assume only integer values; i.e., the unit of time is taken as the stepping interval, and the unit of distance is taken as the station separation (center-to-center). Perfect charge transfer implies

$$q_{x,t} = q_{x-1,t-1} \tag{1}$$

and hence unit signal speed.

Real devices are typically experimentally characterized by the fraction α of the charge which is successfully advanced per step.[4,5] If a fraction ϵ fails to advance and remains in its original station, then the process is described by

$$q_{x,t} = \alpha q_{x-1,t-1} + \epsilon q_{x,t-1} , \tag{2}$$

which reduces to equation (1) in the ideal case of $\alpha = 1$ and $\epsilon = 0$. In other words, equation (2) states that the charge in station x at time t, q_{xt} , is the successfully transferred fraction of the charge in the previous station at the previous time, $\alpha q_{x-1,\ t-1}$, plus a fraction ε of the charge $q_{x,t-1}$ in station x at the previous time which failed to advance. From equation (2) it follows that a fraction $\ell \equiv 1 - \alpha - \epsilon$ of the charge is lost per step. Theoretical expressions for α and/or ε are contained explicitly or implicitly in the work of several authors on different devices.[6–9] [In the appendix, equation (2) is generalized slightly to include an inhomogeneous term and nonconstant values of α and ϵ.] In this context equation (2) was introduced by Berglund to describe incomplete transfer in CCD and IGFET bucket-brigade shift registers, and he showed an approximate equivalence at low frequencies to an (analog) matched transmission line.[6] Traditionally equation (2) is associated with probability theory (Bernoulli trials) where $\alpha + \epsilon = 1$.[10] In the case $\alpha = \epsilon = 1$ (outside our primary interpretation) equation (2) is usually known as Pascal's triangle although he was preceded by Cardan in 1540 who, in turn, cited earlier sources.[11] We give here a number of exact and exact limiting ($\epsilon \to 0$) solutions to equation (2) useful both in image-detection and shift-register contexts. It should be noted that our results can also be applied to Berglund's model[12] of bipolar bucket-brigade shift registers which differs from equation (2) only through an interchange of the physical interpretation of x and t.

III. SINUSOIDAL REPRESENTATION

The dispersion relations can be found from the space and time Fourier transforms of equation (2) or by the separation-of-variables method. Either approach amounts to seeking a running-wave solution of the form

$$\text{Im}\, e^{i(\omega t - kx + \varphi)} \qquad |x|, |t| = 0, 1, 2, \cdots \tag{3}$$

where k and ω may be complex to account for the attenuation, and φ is any constant phase factor. Since x takes on only integer values, equation (3) is not affected by adding multiples of 2π to the real part of k, and thus without loss of generality we require $|\text{Re } k| \leqq \pi$. In the theory of lattice vibrations (where time is continuous but the physical system is discrete) this range limitation on Re k is usually expressed by saying that the wavelength with one atomic species must be at least two lattice spacings.[13] Similarly, since time is discrete, all frequencies outside the fundamental range $|\text{Re } \omega| \leqq \pi$ are redundant. In the theory of sampled-data control systems (where the physical system is usually continuous but time is discrete) this is usually expressed by saying that the sampling frequency must be at least twice the maximum frequency to be detected.[14] (Here the sampling frequency f_s is once per unit time, i.e., $f_s = 1$ or $\omega_s = 2\pi$). Substituting equation (3) into equation (2) yields for the dispersion relation between ω and k

$$e^{i\omega} = \alpha e^{ik} + \epsilon. \tag{4}$$

Two important special cases of equation (4) are discussed in the rest of this section. Since equation (4) shows that the dispersion relations are independent of φ, we take $\varphi = 0$ hereafter.

In the case of image detection (or projection), k is real corresponding to a term in the spatial Fourier representation of the initial image. Equation (4) then reduces to

$$\omega(k) \equiv \omega' + i\omega'' = -\omega^*(-k) \tag{5}$$

$$= k - \tan^{-1} \frac{\sin k}{\cos k + \alpha/\epsilon}$$
$$- i(\ln \alpha + \tfrac{1}{2} \ln [1 + (\epsilon/\alpha)^2 + 2(\epsilon/\alpha) \cos k]); \tag{6}$$

i.e., the wave is attenuated in time but remains spatially sinusoidal. The identity

$$\tan^{-1} \frac{\sin \theta}{c + \cos \theta} + \tan^{-1} \frac{\sin \theta}{c^{-1} + \cos \theta} = \theta$$

was used to express the effect of nonzero ϵ as a separate correction term in equation (6). The real and imaginary parts, ω' and ω'' are plotted in Fig. 1. Although not considered here, it is sometimes useful to regard α and ϵ as k-dependent (i.e., wavelength-dependent) quantities. In the practical case of small ϵ/α, equation (6) is usefully approximated by truncating its expansion in powers of ϵ/α after the linear term, i.e.,

$$\omega(k) \rightarrow k - (\epsilon/\alpha) \sin k - i[\ln \alpha + (\epsilon/\alpha) \cos k], \qquad \epsilon/\alpha \rightarrow 0$$
$$\rightarrow k - \epsilon \sin k + i[\ell + \epsilon(1 - \cos k)], \qquad \epsilon, \ell \rightarrow 0. \tag{7}$$

