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A New Equalizer Structure for Fast Start-Up Digital Communication

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The use of transversal filters for automatic equalization has made possible high-speed data communication over voiceband telephone channels. Recently much attention has been focused on the possible use of the high-speed data sets in private line multiparty polling systems. However, for such applications, it is necessary to reduce the start-up time of the present automatic equalizer drastically. This paper examines the start-up time (settling time) of the transversal filter equalizer for two important classes of data communication systems: Class IV partial-response systems and single-sideband Nyquist systems. (The latter represents the limiting case of vestigial-sideband systems with small roll-off bandwidth.) It is shown that in single-sideband Nyquist systems the input signals to the gain controls of the transversal equalizer may be nearly orthonormal. Consequently the equalizers may have a short settling time. It is also shown that the equalizer settling time is much longer in Class IV partial-response systems, because such systems use controlled intersymbol interference and the input signals to the gain controls are highly correlated.

The possibility of reducing the settling time of the automatic equalizers is examined. A new equalizer structure is developed based on the following principles: (i) Equalizer settling time can be minimized by making the input signals to the gain controls orthonormal, and (ii) Such a minimization does not change the noise power, the mean-square equalization error, the convexity of the gain control adjustment, and the feedback control loops in the equalizer. These principles are general in that they apply regardless of the type of modulation—single- , vestigial- , or double-sideband (SSB, VSB, or DSB)—or the signaling scheme (Nyquist or partial-response). Application of these principles to Class IV partial-response systems is considered. For private line systems and systems where amplitude *distortions in the communication channels are not severe (delay distortions can be arbitrary), the new equalizer can be implemented by simply adding a prefixed weighting matrix to the conventional transversal equalizer. Analysis and computer simulation show that the use of such a new equalizer can result in a significant reduction in the system's start-up time.*

I. INTRODUCTION

In order to meet the needs of the rapidly growing computer and data processing industries, a number of high-speed data sets have been developed in recent years for voiceband telephone channels. Most of these data sets use transversal filters[1] for precise automatic equalization. The transversal equalizer consists of a tapped delay line with variable tap gains. During a start-up period prior to data transmission, the tap gains are adjusted automatically to minimize the peak distortion[1] or the mean-square error[2,3,4] of the received pulses. The time required to adjust the tap gains to nearly their optimum settings is usually called the settling time of the equalizer. Most automatic equalizers have settling times of a few seconds. The start-up time of the system can be longer because there are usually other operations to be performed in the start-up period (operations such as synchronization, carrier recovery, and so forth).

Recently, much attention has been focused on the possible use of the high-speed data sets in private-line multiparty polling systems (such as airline reservation systems, on-line banking systems, and so forth). Such systems are generally real-time information retrieval systems where the inquiry and response are short (the message lengths are usually less than 1000 bits[5]). With a 4800 b/s data set, it takes only 0.208 second to transmit a message of 1000 bits. Consequently the actual transmission time can be much less than the start-up time of the system (which can be five seconds). In order to allow additional stations to be served or to reduce the response time of the system (response time is important in real-time systems), it is necessary[5] to reduce the start-up time of the high-speed data sets drastically. This means that the settling time of the automatic equalizer must be reduced drastically (for example, from a few seconds to tens of milliseconds).

Such a drastic reduction in the start-up time of the high-speed data sets raises a number of theoretical questions. For example, it is no longer sufficient to just prove the convergence of the equalizer adjustment; it must be shown that the convergence is sufficiently fast to meet the requirement. It becomes necessary to examine the dependence of equalizer settling time on modulation and signaling schemes. Instead of considering synchronization, carrier recovery, and automatic equalization separately, one has to consider the mutual dependence of these adjustments and the possibility of minimizing the overall adjustment time. The start-up time requirement also provides a strong motivation to search for a new equalizer that has a settling time shorter than that of the conventional transversal equalizer. Answers to some of these problems are presented in this paper. Because the paper is lengthy, the contents of each section are outlined here; the results are summarized in Section VIII (the reader may read Section VIII first).

Section II includes a description of the mathematical model and reviews some of the fundamental works on automatic equalization (particularly those by Lucky and Gersho). Section III examines the equalizer settling time for two important classes of digital communication systems: the Class IV partial-response system and the SSB Nyquist system. (The latter represents the limiting case of VSB systems with small roll-off bandwidth.) Surprisingly, the results show that the equalizer settling times of these two systems can be very different. The reason for this difference (eigenvalue spread) is explained so that the method of analysis can be extended to other systems. Sections IV and V consider a general data communication system and develop a new equalizer structure for fast start-up purpose. These sections stress the underlying principle (condition of orthogonality) and analyze the various properties of the new equalizer (including convergence rate of equalizer adjustment, residual noise power, minimum mean-square error, and convexity of the adjustment). In Sections VI and VII, application of the new equalizer to the Class IV partial-response system is considered. Most importantly, it is shown that the new equalizer can be implemented by simply adding a prefixed weighting matrix to a conventional transversal equalizer. The related analytical studies and computer simulation are described; fast convergence of the equalizer adjustments is demonstrated. Section VIII is a summary of the results.

II. REVIEW OF FUNDAMENTALS

An amplitude modulation data communication system utilizing a transversal equalizer is depicted in Fig. 1. The equalizer consists of

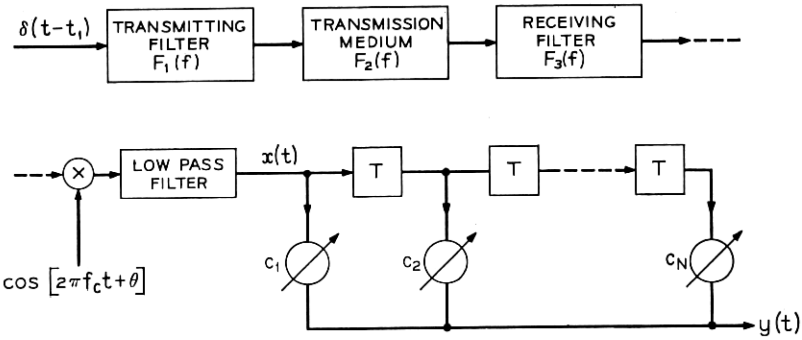

Fig. 1—Block diagram of an amplitude modulation data communication system with a conventional transversal equalizer.

a delay line tapped at T-second intervals, where T is the signaling interval of the system. The ith tap, $i = 1$ to N, is connected through a variable gain control c_i to a summing bus. During data transmission, the transmitter transmits the information digits sequentially at time-instants $t = \cdots, t_1 - T, t_1, t_1 + T, t_1 + 2T, \cdots$. The equalizer output is sampled sequentially at time-instants $t = \cdots, t_2 - T, t_2, t_2 + T, t_2 + 2T, \cdots$, and the time-samples are used to recover the information digits. To simplify the notations, we shall shift the origin of the time-axis to make $t_2 = 0$.

When an impulse $\delta(t - t_1)$ is applied at the transmitter input, the equalizer input and output are, respectively, $x(t)$ and $y(t)$. Since we consider only linear systems, $y(t)$ is the overall impulse response. The desired overall impulse response of the system is $d(t)$. In this study, we adopt the familiar mean-square error criterion[3,4,6] and adjust the gain controls of the equalizer to minimize the mean-square error between $y(t)$ and $d(t)$ at the receiver sampling instants $t = \cdots, -T, 0, T, 2T, \cdots$. The mean-square error can be written as

$$\epsilon = \sum_{i=-\infty}^{\infty} [y(iT) - d(iT)]^2. \tag{1}$$

It can be seen from Fig. 1 that

$$y(t) = \sum_{k=1}^{N} c_k x[t - (k - 1)T]. \tag{2}$$

For the sake of simplicity, we shall use the abbreviations $y_i = y(iT)$, $d_i = d(iT)$, and $x_i = x(iT)$. It can be seen from (2) that (1) can be written in the following matrix form

$$\epsilon = \mathbf{c}'\mathbf{A}\mathbf{c} - 2\mathbf{c}'\mathbf{v} + \sum_{i=-\infty}^{\infty} d_i^2 , \tag{3}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix}, \tag{4}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \tag{5}$$

$$a_{ij} = \sum_{l=-\infty}^{\infty} x_{l-i+1} x_{l-j+1} , \quad \text{all} \quad i, j, \tag{6}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}, \tag{7}$$

$$v_k = \sum_{i=-\infty}^{\infty} x_{i-k+1} d_i , \quad \text{all} \quad k. \tag{8}$$

It can be shown that **A** is positive definite.

Let $\partial\epsilon/\partial c_i$ be the partial derivative of ϵ with respect to c_i, $i = 1$ to N, and let $\partial\epsilon/\partial\mathbf{c}$ represent an $N \times 1$ column vector whose ith element is $\partial\epsilon/\partial c_i$; i.e.,

$$\frac{\partial\epsilon}{\partial\mathbf{c}} = \begin{bmatrix} \dfrac{\partial\epsilon}{\partial c_1} \\ \dfrac{\partial\epsilon}{\partial c_2} \\ \vdots \\ \dfrac{\partial\epsilon}{\partial c_N} \end{bmatrix}.$$

From (1), we obtain

$$\frac{\partial \epsilon}{\partial \mathbf{c}} = 2\mathbf{A}\mathbf{c} - 2\mathbf{v}. \tag{9}$$

The measured $\partial\epsilon/\partial\mathbf{c}$ in actual data sets will deviate[4] from that in (9) due to noise in the receiver. However this deviation is negligible in high signal-to-noise ratio systems. For example, it has been found (Section 7.3) that at a 30-dB signal-to-noise ratio this deviation has only a minor effect on the settling time of the equalizer. In this paper we consider high signal-to-noise ratio systems and neglect such a deviation.