A phase velocity v'_φ can be defined which describes the apparent speed of the crests and troughs of the attenuating sine wave; i.e.,

$$v'_\varphi(k, \epsilon/\alpha) \equiv \frac{\omega'}{k} = 1 - k^{-1} \tan^{-1} \frac{\sin k}{\alpha/\epsilon + \cos k} \tag{8}$$

$$\rightarrow 1 - \frac{\epsilon}{\alpha} \frac{\sin k}{k}, \qquad \epsilon/\alpha \rightarrow 0. \tag{9}$$

In particular

$$v'_\varphi(k, \epsilon/\alpha) + v'_\varphi(k, \alpha/\epsilon) = 1 \tag{10}$$

$$v'_\varphi(k, 1) = \tfrac{1}{2} \tag{11}$$

$$\begin{aligned} v'_\varphi(k, \epsilon/\alpha) &\rightarrow 1 \quad \text{as} \quad k \rightarrow \pi \quad \text{if} \quad \epsilon/\alpha < 1 \\ &\rightarrow 0 \quad \text{as} \quad k \rightarrow \pi \quad \text{if} \quad \epsilon/\alpha > 1. \end{aligned}$$

At long wavelengths (i.e., $k \rightarrow 0$) $\omega' \sim k$ while $\omega'' - \text{const} \sim k^2$. Thus we ignore the attenuation and find an infinite-wavelength group velocity

$$\frac{d\omega'(0)}{dk} = v'_\varphi(0, \epsilon/\alpha) = (1 + \epsilon/\alpha)^{-1}.$$

In the case of shift-register operation ω is real, corresponding to a term in the Fourier representation of a time-dependent signal introduced into one station. Equation (4) then reduces to

$$k(\omega) = k' - ik'' = -k^*(-\omega), \qquad 0 \leqq \omega \leqq \pi \tag{12}$$

$$= \omega + \tan^{-1} \frac{\sin \omega}{\epsilon^{-1} - \cos \omega} + i[\ln \alpha - \tfrac{1}{2} \ln (1 + \epsilon^2 - 2\epsilon \cos \omega)] \tag{13}$$

$$\begin{aligned} &\rightarrow \omega + \epsilon \sin \omega + i(\ln \alpha + \epsilon \cos \omega), \qquad \epsilon \rightarrow 0 \\ &\rightarrow \omega + \epsilon \sin \omega - i[\ell + \epsilon(1 - \cos \omega)], \qquad \epsilon, \ell \rightarrow 0; \end{aligned} \tag{14}$$

i.e., the wave is attenuated as a function of x but is purely sinusoidal in time at a given x. The identity following equation (6) was again used. The quantity ik corresponds to the *propagation function* of (continuous-time) lossy transmission lines; similarly k' and k'' correspond to the *phase function* and *attenuation function* respectively.[15] When Fig. 1 is rotated through 180 degrees in its plane, it becomes a plot of equation (13). Although not considered here it is sometimes useful to regard α and ϵ as ω-dependent quantities.

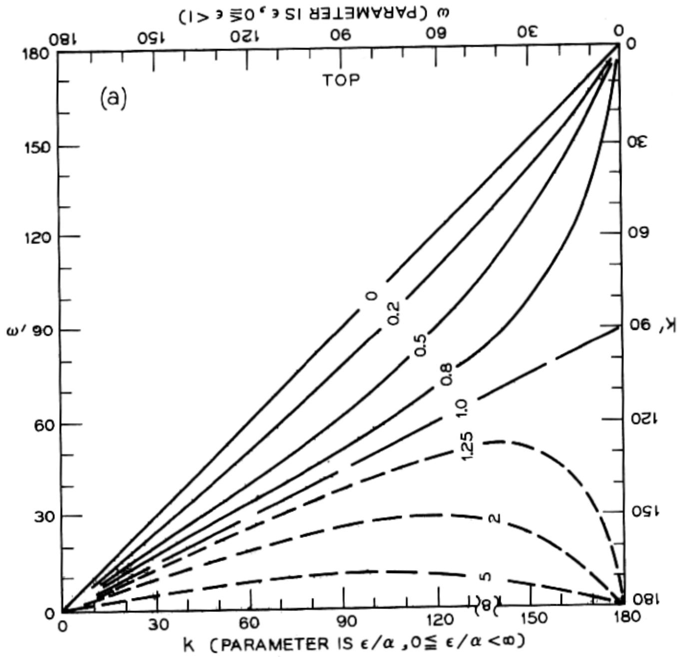

Fig. 1a

Figs. 1a and 1b—For a spatial sine wave of the form $\exp[i(\omega t - kx)]$ with $|x|$, $|t| = 0, 1, 2, \cdots$ the complex angular frequency $\omega = \omega' + i\omega''$ is plotted versus the (real) angular wave number k. After conversion from degrees to radians, k is measured in units of (2π times the stations spacing)$^{-1}$. The curves are parameterized by the ratio ϵ/α where α is the fraction of the charge successfully advanced per stepping operation and ϵ is the fraction of the charge which remains in a station per step [From equation (6)].

Fig. 1a—The real part of ω. (Note that $\omega'(k, \epsilon/\alpha) + \omega'(k, \alpha/\epsilon) = k$.)

Fig. 1b—The imaginary part of ω.

After rotating the figures by 180 degrees: For a sinusoidal signal of the form $\exp[i(\omega t - kx)]$ with $|x|$, $|t| = 0, 1, 2, \cdots$ (temporal sine wave) the complex angular wave number $k = k' - ik''$ is plotted versus the (real) angular signal frequency ω where ω is measured in degrees per stepping interval. The curves are parameterized by ϵ where $0 \leq \epsilon < 1$ is the physically significant range [From equation (13)].

Fig. 1a—The real part of k.

Fig. 1b—The imaginary part of k.

The phase velocity is given by

$$v_\varphi(\omega, \epsilon) = \frac{\omega}{k'} = \left[1 + \omega^{-1} \tan^{-1} \frac{\sin \omega}{\epsilon^{-1} - \cos \omega}\right]^{-1}$$
$$\to 1 - \epsilon \frac{\sin \omega}{\omega}, \qquad \epsilon \to 0. \tag{15}$$

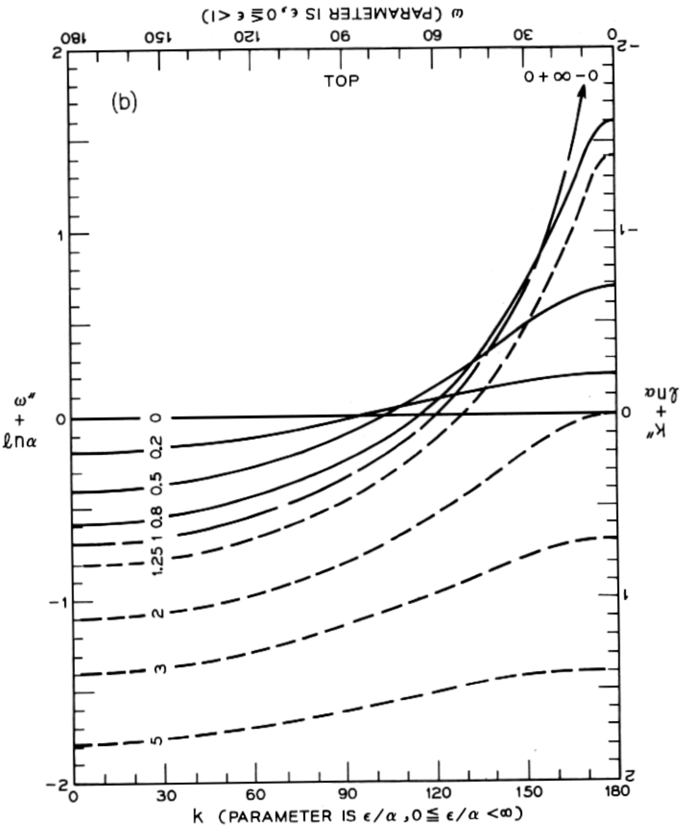

Fig. 1b

k'', and hence the attenuation, is frequency dependent for any $\epsilon > 0$ (just as ω'' is k dependent for any $\varepsilon/\alpha > 0$); however, as one would expect[16] from causality (Hilbert Transform; Kramers-Kronig Relation) in a continuous-time system, the apparent phase velocity ω/k' has no dispersionless case for $0 < \epsilon < 1$ analogous to equation (11). At low frequencies $k' \sim \omega$ while $k'' - \text{const} \sim \omega^2$. Thus we ignore the attenuation and find a zero-frequency group velocity

$$\left(\frac{dk'(0)}{d\omega}\right)^{-1} = v_\varphi(0, \epsilon) = 1 - \epsilon.$$

For a numerical example in the small-ϵ approximation, let a time-dependent signal be introduced into a shift register at $x = 0$ and extracted at station x. If x is large, the product ϵx is not necessarily small compared to unity; consider the case $\epsilon x = 0.2$. From equation (15) the phase transit time through the shift register is $x/v_\varphi = xk'/\omega = x + x\epsilon \sin \omega/\omega$. Thus the highest frequency component of the signal ($\omega = \pi$) is transmitted in the ideal (i.e., $\epsilon = 0$) time x. Lower frequencies take increasingly long with the $\omega = 0$ component arriving $x\epsilon = 0.2$ stepping intervals after the $\omega = \pi$ component. The amplitude is attenuated by the factor $e^{-k''x} = \alpha^x e^{\epsilon x \cos \omega}$ which is a decreasing function of frequency. Thus the cut-off frequency ($\omega = \pi$) is attenuated by an additional factor of $e^{-2\epsilon x} = e^{-0.4} = 0.67$ over the attenuation of the $\omega = 0$ component.