The optimum value of $\mathbf{c}$ that minimizes the mean-square error ϵ will be called $\mathbf{c}_{opt}$. It is clear that $\mathbf{c}$ minimizes ϵ if and only if $\partial\epsilon/\partial\mathbf{c} = \mathbf{0}$; therefore, from (10) the optimum $\mathbf{c}$ is

$$\mathbf{c}_{opt} = \mathbf{A}^{-1}\mathbf{v}. \tag{10}$$

The difference between $\mathbf{c}$ and $\mathbf{c}_{opt}$ is denoted by $\mathbf{e}$: i.e.,

$$\mathbf{e} = \mathbf{c} - \mathbf{A}^{-1}\mathbf{v}. \tag{11}$$

Let ϵ_{min} be the minimum value of ϵ when $\mathbf{c} = \mathbf{c}_{opt}$. From (3) and (10)

$$\epsilon_{min} = \sum_{i=-\infty}^{\infty} d_i^2 - \mathbf{v}'\mathbf{A}^{-1}\mathbf{v}. \tag{12}$$

Now consider the adjustment of the equalizer. As is well known,[3,7] the equalizer can be adjusted in the training period prior to data transmission by transmitting either a succession of isolated test pulses or a sequence of Pseudo-random numbers. Since these two methods differ considerably, we shall consider only the first method (isolated test pulses) in this paper.

In the training period, isolated impulses are applied to the transmitter input. For instance, $\delta(t - t_1)$ in Fig. 1 may be the kth such impulse. The transmission of $\delta(t - t_1)$ produces the test pulse $x(t)$ at the equalizer input. From $x(t)$ the partial derivatives $\partial\epsilon/\partial c_i$, $i = 1$ to N, are computed[3,4,6] and the gain control c_i is changed by an amount proportional to $\partial\epsilon/\partial c_i$. This process is then repeated for the next test pulse.

The adjustment made after the kth test pulse, $k = 1, 2, 3, \cdots$, will be referred to as the kth adjustment. The initial values of $\mathbf{c}$, $\mathbf{e}$, and ϵ (i.e., their values prior to the first adjustment) will be denoted, respectively, by $\mathbf{c}_0$, $\mathbf{e}_0$, and ϵ_0. The values of $\mathbf{c}$, $\mathbf{e}$, and ϵ after the

kth adjustment are denoted, respectively, by $\mathbf{c}_k$, $\mathbf{e}_k$, and ϵ_k. From (11)

$$\mathbf{e}_k = \mathbf{c}_k - \mathbf{A}^{-1}\mathbf{v} \qquad k = 0, 1, 2 \cdots . \tag{13}$$

It can be easily shown from (3), (12), and (13) that

$$\epsilon_k = \epsilon_{\min} + \mathbf{e}_k'\mathbf{A}\mathbf{e}_k, \quad k = 0, 1, 2, \cdots . \tag{14}$$

The kth adjustment is made according to the equation

$$\mathbf{c}_k = \mathbf{c}_{k-1} - \frac{1}{2}\alpha_k\left[\frac{\partial\epsilon}{\partial\mathbf{c}}\right]_k, \tag{15}$$

where α_k, $k = 1, 2, 3, \cdots$, are suitably chosen constants.[4]

The subscript k of $\partial\epsilon/\partial\mathbf{c}$ indicates that the $\partial\epsilon/\partial\mathbf{c}$ is computed from the kth test pulse. It can be shown from (9), (15), and (13) that

$$\mathbf{e}_k = (\mathbf{I} - \alpha_k\mathbf{A})\mathbf{e}_{k-1}, \quad k = 1, 2, 3, \cdots . \tag{16}$$

We now proceed to study the convergence of the mean-square error for several data communication systems.

III. A STUDY OF CONVERGENCE FOR PARTIAL-RESPONSE AND NYQUIST SYSTEMS

We have defined ϵ_k as the value of ϵ after the kth adjustment. Note from (14) that ϵ_k consists of two terms. The first term $\epsilon_{\min}$ is the irreducible value of ϵ. Only the second term $\mathbf{e}_k'\mathbf{A}\mathbf{e}_k$ depends on $\mathbf{c}$ and the adjustments. Thus, we shall study the convergence of the second term in this section.

It can be seen from (6) that $\mathbf{A}$ is a symmetric matrix. Let the eigenvalues of $\mathbf{A}$ be denoted, in the order of increasing magnitude, by λ_i, $i = 1$ to N, so that

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_N, \tag{17}$$

and let $\mathbf{u}_i$, $i = 1$ to N, be a set of orthonormal eigenvectors of $\mathbf{A}$ ($\mathbf{u}_i$ is the eigenvector corresponding to λ_i). It is well known that $\mathbf{A}$ can be represented in the form

$$\mathbf{A} = \mathbf{Q}\mathbf{D}\mathbf{Q}', \tag{18}$$

where $\mathbf{D}$ is an $N \times N$ diagonal matrix whose ith diagonal element is λ_i, and $\mathbf{Q}$ is an $N \times N$ matrix whose ith column is the eigenvector $\mathbf{u}_i$. It is also well known that Q is an orthogonal matrix; i.e.,

$$\mathbf{Q}' = \mathbf{Q}^{-1}. \tag{19}$$

From (18) and (19),

$$\mathbf{I} - \alpha_k \mathbf{A} = \mathbf{Q}[\mathbf{I} - \alpha_k \mathbf{D}]\mathbf{Q}'. \tag{20}$$

By repeated application of (16), one obtains

$$\begin{aligned} \mathbf{e}_k &= (\mathbf{I} - \alpha_k \mathbf{A})(\mathbf{I} - \alpha_{k-1}\mathbf{A}) \cdots (\mathbf{I} - \alpha_2 \mathbf{A})(\mathbf{I} - \alpha_1 \mathbf{A})\mathbf{e}_0 \\ &= \prod_{n=1}^{k} (\mathbf{I} - \alpha_n \mathbf{A})\mathbf{e}_0 , \qquad k = 1, 2, 3, \cdots . \end{aligned} \tag{21}$$

Substituting (20) into (21) and noting that $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$ gives

$$\mathbf{e}_k = \mathbf{Q}\left[\prod_{n=1}^{k} (\mathbf{I} - \alpha_n \mathbf{D})\right]\mathbf{Q}'\mathbf{e}_0 , \qquad k = 1, 2, 3, \cdots . \tag{22}$$

From (22) and (18),

$$\mathbf{e}_k'\mathbf{A}\mathbf{e}_k = \mathbf{e}_0'\mathbf{Q}\left[\prod_{n=1}^{k} (\mathbf{I} - \alpha_n \mathbf{D})\right]\mathbf{D}\left[\prod_{n=1}^{k} (\mathbf{I} - \alpha_n \mathbf{D})\right]\mathbf{Q}'\mathbf{e}_0 , \qquad k = 1, 2, 3, \cdots . \tag{23}$$

Using the properties of **D** and **Q** described one can carry out the matrix multiplications in (23) and obtain

$$\mathbf{e}_k'\mathbf{A}\mathbf{e}_k = \sum_{i=1}^{N} \xi_i(k), \qquad k = 1, 2, 3, \cdots . \tag{24}$$

where

$$\xi_i(k) = (\mathbf{e}_0'\mathbf{u}_i)^2 \lambda_i \left[\prod_{n=1}^{k} (1 - \alpha_n \lambda_i)^2\right], \qquad i = 1 \quad \text{to} \quad N. \tag{25}$$

In a similar manner, we find that

$$\mathbf{e}_0'\mathbf{A}\mathbf{e}_0 = \sum_{i=1}^{N} \xi_i(0), \tag{26}$$

where

$$\xi_i(0) = (\mathbf{e}_0'\mathbf{u}_i)^2 \lambda_i , \qquad i = 1 \text{ to } N. \tag{27}$$

Since **A** is positive definite, $\lambda_i > 0$ for all i. Thus, $\xi_i(k) \geqq 0$ for all i. Consequently, $\mathbf{e}_k'\mathbf{A}\mathbf{e}_k$ converges to zero if and only if $\xi_i(k)$, $i = 1$ to N, all converge to zero (see (24)). For this reason, $\xi_i(k)$ will be called the ith error component. If the error components all converge rapidly to zero as k increases, $\mathbf{e}_k'\mathbf{A}\mathbf{e}_k$ converges rapidly to zero.

It is clear from (25) that $\xi_i(k)$ converges to zero if and only if the factor

$$\left[\prod_{n=1}^{k} (1 - \alpha_n \lambda_i)^2\right]$$

converges to zero. Thus, the convergence of $\xi_1(k)$ *to* $\xi_N(k)$ depend only on two sets of parameters: $\alpha_1, \cdots, \alpha_k$ and $\lambda_1, \cdots, \lambda_N$. The first set of parameters corresponds to the magnitudes of the gain-control adjustments [As can be seen from (15), α_k determines the magnitude of the kth adjustment.]. In the following subsections, we show that the second set of parameters, $\lambda_1, \cdots, \lambda_N$, depend on the modulation scheme and channel characteristics. In some systems, $\lambda_1, \cdots, \lambda_N$ differ only slightly in value. Consequently α_k can be selected such that each adjustment reduces each of the error components by a large factor (such as 100). However, this is not possible in some other systems.

3.1 *Class IV Partial-Response System*

There are several classes of partial-response systems.[8] We shall consider the most important one: Class IV partial response system.[9,10,11,12,13] The results can be easily extended to other classes.

As depicted in Fig. 1, the transfer functions of the transmitting filter, transmission medium, and receiving filter are, respectively, $F_1(f)$, $F_2(f)$, and $F_3(f)$. The amplitude and phase characteristics of $F_i(f)$ will be denoted, respectively, by $|F_i(f)|$ and $\beta_i(f)$, i.e.,

$$F_i(f) = | F_i(f) | e^{J\beta_i(f)}, \qquad i = 1, 2, 3. \tag{28}$$

Note that in this paper J will be used to denote the imaginary number $\sqrt{-1}$ (j is used as an index).