Equation (4) is invariant under the transformation

$$\omega \to k \qquad k \to \omega$$

$$\alpha \to 1/\alpha \qquad \epsilon \to -\epsilon/\alpha$$

which can therefore be used to derive from each other the two parallel sets of relations developed in this section.

If ω is eliminated from equations (3) and (4), equation (25) results; if k is eliminated, equation (33) results.

IV. IMPULSE REPRESENTATION

Having established the connection with the usual basis for characterizing continuous linear systems in terms of their effect on sine waves, we turn to the impulse representation and investigate the solution of a unit charge placed in station $x = 0$ at $t = 0$ with all other stations initially uncharged. By considering the spatial distribution after the first few steppings as shown in Fig. 2, the solution to equation (2) with this boundary condition can be recognized as the binomial expansion of $(\alpha + \varepsilon)^t$; i.e.,

$$q_{x,t} = G_{x,t} \equiv \binom{t}{x} \alpha^x \epsilon^{t-x}, \qquad 0 \leqq x \leqq t$$
$$\equiv 0 \qquad \text{otherwise} \tag{16}$$

where $\binom{t}{x}$ is the widely tabulated binomial coefficient. Later we will regard $G_{x,t}$ as the Green's function for more general initial or boundary conditions. Although our interpretation is deterministic rather than probabilistic, the terminology and results of probability theory will be quite useful. For example, equation (16) is a discrete bivariate

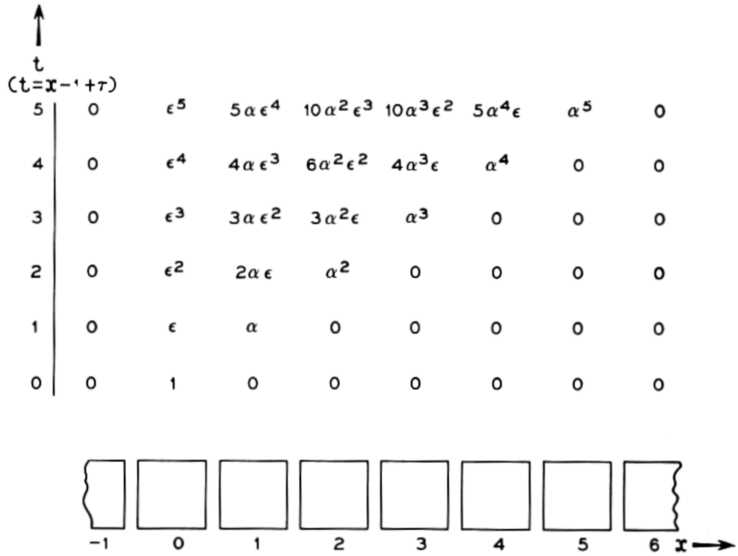

Fig. 2—Evolution of a unit charge initially in station $x = 0$ at $t = 0$. A fraction α of the charge is advanced per step and a fraction ϵ remains behind.

(x and t independent discrete variables) distribution, defined by a parabolic (partial) finite difference equation [Equation (2)].

In the description of practical devices one is usually interested in limiting expressions as ℓ and/or ϵ approach zero. Rather than writing several limiting forms at each point in the development, we emphasize once and for all an approximation which, in one variation or another, is frequently useful,

$$(\alpha + \epsilon)^t = (1 - \ell)^t = e^{t \ln (1-\ell)} \approx e^{-t\ell}, \quad t\ell^2 \ll 1. \tag{17}$$

Although in a real device the loss per transfer ℓ may be orders of magnitude less than unity, the number of stations and hence transfers may be so large that $t\ell$ is comparable to unity, and the further approximation $e^{-t\ell} \approx 1 - t\ell$ is not generally accurate.

Consider first the spatial distribution of charge with time regarded merely as a parameter, i.e., a horizontal row in Fig. 2. In probability theory it is usually assumed that $\alpha + \epsilon = 1$ (probability of heads plus probability of tails are unity). To make direct use of extensively compiled properties define

$$\alpha' = \alpha/(1 - \ell), \quad \epsilon' = \epsilon/(1 - \ell) \tag{18}$$

so that $\alpha' + \epsilon' = 1$. Then equation (16) becomes

$$G_{x,t} = (1 - \ell)^t \binom{t}{x} \alpha'^x \epsilon'^{t-x} = (1 - \ell)^t b_x , \qquad x = 0, 1, \cdots , t \tag{19}$$

which defines b_x the binomial distribution[17] with normalization

$$\mu_0 \equiv \sum_{x=0}^{t} b_x = (\alpha' + \epsilon')^t = 1 \tag{20}$$

and $(1 - \ell)^t$ is just a position-independent scale factor. (The intermediate steps involved in carrying out the numerous sums which follow can be found in many texts on probability theory.[17])