In a Class IV partial-response system, the transmitting and the receiving filters are band-limited; that is,

$$| F_1(f)F_3(f) | = 0,$$

when

$$| f | \leqq f_1 \quad \text{and} \quad | f | \geqq f_2 , \tag{29}$$

where f_1 and f_2 are, respectively, the lower and the upper cutoff frequencies. The demodulating carrier frequency, f_c, is usually equal to f_2. This implies that the system is SSB. According to the previous definition, when an impulse $\delta(t - t_1)$ is applied at the transmitter input, the equalizer input is $x(t)$. Let $X(f)$ denote the Fourier transform of $x(t)$. It can be shown from (28), (29), and the demodulation process that

$$
\begin{aligned}
X(f) &= \tfrac{1}{2}\,|F_1(f-f_c)F_2(f-f_c)F_3(f-f_c)| \\
&\quad \cdot \exp\{J[\beta_1(f-f_c)+\beta_2(f-f_c)+\beta_3(f-f_c)-2\pi(f-f_c)t_1+\theta]\}, \\
&\qquad\qquad 0 \leqq f \leqq f_2 - f_1 \\
&= \tfrac{1}{2}\,|F_1(f+f_c)F_2(f+f_c)F_3(f+f_c)| \\
&\quad \cdot \exp\{J[\beta_1(f+f_c)+\beta_2(f+f_c)+\beta_3(f+f_c)-2\pi(f+f_c)t_1-\theta]\}, \\
&\qquad\qquad -(f_2-f_1) \leqq f \leqq 0 \\
&= 0, \quad \text{all other frequencies.}
\end{aligned}
\tag{30}
$$

Now we can determine the **A** matrix. From (6), the elements of **A** are

$$a_{ij} = \sum_{l=-\infty}^{\infty} x_{l-i+1}x_{l-j+1} \,. \tag{6}$$

In a Class IV partial-response system the signaling interval is

$$T = \frac{1}{2(f_2 - f_1)}. \tag{31}$$

It is clear from (30) that $x(t)$ is band-limited from 0 to $(f_2 - f_1)$ Hz. Thus, the time-samples $x(kT)$, $k = \cdots, 0, 1, 2, \cdots$, are taken at the Nyquist rate. Therefore, according to the sampling theorem

$$\int_{-\infty}^{\infty} x[t - iT + T]x[t - jT + T]\,dt = \frac{1}{2(f_2 - f_1)} \sum_{l=-\infty}^{\infty} x_{l-i+1}x_{l-j+1} \,. \tag{32}$$

Comparing (6) and (32) shows that

$$a_{ij} = 2(f_2 - f_1)\int_{-\infty}^{\infty} x[t - iT + T]x[t - jT + T]\,dt. \tag{33}$$

From Parseval's theorem and (30), one can rewrite (33) as

$$
\begin{aligned}
a_{ij} = (f_2 - f_1)\int_0^{f_2-f_1} &[\cos 2\pi f(i-j)T] \\
&\cdot [|F_1(f-f_c)F_2(f-f_c)F_3(f-f_c)|]^2\,df.
\end{aligned}
\tag{34}
$$

It can be seen from (34) that a_{ij} is independent of the following parameters: demodulating carrier phase θ, system timing t_1, phase characteristics of the transmitting and receiving filters, and phase characteristic of the transmission medium. To evaluate a_{ij}, we need to specify only the amplitude characteristics $|F_1(f - f_c)|$, $|F_2(f - f_c)|$, and $|F_3(f - f_c)|$. In a Class IV partial-response system, the transmitting and receiving filters are designed such that

$$|F_1(f)|\,|F_3(f)| = \sin \pi\left[\frac{f - f_1}{f_2 - f_1}\right], \qquad f_1 \leqq f \leqq f_2$$
$$= \sin \pi\left[\frac{-f - f_1}{f_2 - f_1}\right], \qquad -f_2 \leqq f \leqq -f_1 . \tag{35}$$

Substituting (35) into (34) gives

$$a_{ij} = (f_2 - f_1)\int_0^{f_2 - f_1} [\cos 2\pi f(i - j)T] \cdot\left[\left(\sin \pi \frac{f}{f_2 - f_1}\right)|F_2(f - f_c)|\right]^2 df. \tag{36}$$

We first consider the case where the transmission medium has a constant amplitude characteristic

$$|F_2(f)| = 1 \tag{37}$$

in the pass band $f_1 \leqq f \leqq f_2$. Substituting (37) into (36) and evaluating the integral, we obtain

$$a_{ij} = -\frac{(f_2 - f_1)^2}{4}, \qquad i - j = -2 \quad \text{or} \quad 2$$
$$= \frac{(f_2 - f_1)^2}{2}, \qquad i - j = 0$$
$$= 0, \qquad \text{all other} \quad i - j. \tag{38}$$

The constant term $(f_2 - f_1)^2/2$ in a_{ij} may be dropped (this corresponds to introducing a gain of $\sqrt{2}/(f_2 - f_1)$ in the channel). Then (38) becomes

$$a_{ij} = -\tfrac{1}{2}, \qquad i - j = -2 \quad \text{or} \quad 2$$
$$= 1, \qquad i - j = 0$$
$$= 0, \qquad \text{all other} \quad i - j. \tag{39}$$

Thus, the matrix $\mathbf{A}$ is of the following form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \cdots & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{40}$$

In particular the main diagonal elements are 1, the second off-diagonal elements are $-1/2$, and the other elements are all 0 (notice that $\mathbf{A}$ belongs to a class of matrixes known as the "Toeplitz"[14]). We have defined λ_1 to λ_N as the eigenvalues of $\mathbf{A}$, and $\mathbf{u}_1$ to $\mathbf{u}_N$ as a set of orthonormal eigenvectors of $\mathbf{A}$. These eigenvalues and eigenvectors are determined in Appendix A. Since transversal equalizers usually have an odd number of taps, we shall assume that N is odd in the following discussion. From Appendix A, the N eigenvalues of $\mathbf{A}$ are

$$1 - \cos \frac{k\pi}{\frac{N+1}{2} + 1}, \qquad k = 1, 2, \cdots, \frac{N+1}{2}, \tag{41}$$

and

$$1 - \cos \frac{k\pi}{\frac{N-1}{2} + 1}, \qquad k = 1, 2, \cdots, \frac{N-1}{2}. \tag{42}$$

Thus, the eigenvalues of $\mathbf{A}$ are points on the curve $1 - \cos \theta$, $0 < \theta < \pi$ (see Fig. 2). The minimum eigenvalue

$$1 - \cos \frac{\pi}{\frac{N+1}{2} + 1}$$

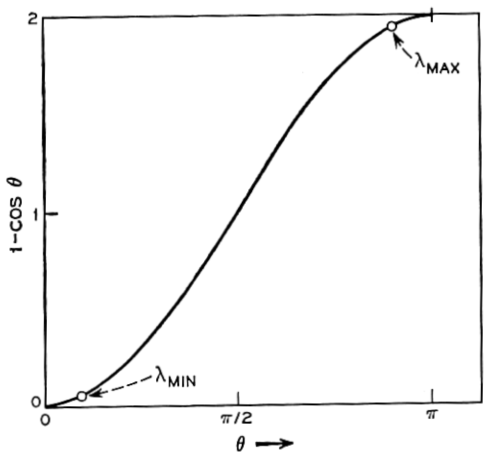

Fig. 2—Distribution of the eigenvalues of $\mathbf{A}$($\lambda_{\min}$ and $\lambda_{\max}$ shown are for $N = 17$).

and the maximum eigenvalue

$$1 - \cos \frac{\frac{N+1}{2}\pi}{\frac{N+1}{2}+1}$$

are also shown in Fig. 2. Other eigenvalues lie between these two points. It can be seen from this figure that, for typical values of N (13, 17, etc.), the minimum eigenvalue is very close to 0 and the maximum eigenvalue is very close to 2. For example, consider a transversal equalizer with 17 taps ($N = 17$). From (41) and (42), the 17 eigenvalues of **A** are 0.0489, 0.0603, 0.1910, 0.2340, 0.4122, 0.5, 0.6910, 0.8263, 1, 1.174, 1.309, 1.5, 1.588, 1.766, 1.809, 1.940, and 1.951.

When the amplitude characteristic of the transmission medium is not exactly a constant in the passband $f_1 \leqq f \leqq f_2$, the calculations from (38) to (42) will change slightly. Consequently the eigenvalues will differ slightly from those above.