If the speed of the pulse is defined as the speed of the center of mass of that charge remaining at any time, then from

$$\langle x \rangle \equiv \sum_{x=0}^{t} x b_x = \alpha' t \tag{21}$$

it follows that the speed is $\alpha' = (1 + \epsilon/\alpha)^{-1}$, i.e., the same as the infinite-wavelength group velocity following equation (11). In general, the pulse can be characterized by its central moments μ of which the rth

$$\mu_r = \sum_{x=0}^{t} (x - \langle x \rangle)^r b_x , \qquad r = 0, 1, \cdots , \infty \tag{22}$$

can be obtained explicitly as the coefficient of $s^r/r!$ in the Taylor-series expansion in s of the *central-moment-generating function*[17]

$$e^{-s\langle x \rangle} \sum_{x=0}^{t} e^{sx} b_x = e^{-s\langle x \rangle} (\epsilon' + \alpha' e^s)^t . \tag{23}$$

Thus in particular the variance of the pulse, a useful measure of its spread, is $\mu_2 = t\alpha' \epsilon' = t(2 + \alpha/\epsilon + \epsilon/\alpha)^{-1}$ where the last form again explicitly exhibits the fact that normalization-independent properties depend on α and ϵ only through their ratio. Higher central moments are conveniently obtained from Romanovsky's recursion relation.[17,18]

$$\mu_{r+1} = \alpha' \epsilon' (tr\mu_{r-1} + d\mu_r/d\alpha'). \tag{24}$$

Some further useful relations are as follows: When the pulse spreads excessively for digital applications, one alternative is to utilize $2p + 1$ consecutive stations for one pulse. It then follows directly from the parallel-axis theorem for the moment of inertia that the variance is increased by (only) the additive term $p(p + 1)/3$ over the previous result.

At time t the ratio of the charge in the next-to-leading station to that in the leading station of the pulse is $t\epsilon/\alpha$. Thus the leading station of the pulse is also the most heavily charged until time $t = \alpha/\epsilon$ when the peak starts to drop back from the lead. At all times the charge distribution falls off monotonically from the peak. At any time the fraction of the remaining charge confined to the leading station is $(1 + \epsilon/\alpha)^{-t}$.

The results in Section III can be derived as consequences of those of Section IV, and conversely. For example the single-spatial-Fourier-transform solution of equation (2) can be obtained via the binomial distribution regarded as a Green's function (in the terminology of mathematical physics). Thus the initial sinusoidal spatial charge distribution $q_{x,0} = \text{Im } e^{-ikx}$ evolves into $q_{x,t}$ at time t where

$$\begin{aligned} q_{x,t} &= \text{Im} \sum_{x=0}^{t} G_{x',t} e^{-ik(x-x')} \\ &= \text{Im } (\epsilon + \alpha e^{ik})^t e^{-ikx} = \text{Im } (\alpha + \epsilon e^{-ik})^t e^{ik(t-x)}. \end{aligned} \tag{25}$$

Equation (25) is equivalent to equations (3) and (4) with ω eliminated. Equation (6) results if the real and imaginary parts of equation (25) are extracted. Apart from the normalization, the evolution factor which carries the initial spatial distribution e^{-ikx} into the later distribution can be recognized as $(\epsilon' + \alpha' e^{ik})^t$, the *characteristic function* (Fourier transform) of the binomial distribution in probability theory.[17] In the last form of equation (25), $(\alpha + \epsilon e^{-ik})^t$ evolves the unperturbed or ideal ($\epsilon = 0$, $\alpha = 1$) solution $e^{ik(t-x)}$ into the actual solution. Finally $(1 + (\epsilon/\alpha)e^{-ik})^t$ ($\rightarrow \exp(e^{-ik}t\epsilon/\alpha)$ as $\epsilon/\alpha \rightarrow 0$) evolves the $\epsilon = 0$ solution $\alpha^t e^{ik(t-x)}$.

As before [cf. equation (7)] we seek an approximation of the Green's function in the limit that ϵ/α, but not necessarily $t\epsilon/\alpha$, is small compared to unity. If ℓ is also small, repeated use of equation (17) yields

$$G_{x,t} \rightarrow e^{-t\ell} p(t - x), \qquad \epsilon, \ell \rightarrow 0, \quad t \rightarrow \infty, \quad t\epsilon, t\ell = \text{const.}$$

$$\lambda \equiv t\epsilon, \quad x = t, t - 1, \cdots \tag{26}$$

where $p(u) = e^{-\lambda}\lambda^u/u!$ $(u = 0, 1, \cdots)$ is the Poisson distribution.[17] Sum rules similar to those illustrated for the binomial distribution can be carried out yielding in particular a pulse speed $\langle x \rangle/t = 1 - \lambda/t = 1 - \epsilon$ and a variance $\lambda = t\epsilon$. As long as the pulse peaks near or at the leading station, the Poisson distribution is usefully accurate for the more strongly charged stations even when t is only modestly large. A comparison of the binomial distribution and its Poisson approximation are given in Table I.

It is impractical to summarize the very many relevant exact and approximate properties of the binomial distribution here, particularly since a thorough compilation is already available.[17] We have, however, given a few of the most important properties to illustrate the usefulness of this connection with the extensive literature of probability theory.