Having determined the distribution of the eigenvalues, we now consider the convergence of the error components $\xi_i(k)$, $i = 1$ to N. Prior to the kth adjustment, the ith error component is

$$\xi_i(k-1) = (\mathbf{e}_0'\mathbf{u}_i)^2\lambda_i\left[\prod_{n=1}^{k-1}(1-\alpha_n\lambda_i)^2\right]. \tag{43}$$

After the kth adjustment, the ith error component is

$$\xi_i(k) = (\mathbf{e}_0'\mathbf{u}_i)^2\lambda_i\left[\prod_{n=1}^{k}(1-\alpha_n\lambda_i)^2\right].$$

Substituting (43) into the above equation gives

$$\xi_i(k) = (1-\alpha_k\lambda_i)^2\,\xi_i(k-1), \quad k = 1, 2, 3, \cdots. \tag{44}$$

It is easily seen from (44) that the kth adjustment will reduce each of the error components by a large factor if

$$(1-\alpha_k\lambda_i)^2 \ll 1, \qquad \text{for all } i$$

or if

$$\alpha_k \cong \frac{1}{\lambda_i}, \qquad \text{for all} \quad i. \tag{45}$$

Unfortunately, α_k cannot satisfy (45) because $\lambda_1, \cdots, \lambda_N$ are very different in value. Therefore, the kth adjustment cannot reduce each of the error-components by a large factor. To illustrate this quantitatively, consider again the 17-tap transversal equalizer ($\lambda_1 = 0.0489$ and $\lambda_{17} = 1.951$). In order to reduce the first error-component $\xi_1(k)$, we set $(1 - \alpha_k\lambda_1)^2 \ll 1$ or $\alpha_k \cong 1/\lambda_1 = 20.4$. However, this α_k will cause a large increase in some other error-components. For example, the 17th error-component will increase about 1505 times because $(1 - \alpha_k\lambda_{17})^2 = (1 - 20.4 \times 1.951)^2 = 1505$. On the other hand, if we wish to reduce the 17th error-component, we should set $(1 - \alpha_k\lambda_{17})^2 \ll 1$, or $\alpha_k \cong 1/\lambda_{17} = 0.512$. However, this α_k is too small to reduce some other error-components rapidly. For instance, the first error-component will reduce only five percent because $(1 - \alpha_k\lambda_1)^2 = (1 - 0.512 \times 0.0489)^2 = 0.95$. These examples clearly demonstrate that the kth adjustment cannot reduce each of the error-components by a large factor.

We now summarize the results in this subsection:

(*i*) In a Class IV partial-response system, the eigenvalues λ_1 to λ_N of the **A** matrix depend only on the amplitude characteristic of the channel, but not on carrier phase, system timing, and phase characteristic of the channel.

(*ii*) These eigenvalues are very different in value. For typical values of N (13, 17, etc.), the minimum eigenvalue λ_1 is very close to 0, while the maximum eigenvalue λ_N is very close to 2.

(*iii*) Because of the large differences in the eigenvalues, each adjustment of the gain controls cannot reduce each of the error-components by a large factor.

3.2 *SSB Nyquist Systems*

In this section, we study SSB data communication systems which transmit at the Nyquist rate with sin x/x pulses (hereafter referred to as SSB Nyquist systems). Such a system has the same configuration as the Class IV partial-response system except that its transmitting and receiving filters have constant amplitude characteristics in the passband. The main advantage of an SSB Nyquist system is that it transmits at the maximum possible bauds (Nyquist rate) without noise penalty. Saltzberg[15] has shown that this signaling method is not as sensitive to timing error as is commonly believed. In fact we show in this section that this scheme exhibits fast start-up advantages.

The transmitting and receiving filters are specified by

$$\begin{aligned} |F_1(f)| &= 1, \qquad f_1 \leqq |f| \leqq f_2 \\ &= 0, \qquad \text{other } f. \end{aligned} \tag{46}$$

and

$$|F_3(f)| = 1, \qquad f_1 \leqq |f| \leqq f_2$$
$$= 0, \qquad \text{other } f. \tag{47}$$

The signaling interval T is again given by (31), and a_{ij} by (34). Substituting (46) and (47) into (34), we have

$$a_{ij} = (f_2 - f_1) \int_0^{f_2 - f_1} [\cos 2\pi f(i - j)T] \, |F_2(f - f_c)|^2 \, df. \tag{48}$$

Note again that a_{ij} (and hence the eigenvalues λ_1 to λ_N) depends only on the amplitude characteristics of the transmission medium and the filters. When the transmission medium has a constant amplitude characteristic

$$| F_2(f) | = 1$$

in the passband $f_1 \leqq f \leqq f_2$, (48) yields

$$a_{ij} = (f_2 - f_1)^2, \qquad i - j = 0$$
$$= 0, \qquad i - j \neq 0. \tag{49}$$

Neglecting the constant $(f_2 - f_1)^2$ above, we see that $\mathbf{A}$ is simply the identifying matrix and the eigenvalues of $\mathbf{A}$ are

$$\lambda_i = 1, \qquad i = 1 \text{ to } N. \tag{50}$$

Now consider the convergence of the error components $\xi_i(k)$, $i = 1$ to N. Prior to the first adjustment, the ith error component is

$$\xi_i(0) = (\mathbf{e}_0'\mathbf{u}_i)^2 \lambda_i . \tag{51}$$

After the first adjustment, the ith error component is

$$\xi_i(1) = (\mathbf{e}_0'\mathbf{u}_i)^2 \lambda_i (1 - \alpha_1 \lambda_i)^2. \tag{52}$$

Substituting (50) and (51) into (52) yields

$$\xi_i(1) = (1 - \alpha_1)^2 \xi_i(0). \tag{53}$$

If we set $\alpha_1 = 1$, $\xi_i(1) = 0$ for all i. In other words, when $\alpha_1 = 1$, the first adjustment reduces all the error-components to zero. Consequently the mean-square error ϵ is reduced to its irreducible value $\epsilon_{\min}$ after only one adjustment. This fastest possible convergence is obtained regardless of the initial equalizer settings, the carrier phase and system timing, the phase characteristic of the transmission medium, and the phase characteristics of the transmitting and receiving filters (because the eigenvalues λ_1 to λ_N are independent of all these parameters).

From (48), a_{ij} can be computed for any given $|F_2(f)|$. When $|F_2(f)|$ is not exactly a constant in the passband, **A** will not be an identity matrix and the eigenvalues will not be all equal. However, fast convergence is obtainable as long as the differences between the eigenvalues are small. For example, consider the case where the minimum eigenvalue is 0.9 and the maximum eigenvalue is 1.1. If we use $\alpha_1 = 1$, the maximum value of $(1 - \alpha_1\lambda_i)^2$ will be $(1 - 0.9)^2 = 0.01$. Consequently the first adjustment will reduce each of the error components by at least a factor of 100 (as will each following adjustment). The equalizer adjustment therefore will be completed after only a few adjustments.

IV. A NEW EQUALIZER STRUCTURE

In the preceding sections, we have clearly shown that, when the eigenvalues of **A** have close magnitudes, each adjustment of the equalizer reduces each of the error-components by a large factor (such as 100). We have also shown that such a fast convergence is not possible when the eigenvalues are very different in magnitude (such as in the case of a Class IV partial-response system). These results suggest that in order to improve the convergence rate we should attempt to reduce the differences between the eigenvalues.

Now we ask: *What causes the eigenvalues to be different? Is it possible to reduce such differences (and hence increase the convergence rate) by changing the equalizer structure? Does the use of the new equalizer structure alter the system's performance otherwise?* We shall consider the first two questions in this section, and derive a new equalizer structure. The last question will be considered in the next section. Application of the results to a Class IV partial-response system and the various related problems will be studied in Sections VI and VII.

It may appear that the new equalizer structure derived in this section requires complicated mathematical operations (computation of eigenvalues and eigenvectors of a matrix). However, it will be shown in Section VI that such mathematical operations can be completely eliminated for the systems of interest here.

Now consider the first question: What causes the eigenvalues to be different? From the definition in (6)

$$a_{ij} = \sum_{l=-\infty}^{\infty} x[(l-i+1)T]x[(l-j+1)T].$$

Since the input to the ith gain control is $x[t-(i-1)T]$, $x[(l-i+1)T]$ are time-samples of the input to the ith gain control, while $x[(l-j+1)T]$ are time-samples of the input to the jth gain control. Therefore, a_{ij} is simply the cross-correlation between the inputs to the ith and the jth gain control, and **A** has the interpretation of a correlation matrix. Therefore, we might state that it is the correlations between the inputs to the gain controls that result in the differences between the eigenvalues. When the inputs to the gain controls are orthonormal, **A** is an identity matrix and the eigenvalues are equal. When the inputs to the gain controls are correlated, the eigenvalues are different.

The above discussion suggest that, if we could construct a new equalizer such that the inputs to the gain controls are orthonormal, the eigenvalues of the correlation matrix of the new equalizer might be all equal and it might be possible to minimize the mean-square error in a single adjustment. Following this line of thinking, we obtain the generalized equalizer structure depicted in Fig. 3. As shown in Fig. 3, the equalizer input $x(t)$ is connected to a bank of filters. The output of the ith filter is connected through a variable gain control c_i to the summing bus. The equalizer output is

$$y(t) = \sum_{i=1}^{N} c_i z_i(t), \tag{54}$$

where $z_i(t)$ is the input to the ith gain control c_i. These signals $z_i(t)$, $i = 1$ to N, are orthonormal when

$$\sum_{l=-\infty}^{\infty} z_i(lT)z_j(lT) = \delta_{ij} . \tag{55}$$

In the following, we show that, for any given channel, the filters in Fig. 3 can be designed to satisfy (55). We also show that, when (55) is satisfied, the mean-square error can be minimized by a single adjustment of the gain controls.

Because of the press toward digitalization and integrated circuits, we shall specify the filters in Fig. 3 directly in digital filter form (instead of specifying them in analog filter form and then approximating them with digital filters). We shall use nonrecursive digital

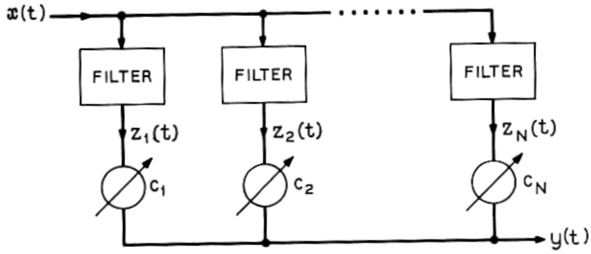

Fig. 3—Generalized equalizer structure.

filters[16] because they do not have stability problems and can be implemented most easily.

A realization of the generalized equalizer using nonrecursive digital filters is depicted in Fig. 4. The N digital filters share the same tapped delay line. The ith digital filter, $i = 1$ to N, consists of the tapped delay line and the N coefficients P_{i1} to P_{iN}. Its output is

$$z_i(t) = \sum_{j=1}^{N} P_{ij}x[t - (j - 1)T], \qquad i = 1 \quad \text{to} \quad N. \tag{56}$$

The gain controls c_1 to C_N are again adjusted to minimize the mean-square error ϵ defined in (1). From (54) and (56), we can rewrite (1) into the following matrix form

$$\epsilon = \mathbf{c'PAP'c} - 2\mathbf{c'Pv} + \sum_{k=-\infty}^{\infty} d_k^2 \tag{57}$$

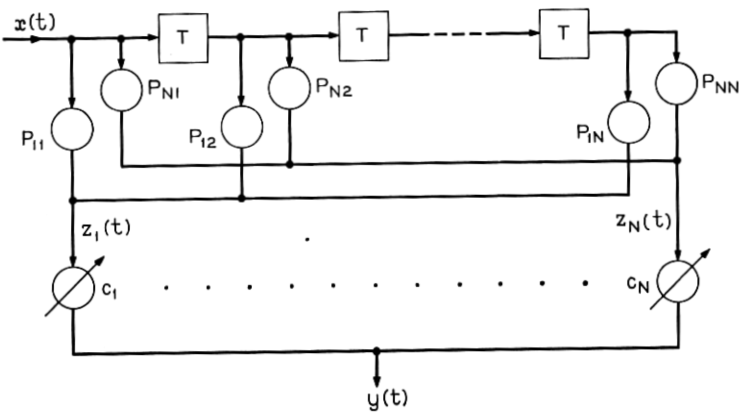

Fig. 4—A new equalizer structure (a realization of the generalized equalizer structure using nonrecursive digital filters).

where $\mathbf{c}$, $\mathbf{A}$, and $\mathbf{v}$ have been defined previously [see (4), (5), and (7)]. The matrix $\mathbf{P}$ is defined by

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & \vdots & & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix}. \tag{58}$$

As in Section II, we define $\partial\epsilon/\partial\mathbf{c}$ as the $N \times 1$ column vector whose ith element is $\partial\epsilon/\partial c_i$. It can be shown from (57) that

$$\frac{\partial\epsilon}{\partial\mathbf{c}} = 2\mathbf{PAP'c} - 2\mathbf{Pv}. \tag{59}$$

It can also be shown from (57) that $\mathbf{c}$ minimizes ϵ if and only if $\partial\epsilon/\partial\mathbf{c} = \mathbf{0}$; therefore, from (59) the optimum value of $\mathbf{c}$, $\mathbf{c}_{\text{opt}}$, is

$$\mathbf{c}_{\text{opt}} = (\mathbf{P}')^{-1}\mathbf{A}^{-1}\mathbf{v}. \tag{60}$$

We again define $\mathbf{e}$ as the difference between $\mathbf{c}$ and $\mathbf{c}_{\text{opt}}$; hence, from (60)

$$\mathbf{e} = \mathbf{c} - (\mathbf{P}')^{-1}\mathbf{A}^{-1}\mathbf{v}. \tag{61}$$

As in Section II, the initial values of $\mathbf{c}$, $\mathbf{e}$, and ϵ are denoted, respectively, by $\mathbf{c}_0$, $\mathbf{e}_0$, and ϵ_0. The values of $\mathbf{c}$, $\mathbf{e}$, and ϵ after the kth adjustment are denoted, respectively, by $\mathbf{c}_k$, $\mathbf{e}_k$, and ϵ_k. From (61)

$$\mathbf{e}_k = \mathbf{c}_k - (\mathbf{P}')^{-1}\mathbf{A}^{-1}\mathbf{v}, \qquad k = 0, 1, 2, \cdots. \tag{62}$$

It can be shown from (57) and (62) that

$$\epsilon_k = \sum_{k=-\infty}^{\infty} d_k^2 - \mathbf{v}'\mathbf{A}^{-1}\mathbf{v} + \mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k \qquad k = 0, 1, 2, \cdots. \tag{63}$$

The kth adjustment, $k = 1, 2, 3, \cdots$, of the gain control is again made according to (15). From (62), (15), and (59),

$$\begin{aligned} \mathbf{e}_k &= \mathbf{c}_{k-1} - \frac{1}{2}\alpha_k\left[\frac{\partial\epsilon}{\partial\mathbf{c}}\right]_k - (\mathbf{P}')^{-1}\mathbf{A}^{-1}\mathbf{v} \\ &= \mathbf{c}_{k-1} - (\mathbf{P}')^{-1}\mathbf{A}^{-1}\mathbf{v} - \alpha_k[\mathbf{PAP}'\mathbf{c}_{k-1} - \mathbf{Pv}]. \end{aligned} \tag{64}$$

From (62), we have

$$\mathbf{e}_{k-1} = \mathbf{c}_{k-1} - (\mathbf{P}')^{-1}\mathbf{A}^{-1}\mathbf{v}. \tag{65}$$

Using (65), we can rewrite (64) as

$$\begin{aligned}\mathbf{e}_k &= \mathbf{e}_{k-1} - \alpha_k[\mathbf{PAP}'\mathbf{c}_{k-1} - \mathbf{Pv}] \\ &= \mathbf{e}_{k-1} - \alpha_k\mathbf{PAP}'[\mathbf{c}_{k-1} - (\mathbf{P}')^{-1}\mathbf{A}^{-1}\mathbf{v}] \\ &= [\mathbf{I} - \alpha_k\mathbf{PAP}']\mathbf{e}_{k-1}\ ,\end{aligned}$$

$$k = 1, 2, \cdots . \tag{66}$$

Now we can prove the first statement after (55) [that is, for any given channel, **P** can be chosen to satisfy (55)]. It is easily shown from (58), (5), (6), and (56) that **PAP′** can be written in the form

$$\mathbf{PAP}' = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & \vdots & & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix}. \tag{67}$$

where

$$b_{ij} = \sum_{l=-\infty}^{\infty} z_i(lT)z_j(lT). \tag{68}$$

Equation (55) is therefore equivalent to $b_{ij} = \delta_{ij}$, or

$$\mathbf{PAP}' = \mathbf{I}. \tag{69}$$

From (18),

$$\mathbf{A} = \mathbf{QDQ}'. \tag{18}$$

Let **H** be an $N \times N$ diagonal matrix whose ith diagonal element is $\sqrt{\lambda_i}$, that is,

$$\mathbf{H} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_N} \end{bmatrix}. \tag{70}$$

Then $\mathbf{D} = \mathbf{HH}'$ and (18) becomes

$$\mathbf{A} = \mathbf{QHH}'\mathbf{Q}'. \tag{71}$$

Substituting (71) into (69) yields

$$\mathbf{PQH}(\mathbf{PQH})' = \mathbf{I}. \tag{72}$$

Equation (72) holds if and only if **PQH** is an orthogonal matrix **G**.

From $\mathbf{Q'Q} = \mathbf{I}$, we can rewrite

$$\mathbf{PQH} = \mathbf{G} \tag{73}$$

as

$$\mathbf{P} = \mathbf{GH}^{-1}\mathbf{Q}'. \tag{74}$$

For any given channel [i.e., for any given $x(t)$], we can compute a_{ij} from (6) and determine the eigenvalues λ_1 to λ_N and the eigenvectors $\mathbf{u}_1$ to $\mathbf{u}_N$. This determines $\mathbf{D}$, $\mathbf{Q}$, and $\mathbf{H}$. There are an infinite number of $N \times N$ orthogonal matrices. Any of them can be used as $\mathbf{G}$ (the choice of $\mathbf{G}$ will be discussed later). Then from (74) $\mathbf{P}$ can be determined, and this $\mathbf{P}$ satisfies (72), (69), and (55). This proves that, for any given channel, the digital filters in Fig. 4 can be designed to satisfy (55).

Next we prove the second statement after (55). As in Section II, we use $\epsilon_{\min}$ to denote the minimum value of ϵ when $\mathbf{c} = \mathbf{c}_{\text{opt}}$. Substituting $\mathbf{c}_{\text{opt}}$ in (60) into (57) yields

$$\epsilon_{\min} = \sum_{k=-\infty}^{\infty} d_k^2 - \mathbf{v}'\mathbf{A}^{-1}\mathbf{v}. \tag{75}$$

Combining (63) and (75) gives

$$\epsilon_k = \epsilon_{\min} + \mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k \qquad k = 0, 1, 2, \cdots. \tag{76}$$

From (76), the initial value of ϵ is

$$\epsilon_0 = \epsilon_{\min} + \mathbf{e}_0'\mathbf{PAP}'\mathbf{e}_0, \tag{77}$$

and the value of ϵ after the first adjustment is

$$\epsilon_1 = \epsilon_{\min} + \mathbf{e}_1'\mathbf{PAP}'\mathbf{e}_1. \tag{78}$$

From (66), $\mathbf{e}_1 = [\mathbf{I} - \alpha_1\mathbf{PAP}']\mathbf{e}_0$. Substituting this into (78), we have

$$\epsilon_1 = \epsilon_{\min} + \mathbf{e}_0'[\mathbf{I} - \alpha_1\mathbf{PAP}']\mathbf{PAP}'[\mathbf{I} - \alpha_1\mathbf{PAP}']\mathbf{e}_0. \tag{79}$$

when $\mathbf{P}$ satisfies (74), (69) holds; i.e., $\mathbf{PAP}' = \mathbf{I}$. If we set $\alpha_1 = 1$, $\mathbf{I} - \alpha_1\mathbf{PAP}' = 0$. Then from (79) $\epsilon_1 = \epsilon_{\min}$. This proves that, when $\mathbf{P}$ satisfies (74) and $\alpha_1 = 1$, the mean-square error ϵ reduces to its minimum value $\epsilon_{\min}$ after only one adjustment of the gain controls.