Turning now to a shift-register interpretation of equation (16), we regard x as a parameter (the number of the stations which are used as the register) and study the time-dependence of the charge in station $x - 1$, i.e., a vertical column in Fig. 2. The development proceeds in close analogy to that of the binomial distribution and most of the motivating remarks need not be repeated.

TABLE I—A COMPARISON OF THE BINOMIAL AND NEGATIVE BINOMIAL DISTRIBUTIONS WITH THEIR POISSON APPROXIMATION

1	2	3	4	5	6
x					τ
9	0	0	0	0	−1
8	0	0.430	0.411	0.430	0
7	0	0.383	0.365	0.344	1
6	0	0.149	0.162	0.155	2
5	0	0.033	0.048	0.052	3
4	0	0.0046	0.011	0.014	4
3	0	4.1×10^{-4}	1.9×10^{-3}	3.4×10^{-3}	5
2	0	2.3×10^{-5}	2.8×10^{-4}	7.4×10^{-4}	6
1	0	7.2×10^{-7}	3.6×10^{-5}	1.5×10^{-4}	7
0	1	1×10^{-8}	4.0×10^{-6}	2.8×10^{-5}	8
−1	0	0	3.9×10^{-7}	4.9×10^{-6}	9
−2	0	0	3.5×10^{-8}	8.4×10^{-7}	10
−3	0	0	—	—	11

For the case $\alpha = 0.9$ (fraction of charge successfully advanced per step), $\epsilon = 0.1$ (fraction which remains in its station per step), and $\ell = 0$ (fraction of charge lost per step).

Column 1: Station number x for columns 2, 3, and 4.
Column 2: Initial distribution of charge among the stations.
Column 3: Binomial distribution of charge among stations eight steps later [from equation (16) and (19)].
Column 6: Detection time τ for Columns 5 and 4 as measured from time of initial detection, $\tau = 0$.
Column 5: α times the negative binomial distribution of charge nondestructively observed in station 7 (α times equation (27)) as a result of initial spatial distribution of Column 2 = distribution of charge observed by a charge-removing detector in station 8 (cf. Appendix).
Column 4: Poisson approximation to Columns 3 and 5 [from equation (26) or (34)].

It is notationally convenient to use a new time τ which starts at time $t = x - 1$ when some of the charge placed initially in station zero is first observed in station $x - 1$, i.e., $\tau = t - x + 1$. Then equation (16) becomes

$$G_{x-1,\tau} = \binom{\tau + x - 1}{x - 1} \alpha^{x-1} \epsilon^{\tau}, \qquad \tau = 0, 1, \cdots, \infty. \tag{27}$$

If $\alpha^{\dagger}$ is defined as $1 - \epsilon$ (so that $\alpha^{\dagger} + \epsilon = 1$), equation (27) becomes

$$G_{x-1,\tau} = \alpha^{-1}\left(\frac{\alpha}{1 - \epsilon}\right)^{x} \binom{\tau + x - 1}{x - 1} \alpha^{\dagger x} \epsilon^{\tau} \equiv \alpha^{-1}\left(\frac{\alpha}{1 - \epsilon}\right)^{x} n_{\tau} \tag{28}$$

and defines n_{τ} the negative binomial distribution[17] with normalization

$$\mu_0 \equiv \sum_{\tau=0}^{\infty} n_{\tau} = \alpha^{\dagger x}(1 - \epsilon)^{-x} = 1. \tag{29}$$

Since charge which fails to advance is observed again, the sum of all charge (nondestructively) observed in station $x - 1$, i.e., $\alpha^{x-1}(1 - \epsilon)^{-x}$, may exceed unity even if $\ell \geqq 0$ (cf. the discussion of boundary conditions in the appendix). Whereas all normalization-independent properties of the binomial distribution depend on α and ϵ only through their ratio ϵ/α, here all normalization-independent properties depend only on ϵ (cf. Figs. 1 and 2).

The charge in station $x - 1$ is observed at a mean time $\langle t \rangle = x - 1 + \langle \tau \rangle$ where

$$\langle \tau \rangle = \sum_{\tau=0}^{\infty} \tau n_{\tau} = x\epsilon(1 - \epsilon)^{-1} \tag{30}$$

and thus in the shift-register context the pulse speed can be defined as $(x - 1)/\langle t \rangle$ which approaches $1 - \epsilon$ for large x; i.e., the same as the zero-frequency group velocity following equation (15) but faster than the definition following equations (11) and (21) except in the case of no charge loss ($\ell = 0$) when all definitions agree.

The rth central moment of the charge sequence in station $x - 1$ is

$$\mu_r = \sum_{\tau=0}^{\infty} (\tau - \langle \tau \rangle)^r n_{\tau} \tag{31}$$

which can be obtained by direct calculation or as the coefficient of $s^r/r!$ in the Taylor expansion in powers of s of the central-moment generating function[17]

$$e^{-s(\tau)} \sum_{\tau=0}^{\infty} e^{s\tau} n_\tau = e^{-s(\tau)}(1 - \epsilon e^s)^{-x}(1 - \epsilon)^x. \tag{32}$$

In particular, the variance is $\mu_2 = x\epsilon\,(1 - \epsilon)^{-2}$.

The first charge observed in station $x - 1$ is a fraction $(1 - \epsilon)^x$ of all that will be observed in that station. It is also the largest charge observed in $x - 1$ if $x\epsilon$, the ratio of the second amount observed to the first, is less than unity. Thus the input, $x\epsilon$, for the example at the end of Section III can be obtained experimentally from the ratio of the first two nonzero charges present in $x - 1$ due to a single initial charge in $x = 0$. More accurate graphical methods actually developed for statistical contexts can be directly applied here to infer the parameters from the final distribution resulting from an initial pulse.[17]

According to equation (2) a sinusoidal signal $q_{0,t} = \text{Im } e^{i\omega t}$ present in station zero at time t adds an additional charge $\Delta q_{1,t+1}$ to station 1 at $t + 1$ where $\Delta q_{1,t+1} = \alpha q_{0,t}$ (cf. the Appendix). Thus from equation (28) the charge present in station x at time t is

$$\begin{aligned} q_{x,t} &= \text{Im} \sum_{\tau=0}^{\infty} G_{x-1,\tau}\alpha e^{i\omega(t-x-\tau)} \\ &= \text{Im}\left(\frac{\alpha}{e^{i\omega} - \epsilon}\right)^x e^{i\omega t} = \text{Im}\left(\frac{\alpha}{1 - \epsilon e^{-i\omega}}\right)^x e^{i\omega(t-x)} \end{aligned} \tag{33}$$

which is equivalent to equations (3) and (4) with k eliminated. Equation (13) results if the real and imaginary parts of equation (33) are extracted. Apart from the normalization the transfer factor which multiplies the initial signal $e^{i\omega t}$ in equation (33) is the characteristic function of the negative binomial distribution.

In the limit of small ϵ and ℓ, but not necessarily small ϵx or $\epsilon\ell$, $G_{x,\tau}$ becomes $\exp(-\ell x)$ times the Poisson distribution; i.e.,

$$G_{x,\tau} \to e^{-\ell x} e^{-\Lambda} \Lambda^\tau / \tau!, \qquad \ell, \epsilon \to 0;\ x \to \infty;\ \ell x,\ \epsilon x = \text{const}\ \Lambda = x\epsilon \tag{34}$$

which is functionally equivalent to equation (26). Thus as illustrated in Table I, the distinction between the spatial and temporal projections of the impulse solution disappears in the Poisson limit [cf. equations (7) and (14)].

V. SUMMARY

Without reviewing any one of the three specialized fields individually, we have emphasized by way of an explicit case of current interest the connection between, on the one hand, the present practice of characterizing discrete charge-transfer devices in terms of their charge-transfer efficiencies and, on the other hand, the two well-developed fields of (analog) sinusoidal signal analysis and probability theory (interpreted deterministically).

Parenthetically we note that the model of this paper [equation (2)] is of some tutorial interest in that it permits the basic ideas of a traveling-wave description of discrete-space-and-time linear systems (including therefore as special cases the limits of continuous space and time) to be exemplified immediately in a unified manner via a simple device. The usual textbook vehicles such as sampled-data control systems, lattices, or transmission lines suffer tutorially in varying degrees from being too restricted (discrete in only space or time; no attenuation) and requiring a lengthy and specialized physical explanation of the origin of the basic equation in a less easily visualized system.

VI. ACKNOWLEDGMENTS

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APPENDIX

In the main portion of this paper the boundary conditions were deemphasized. For example, in shift-register operation the charge distribution was assumed to be unaffected by its own detection. Let $q_{x,t}$ be the nondestructively observed charge as distinguished from $q'_{x,t}$, the charge measured by a destruction process that removes all of the charge from station x. Then by equation (2) these two limiting cases are simply related as follows

$$q'_{x,t} = \alpha q_{x-1,t-1} ; \tag{35}$$

i.e., the expressions given for a nondestructively observed shift-register need only be multiplied by α to obtain their destructively measured counterparts in a register with one more station. In the case of a detection process which removes a fixed fraction of the charge, the signal can be obtained from the difference between successive measurements.

Similarly the two limiting cases at the input station of the register are Dirichlet (voltage) and Neumann (charge) boundary conditions. In the Neumann case a fixed amount of charge is added to whatever charge may already be in the station. In the Dirichlet case a fixed applied voltage adds charge as necessary to what is already in the station thereby achieving a total charge predetermined by the voltage level. By equation (2), and in analogy to equation (35), these two cases are simply related because a fixed total charge $q_{x,t}$ in x at t injects a fixed additional charge $\alpha q_{x,t}$ into $x + 1$ at $t + 1$ (cf. equation (33)).