In general it can be seen from (66) that, when $\mathbf{P}$ satisfies (74) and $\alpha_k = 1$, $\mathbf{e}_k = [\mathbf{I} - \alpha_k\mathbf{PAP}']\mathbf{e}_{k-1} = 0$, regardless of the value of $\mathbf{e}_{k-1}$. Consequently $\epsilon_k = \epsilon_{\min}$, regardless of the value of ϵ_{k-1}. This means that each of the adjustments is capable of reducing the mean-square error to its minimum value ϵ_{min}, regardless of what is the value of ϵ prior to that adjustment.

Now we summarize this section. It is emphasized at the beginning of this section that it is the correlations between the inputs to the gain controls that determine the differences between the eigenvalues and consequently the rate of convergence of ϵ. When the inputs to the gain controls are orthonormal, ϵ can be minimized in one adjustment. The inputs to the gain controls can be made orthonormal by using the generalized equalizer structure depicted in Fig. 3. The generalized equalizer can be realized with digital or analog filters. A realization using nonrecursive digital filters is depicted in Fig. 4, and is analyzed in detail. It is shown that, for any given channel, one can design the digital filters from the simple equation (74) to orthonormalize the inputs to the gain controls. When the digital filters are so designed, each adjustment of the gain controls can reduce ϵ to its minimum value, regardless of the value of ϵ prior to that adjustment. Therefore, ϵ can be minimized by only one adjustment of the gain controls.

V. FURTHER PROPERTIES OF THE NEW EQUALIZER

We have seen in the preceding section that it is possible to minimize ϵ in one adjustment. In this section we show that such an improvement in convergence rate is obtained without changing the residue noise power, the minimum mean-square error, and the convexity of the adjustments, and without complicating the gain-control adjustment loop.

5.1 *Residual Noise Power and Minimum Mean-Square Error*

As described in Section II, during data transmission the equalizer output is sampled sequentially at time-instants $t = \cdots, t_2 - T, t_2, t_2 + T, t_2 + 2T, \cdots$, and the time-samples are used to recover the transmitted information digits. Each of these time-samples consists of a signal and a noise component. The variance of this noise component (that is, the residue noise power) can be determined for both the new and the conventional equalizer.

Consider first the new equalizer (Fig. 4). Let the noise at the input of the equalizer be denoted by $n(t)$. The resulting noise at the input of c_i is denoted by $\mu_i(t)$, while the resulting noise at the equalizer output is denoted by $\nu(t)$. Clearly

$$\mu_i(t) = \sum_{j=1}^{N} P_{ij} n[t - (j - 1)T], \qquad i = 1 \quad \text{to} \quad N \tag{80}$$

and

$$\nu(t) = \sum_{i=1}^{N} c_i \mu_i(t). \tag{81}$$

We assume that the noise $n(t)$ is a zero-mean stationary Gaussian process. Then $\mu_i(t)$, $i = 1$ to N, and $\nu(t)$ are also zero-mean stationary Gaussian processes. Consequently, the variance of $\nu(t_k)$ is the same for all t_k. It is therefore sufficient to consider a single t_k (for example, $t_k = 0$). From (80) and (81)

$$\nu(0) = \sum_{i=1}^{N} c_i \sum_{j=1}^{N} P_{ij} n[(1 - j)T]. \tag{82}$$

Let $\mathbf{n}$ be an $N \times 1$ vector whose ith element is $n[(1 - i)T]$. It can be easily shown from (82) that

$$\nu(0) = \mathbf{c}'\mathbf{P}\mathbf{n}. \tag{83}$$

The noise vector $\mathbf{n}$ is distributed normally with zero mean. Let the covariance matrix of $\mathbf{n}$ be denoted by $\mathbf{\Lambda}$. Then $\nu(0)$ is distributed as the normal distribution with zero mean and variance $\mathbf{c}'\mathbf{P}\mathbf{\Lambda}\mathbf{P}'\mathbf{c}$. Since $\mathbf{c}$ can be optimized in the training period, we assume that during data transmission $\mathbf{c} = \mathbf{c}_{\text{opt}}$. Thus, during data transmission the variance of $\nu(0)$ is $\text{Var}\,[\nu(0)] = \mathbf{c}'_{\text{opt}}\mathbf{P}\mathbf{\Lambda}\mathbf{P}'\mathbf{c}_{\text{opt}}$. Substituting (60) into above gives

$$\text{Var}\,[\nu(0)] = \mathbf{v}'\mathbf{A}^{-1}\mathbf{\Lambda}\mathbf{A}^{-1}\mathbf{v}. \tag{84}$$

Note from (84) that $\text{Var}[\nu(0)]$ is independent of $\mathbf{P}$.

Next consider the conventional equalizer. It can be easily seen that when $\mathbf{P} = \mathbf{I}$ the new equalizer in Fig. 4 is reduced to the conventional equalizer in Fig. 1. Since $\text{Var}[\nu(0)]$ in (84) remains unchanged when $\mathbf{P}$ is set to $\mathbf{I}$, $\text{Var}[\nu(0)]$ of the new equalizer is equal to $\text{Var}[\nu(0)]$ of the conventional equalizer. In other words, the use of the new equalizer does not change the residue noise power of the system.

The minimum mean-square error, $\epsilon_{\min}$, has already been determined in the preceding sections. Comparing (12) with (75) shows that the minimum mean-square error does not change when the new equalizer is used instead of the conventional equalizer.

5.2 *Convexity of the Adjustment*

It has been shown[3,4] that, when the conventional equalizer is used, ϵ is a strict convex function of the gain controls $\mathbf{c}$. This ensures that ϵ converges for all initial settings of $\mathbf{c}$, and that ϵ converges when the gradient method [equation (15)] is approximated by certain other iterative techniques. In this subsection we show that ϵ remains as a strict convex function of $\mathbf{c}$ when the new equalizer is used.

Let $\mathcal{K}$ denote the open convex set $\{\mathbf{c}: -\infty < c_i < \infty, i = 1 \text{ to } N\}$. Let $\mathbf{h}$ be an $N \times 1$ vector whose ith element is h_i. Clearly,[17] ϵ is a function of class $c^{(2)}$ on the set $\mathcal{K}$. If we can show that

$$\sum_{i=1}^{N} \sum_{j=1}^{N} h_i h_j \frac{\partial^2 \epsilon}{\partial c_i \, \partial c_j} > 0 \tag{85}$$

for every $\mathbf{c} \in \mathcal{K}$ and every $\mathbf{h} \neq \mathbf{0}$, then [17] ϵ is strictly convex on $\mathcal{K}$.

It can be shown from (57) that

$$\frac{\partial^2 \epsilon}{\partial c_i \, \partial c_j} = 2b_{ij}, \quad \text{all} \quad i \quad \text{and} \quad j \tag{86}$$

where b_{ij} has been specified in (68). From (86) and (67), we have

$$\sum_{i=1}^{N} \sum_{j=1}^{N} h_i h_j \frac{\partial^2 \epsilon}{\partial c_i \, \partial c_j} = 2\mathbf{h'PAP'h}. \tag{87}$$

It can be seen from (87) that (85) holds for every $\mathbf{c} \in \mathcal{K}$ and every $\mathbf{h} \neq \mathbf{0}$ if and only if $\mathbf{PAP'}$ is positive definite. Since $\mathbf{P}$ is a nonsingular matrix [see (74)], and $\mathbf{A}$ is positive definite, $\mathbf{PAP'}$ is positive definite. Therefore, (85) holds for every $\mathbf{c} \in \mathcal{K}$ and every $\mathbf{h} \neq \mathbf{0}$, and consequently ϵ is strictly convex on $\mathcal{K}$. This proves that ϵ remains as a strict convex function of $\mathbf{c}$ when the new equalizer is used.

5.3 *Gain-Control Adjustment Loop*

In the case of the conventional equalizer, the gain controls $\mathbf{c}$ are adjusted according to (15). The partial derivative $\partial\epsilon/\partial c_i$ used in (15) is obtained[3] by correlating the time-samples of the tap signal and the time-samples of the error signal $y(t) - d(t)$.

In the case of the new equalizer, the gain controls $\mathbf{c}$ are again adjusted according to (15). Since

$$\begin{aligned} y(t) &= \sum_{i=1}^{N} c_i z_i(t), \\ \frac{\partial \epsilon}{\partial c_i} &= \frac{\partial}{\partial c_i} \left\{ \sum_{k=-\infty}^{\infty} [y(kT) - d(kT)]^2 \right\} \\ &= 2 \sum_{k=-\infty}^{\infty} [z_i(kT)][y(kT) - d(kT)], \\ & \qquad i = 1 \quad \text{to} \quad N. \end{aligned} \tag{88}$$

It is seen from (88) that $\partial\epsilon/\partial c_i$ can be obtained similarly by correlating the time-samples of $z_i(t)$ and the time-samples of $y(t) - d(t)$. Therefore, the gain-control adjustment loop of the new equalizer is essentially the same as that of the conventional equalizer.

VI. APPLICATION OF THE NEW EQUALIZER STRUCTURE

In this and the next sections, we consider how to use the new equalizer structure for a Class IV partial-response system. It will be shown, both analytically and by computer simulation, that when the amplitude characteristic of the transmission medium does not vary appreciably from channel to channel (such as in the case of multiparty private line polling systems), we can use a fixed **P** matrix for the new equalizer (that is, P_{ij} need not be adjusted in each training period). Such a simplification is possible even though the system timing, the demodulating carrier phase, and the phase characteristic of the transmission medium change from channel to channel.

It has been shown in Section IV that ϵ can be minimized in one adjustment if $\alpha_k = 1$ and **P** satisfies the equation

$$\mathbf{PAP'} = \mathbf{I}. \tag{69}$$

As pointed out in Section 3.1, the elements a_{ij} of the **A** matrix are independent of the demodulating carrier phase, the system timing, the phase characteristics of the transmitting and receiving filters, and the phase characteristic of the transmission medium. Therefore, from (69), **P** is independent of all these parameters. It can also be seen from (36) in Section 3.1 that a_{ij} depends only on the amplitude characteristic $|F_2(f)|$ of the transmission medium (amplitude distortions of the transmitting and receiving filters are usually negligible). In some systems, such as private-line polling systems, $|F_2(f)|$ does not vary appreciably from channel to channel. For such systems, a fixed **P** matrix may approximately satisfy (69) for all channels. Therefore, in designing such systems, we may estimate $|F_2(f)|$ and compute from this an estimate of **A** (this estimate of **A** will be denoted by **S**). We can then estimate **P** from the equation

$$\mathbf{PSP'} = \mathbf{I}. \tag{89}$$

and use this estimated **P** for all channels. When the estimated $|F_2(f)|$ agrees with the actual $|F_2(f)|$, **S** agrees with **A**. Consequently the estimated **P** satisfies (69), and ϵ is minimized in one adjustment. When the estimated $|F_2(f)|$ differs from the actual $|F_2(f)|$, the estimated **P** will not satisfy (69) and the convergence rate of ϵ will be reduced. In order to see whether the reduced convergence rate is satisfactory, we need formulas to relate the convergence rate to $|F_2(f)|$. Such formulas will be derived in this section, and will be used in the next section for a hypothetical data communication system.

Let the difference between the estimate of **A** and **A** be denoted by **R**, that is,

$$\mathbf{R} = \mathbf{S} - \mathbf{A}. \tag{90}$$

Since **S** is an correlation matrix, it is symmetric. Therefore, **R** is also a symmetric matrix.

From (66) and (90), we have

$$\mathbf{e}_k = [\mathbf{I} - \alpha_k \mathbf{PSP}' + \alpha_k \mathbf{PRP}']\mathbf{e}_{k-1}, \qquad k = 1, 2, \cdots . \tag{91}$$

As in Section IV, we set

$$\alpha_k = 1, \qquad \text{all } k. \tag{92}$$

Since the estimated **P** is computed from (89), we can substitute (89) and (92) into (91) to obtain

$$\mathbf{e}_k = \mathbf{PRP}'\mathbf{e}_{k-1}, \qquad k = 1, 2, \cdots . \tag{93}$$

It can be easily seen from the recursive equation (93) that

$$\mathbf{e}_k = (\mathbf{PRP}')^k \mathbf{e}_0, \qquad k = 1, 2, \cdots . \tag{94}$$

From (76), the mean-square error after the kth adjustment is

$$\epsilon_k = \epsilon_{\min} + \mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k . \tag{76}$$

We now evaluate the last term $\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k$ in the equation above. It can be shown from (89) that

$$\mathbf{P}'\mathbf{P} = \mathbf{S}^{-1}. \tag{95}$$

From (95), (94) can be rewritten as

$$\mathbf{e}_k = \mathbf{PR}(\mathbf{S}^{-1}\mathbf{R})^{k-1}\mathbf{P}'\mathbf{e}_0, \qquad k = 1, 2 \cdots . \tag{96}$$

From (96) and (95), we obtain

$$\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k = \mathbf{e}_0'\mathbf{P}(\mathbf{RS}^{-1})^k\mathbf{A}(\mathbf{S}^{-1}\mathbf{R})^k\mathbf{P}'\mathbf{e}_0, \quad k = 1, 2, \cdots . \tag{97}$$

Using $\mathbf{A} = \mathbf{S} - \mathbf{R}$ and (89), we can rearrange (97) into the following form

$$\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k = \mathbf{e}_0'\mathbf{P}(\mathbf{RS}^{-1})^{2k}(\mathbf{I} - \mathbf{RS}^{-1})\mathbf{P}^{-1}\mathbf{e}_0, \qquad k = 1, 2, \cdots . \tag{98}$$

From (77), the initial value of mean-square error is

$$\epsilon_0 = \epsilon_{\min} + \mathbf{e}_0'\mathbf{PAP}'\mathbf{e}_0 . \tag{77}$$

It can be easily shown from (90) and (89) that

$$\mathbf{e}_0'\mathbf{PAP}'\mathbf{e}_0 = \mathbf{e}_0'\mathbf{P}[\mathbf{I} - \mathbf{RS}^{-1}]\mathbf{P}^{-1}\mathbf{e}_0 . \tag{99}$$

Since **A** is symmetric, **PAP′** is symmetric. Let $\zeta_1, \zeta_2, \cdots, \zeta_N$ be the eigenvalues of **PAP′**, and let $\mathbf{w}_1, \mathbf{w}_2, \cdots \mathbf{w}_N$ be a set of orthonormal eigenvectors of **PAP′** ($\mathbf{w}_i$ corresponds to ζ_i). From

$$\mathbf{PAP}'\mathbf{w}_i = \zeta_i\mathbf{w}_i \tag{100}$$

and [see (99)]

$$\mathbf{PAP}' = \mathbf{P}[\mathbf{I} - \mathbf{RS}^{-1}]\mathbf{P}^{-1} \tag{101}$$

we have

$$\mathbf{RS}^{-1}\mathbf{P}^{-1}\mathbf{w}_i = (1 - \zeta_i)\mathbf{P}^{-1}\mathbf{w}_i . \tag{102}$$

It is seen from (102) that the ith eigenvalue of $\mathbf{RS}^{-1}$ is $(1 - \zeta_i)$, and the ith eigenvector of $\mathbf{RS}^{-1}$ is $\mathbf{P}^{-1}\mathbf{w}_i$. We shall denote the ith eigenvalue of $\mathbf{RS}^{-1}$ by $\hat{\lambda}_i$ so

$$\hat{\lambda}_i = 1 - \zeta_i , \qquad i = 1 \text{ to } N. \tag{103}$$

Now consider the matrix product $\mathbf{P}(\mathbf{RS}^{-1})^{2k}(\mathbf{I} - \mathbf{RS}^{-1})\mathbf{P}^{-1}$ in (98). To facilitate writing, we shall denote this product by $\mathbf{\Phi}$, that is

$$\mathbf{\Phi} = \mathbf{P}(\mathbf{RS}^{-1})^{2k}(\mathbf{I} - \mathbf{RS}^{-1})\mathbf{P}^{-1}. \tag{104}$$

From the eigenvalue and eigenvectors of $\mathbf{RS}^{-1}$, we see that the ith eigenvalue of $\mathbf{\Phi}$ is $\hat{\lambda}_i^{2k}(1 - \hat{\lambda}_i)$, and the ith eigenvector of $\mathbf{\Phi}$ is $\mathbf{w}_i$. We shall denote the ith eigenvalue of $\mathbf{\Phi}$ by δ_i , so

$$\delta_i = \hat{\lambda}_i^{2k}(1 - \hat{\lambda}_i). \tag{105}$$

We have shown in the above that the orthonormal eigenvectors $\mathbf{w}_1$ to $\mathbf{w}_N$ of **PAP′** are also eigenvectors of $\mathbf{\Phi}$. Hence, **PAP′** and $\mathbf{\Phi}$ can be simultaneously diagonalized by $\mathbf{w}_1$ to $\mathbf{w}_N$. By such diagonalizations we can reduce (99) to the form

$$\mathbf{e}_0'\mathbf{PAP}'\mathbf{e}_0 = \sum_{i=1}^{N} (1 - \hat{\lambda}_i)(\mathbf{e}_0'\mathbf{w}_i)^2 \tag{106}$$

and reduce (98) to the form

$$\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k = \sum_{i=1}^{N} \hat{\lambda}_i^{2k}(1 - \hat{\lambda}_i)(\mathbf{e}_0'\mathbf{w}_i)^2, \qquad k = 1, 2, \cdots . \tag{107}$$

It can be seen from (106) and (107) that the convergence rate of ϵ depends on $\hat{\lambda}_1$ to $\hat{\lambda}_N$. Let $|\hat{\lambda}_i|_{\max}$ denote the largest $|\hat{\lambda}_i|$ among $|\hat{\lambda}_1|$ to $|\hat{\lambda}_N|$, that is,

$$|\hat{\lambda}_i|_{\max} \geqq |\hat{\lambda}_i|, \qquad i = 1 \text{ to } N. \tag{108}$$

Since $\mathbf{PAP'}$ is positive definite, its eigenvalues must all be positive. Hence,

$$\zeta_i = 1 - \hat{\lambda}_i > 0, \qquad i = 1 \text{ to } N. \tag{109}$$

From (106) to (109), the following bound is obtained

$$\mathbf{e}_k'\mathbf{PAP'e}_k \leqq [|\hat{\lambda}_i|_{\max}]^{2k}\mathbf{e}_0'\mathbf{PAP'e}_0, \quad k = 1, 2, \cdots. \tag{110}$$

Based on (110), we have the first method of estimating the convergence rate of $\mathbf{e}_k'\mathbf{PAP'e}_k$:

First Method. From each $|F_2(f)|$, compute $\hat{\lambda}_1$ to $\hat{\lambda}_N$. This gives $|\hat{\lambda}_i|_{\max}$. According to (110), $\mathbf{e}_k'\mathbf{PAP'e}_k$ reduces by at least a factor of $[|\hat{\lambda}_i|_{\max}]^{-2}$ after each adjustment. This completes the estimation.