To generalize equation (2) consider a constant inhomogeneous term q_d which might, for example, represent the amount of dark-current charge added per station per stepping interval. Then (2) becomes

$$q_{x,t} = \alpha q_{x-1,t-1} + \epsilon q_{x,t-1} + q_d . \tag{36}$$

An important particular solution of equation (36) is the spatially-uniform rising-in-time solution which starts from a spatially uniform distribution q_0 at $t = 0$:

$$\begin{aligned} q_{x,t} &= \beta^t q_0 + \frac{1 - \beta^t}{1 - \beta} q_d, \qquad \beta = \alpha + \epsilon, \qquad t = 0, 1, \cdots \\ &= q_0 + q_d t, \qquad \ell = 0. \end{aligned} \tag{37}$$

This could describe a recirculating memory where the output of the last station is fed back into the initial station. The constant-in-time rising-in-space solution is

$$\begin{aligned} q_{x,t} &= \gamma^x q_0 + \frac{1}{1 - \epsilon} \frac{1 - \gamma^x}{1 - \gamma} q_d, \qquad \gamma \equiv \frac{\alpha}{1 - \epsilon}, \qquad x = 0, 1, \cdots \\ &= q_0 + \frac{x}{1 - \epsilon} q_d, \qquad \ell = 0, \end{aligned} \tag{38}$$

where q_0 is now the charge in the (initial) station $x = 0$. This steady-state distribution could be maintained in a shift register by removing a charge $q_d - (1 - \epsilon) q_0$ from the initial station per step. Equation (2) is now understood to describe a signal (homogeneous solution) imposed on some particular solution to equation (36) describing the background charge.

Next assume, as is certainly true in some devices, that the transfer fractions α and ϵ depend upon at least some of the q's; i.e., the process is nonlinear. Then in the small-signal case (whether or not q_d is further generalized to be a function of x and t) α and ϵ in equation (2) would still be independent of the magnitude of the signal charge but no longer independent of x and t even though all stations were physically equivalent apart from their momentary charges. In the case of equation (37) α and ϵ would still be x independent; in the case of equation (38) still t independent[19] (cf. nonuniform transmission lines).

Before proceeding we switch to the suitably developed terminology of probability theory (without necessarily implying a probabilistic interpretation). Actually, this has some plausible justification. To the extent that equation (2) applies independently to each of the indivisible electrons or holes in the charge, α and ϵ must be interpreted as transition probabilities, and our deterministic interpretation results only because the large number of electrons or holes permits fluctuations to be ignored. In this probabilistic sense one can define the information content of a discrete charge distribution.[20] Similarly in equation (36), $q_d(x, t)$ could be a random variable describing the introduction of extraneous noise.

To complete the small-signal or linear model for the signal we note that in practice the stations are driven by n-phase time-dependent voltages such that in a coordinate system y which moves at the ideal signal speed $(y = x - t)$ the voltage distribution among the stations appears constant in discrete time and periodic in discrete space with a period of n stations. In each cycle of n stations one station is the designated potential well which, in the ideal case, carries all of the charge in that spatial cycle. The distortion of the signal corresponds to the transitions which the electrons or holes make to neighboring stations and, eventually, to neighboring wells. In a causal system the homogeneous equation for the signal charge takes the form

$$q_{y,t} = \sum_{t'=-\infty}^{t-1} \sum_{y'=-\infty}^{\infty} T_{y,y',t,t'}\, q_{y',t'} \tag{39}$$

where the transition elements of the T matrix are q-independent in a small-signal theory. (In some cases, the upper limit of Σ is t rather than $t - 1$.)

If the system is memoryless such that the spatial charge distribution at one time is sufficient to determine the distribution at the next time and hence for all time, then equation (39) reduces to the Markoffian form

$$q_{y,t} = \sum_{y'=-\infty}^{\infty} T_{y,y',t,t-1} q_{y',t-1} \; . \tag{40}$$

This simplification corresponds, for example, to neglecting traps which accept signal charge at a rate proportional to the local signal and then emit the charges into the signal over a time scale long compared to the stepping interval.

When the background charge distribution, if any, is time independent, $T_{y,y't,t'}$ depends on t and t' only through their difference. If the background distribution is spatially constant, T is a periodic function of $y - y'$; i.e., T describes a random-walk[21] problem on a one-dimensional lattice with anisotropic and spatially periodic transition probabilities (cf. Floquet's theorem[13]).

Finally, of course, large signals under nonlinear conditions are of interest. However, it is not yet clear that there is any dependence of T on the q's which is more general than the specific device where it arises.

In the moving coordinate system equation (2) becomes

$$q_{y,t} = \alpha q_{y,t-1} + \epsilon q_{y+1,t-1} \tag{41}$$

which is functionally equivalent to equation (2) and distinct only in that the roles of α and ϵ, as well as the sense of spatial direction, are reversed. If the transition probabilities (T matrix elements) are small out of the wells and large out of the other stations between wells, then this 1-phase equation can be used to approximate an n-phase system. The unit of time is taken as the effective stepping interval equal to n actual stepping intervals, and successive integer values of x or y label the n-cycle groups or the wells. The matrix elements ϵ and α in equation (41) then describe a net transfer of charge between, or retention of charge by, the n-cycle moving groups during one (effective) time unit.

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