This method will be illustrated in the next section. In certain applications, measurements of $|F_2(f)|$ may not be available. It may only be specified that $|F_2(f)|$ varies within certain bounds. In such cases, $|\hat{\lambda}_i|_{\max}$ cannot be determined. However, it can be bounded from the bounds of $|F_2(f)|$. It is shown in Appendix B that

$$1 - [\eta(F)]_{\max} \leqq \hat{\lambda}_i \leqq 1 - [\eta(f)]_{\min}, \qquad i = 1 \text{ to } N. \tag{111}$$

where

$$\eta(f) = \frac{[|F_2(f - f_c)|_{\text{act}}]^2}{[|F_2(f - f_c)|_{\text{est}}]^2}, \qquad 0 \leqq f \leqq f_2 - f_1. \tag{112}$$

As explained in Appendix B, $|F_2(f - f_c)|_{\text{act}}$ is the actual value of $|F_2(f - f_c)|$, $|F_2(f - f_c)|_{\text{est}}$ is the estimated value of $|F_2(f - f_c)|$, $[\eta(f)]_{\max}$ is the maximum value of $\eta(f)$ in the frequency range $0 \leqq f \leqq f_2 - f_1$, *and* $[\eta(f)]_{\min}$ is the minimum value of $\eta(f)$ in the frequency range of $0 \leqq f \leqq f_2 - f_1$.

Based on (110) to (112), we have the second method of estimating the convergence rate of $\mathbf{e}_k'\mathbf{PAP'e}_k$:

Second Method. From the bounds of $|F_2(f)|$, determine $[\eta(f)]_{\max}$ and $[\eta(f)]_{\min}$ from (112). If

$$|1 - [\eta(f)]_{\min}| > |1 - [\eta(f)]_{\max}|,$$

use $|1 - [\eta(f)]_{\min}|$ as the upper bound of $|\hat{\lambda}_i|_{\max}$. If

$$|1 - [\eta(f)]_{\max}| > |1 - [\eta(f)]_{\min}|,$$

use $| 1 - [\eta(f)]_{\max} |$ as the upper bound of $| \hat{\lambda}_i |_{\max}$. Use the upper bound of $| \hat{\lambda}_i |_{\max}$ as $| \hat{\lambda}_i |_{\max}$ to obtain from (110) an estimate of the minimum convergence rate of $\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k$.

As illustrated in the next section, the second method requires only some simple hand-calculations, and hence may be applied first even if measurements of $| F_2(f) |$ are available. However, it should be warned that the second method is usually too pessimistic because the upper bound of $| \hat{\lambda}_i |_{\max}$, obtained in the above fashion, can be several times larger than the actual $| \hat{\lambda}_i |_{\max}$. (Consequently, the convergence rate may appear to be unsatisfactory when it is actually satisfactory.) Therefore, for borderline cases, the first method should be used.

We illustrate these two methods in the next section.

VII. NUMERICAL EXAMPLES AND COMPUTER SIMULATION

In order to illustrate the methods we consider a hypothetical private-line data communication system (hereafter referred to as the hypothetical system). We assume that the system uses a Class IV partial-response signal and transmits over voiceband at a baud rate of 4800 bauds. The cutoff frequencies f_1 and f_2 defined in Section 3.1 will be 400 and 2800 Hz respectively. The demodulating carrier frequency f_c is equal to f_2 . It is assumed that the new equalizer in Fig. 4 is used with $N = 13$ and a prefixed $\mathbf{P}$ matrix computed from (89) (see footnote in Section 7.1). The two methods will be used in Sections 7.1 and 7.2, which follow, to estimate the convergence rate of $\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k$ {the convergence rate of the mean-square error ϵ is identical to the convergence rate of $\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k$ [see (76) in Section IV]}. A third and very elaborate method, computer simulation of the data communication system, will be used in Section 7.3 to demonstrate the convergence rate.

7.1 *Estimation by the First Method*

Since $| F_2(f) |$ of private lines does not deviate considerably from a constant in the frequency band 400 to 2800 Hz, we shall simply use a constant as the estimate of $| F_2(f) |$ in this frequency band, and compute a $\mathbf{P}$ matrix accordingly from (89). This prefixed $\mathbf{P}$ matrix will be used for all lines.

Now consider what happens to the convergence rate when the private line has a $| F_2(f) |$ as depicted in Fig. 5. It can be seen that this $| F_2(f) |$ varies from -4.8 dB to 0.4 dB, and then to -6.3 dB in the frequency band 400 to 2800 Hz. Such a variation is large for private lines. However, we show that even for this large variation the prefixed $\mathbf{P}$ matrix

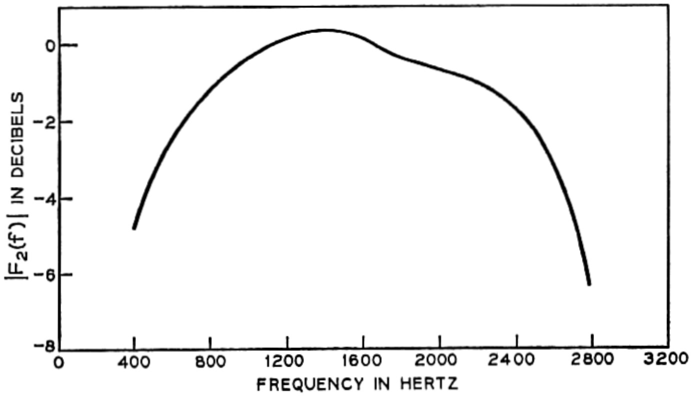

Fig. 5—The $|F_2(f)|$ used in Section 7.1.

can be used and the convergence rate of the mean-square error is satisfactory.

In order to use (110), we compute the eigenvalues $\hat{\lambda}_1$ to $\hat{\lambda}_N$ in the following steps:

Step 1. Compute the elements a_{ij} of the **A** matrix from (36) and the $| F_2(f) |$ in Fig. 5. It is sufficient to compute only the first row of **A**. To see this, note from (6) that a_{ij} satisfies the condition

$$a_{ij} = a_{ji} \tag{113}$$

and the condition

$$a_{ij} = a_{(i+h)(j+h)} . \tag{114}$$

From (113) and (114), we see that **A** can be written in the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1N} \\ a_{12} & a_{11} & a_{12} & a_{13} & \cdots & a_{1(N-1)} \\ a_{13} & a_{12} & a_{11} & a_{12} & \cdots & a_{1(N-2)} \\ a_{14} & a_{13} & a_{12} & a_{11} & \cdots & a_{1(N-3)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{1N} & a_{1(N-1)} & a_{1(N-2)} & a_{1(N-3)} & \cdots & a_{11} \end{bmatrix}, \tag{115}$$

where the elements on the diagonal line are all equal to a_{11}, and the elements on each off-diagonal line are equal (for example, the elements on the second off-diagonal line are all equal to a_{13}). Thus, it is sufficient to compute only the first row of **A**.

Step 2. Compute the **S** matrix. It can be seen from Section 3.1 (from (37) to (40)) that, when

$$| F_2(f) | = \text{a constant}, \qquad f_1 \leqq f \leqq f_2 , \tag{116}$$

we have

$$\mathbf{A} = a_{11}\mathbf{\Delta}, \tag{117}$$

where

$$\mathbf{\Delta} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \cdots & 0 \\ -\frac{1}{2} & 0 & 1 & 0 & \cdots & 0 \\ 0 & -\frac{1}{2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{118}$$

Since we have used a constant as the estimate of $|F_2(f)|$, we have

$$\mathbf{S} = a_{11}\mathbf{\Delta}. \tag{119*}$$

Step 3. Compute $\mathbf{R} = \mathbf{S} - \mathbf{A}$, and compute the eigenvalues $\hat{\lambda}_1$ to $\hat{\lambda}_N$ of $\mathbf{RS}^{-1}$.

The computations in the above three steps can best be carried out by a computer program. Such a program has been written. For the $| F_2(f) |$ in Fig. 5, the eigenvalues of $\mathbf{RS}^{-1}$ are found to be: -0.1573, -0.1434, -0.0902, -0.0615, -0.0034, 0.0242, 0.0716, 0.1084, 0.1834, 0.1945, 0.2954, 0.3797, and 0.4651. The largest magnitude of these eigenvalues is 0.4651; therefore,

$$| \hat{\lambda}_i |_{\max} = 0.4651. \tag{120}$$

Substituting (120) into (110) gives

$$\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k \leqq \left[\frac{1}{4.62}\right]^k \mathbf{e}_0'\mathbf{PAP}'\mathbf{e}_0 . \tag{121}$$

It can be seen from (121) that $\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k$ reduces at least 4.62 times

* From (119) and (89), we may determine the **P** matrix. Since **P** is not needed for the following computation, we do not carry out such calculations here. However, it should be pointed out that the method from (69) to (74) may be used for such calculations. It should also be noted that by setting the **G** matrix in (74) to **I**, we can reduce most of the elements in **P** to zero. For example, when N is a multiple of four, it is sufficient to implement only $N^2/8$ of the P_{ij} s in Fig. 4. This greatly simplifies the implementation.

after each adjustment. In other words, each adjustment reduces the mean-square error by at least 6.65 dB. This convergence rate is satisfactory. For example, let ϵ_{min} be 0.01 and let the initial mean-square error ϵ_0 be as large as 4 (see computer simulation in Section 7.3). From (76), (77), and (121), we see that after four adjustments (approximately 16 milliseconds) the mean-square error ϵ reduces to less than 0.0187. Clearly, this convergence rate is satisfactory.

7.2 *Estimation by the Second Method*

When measurements of $|F_2(F)|$ are not available, we may estimate the convergence rate from the bounds of $|F_2(f)|$. For illustrative purposes, let us assume that $|F_2(f)|$ does not deviate from unity by more than -2.5 dB or 1.5 dB, that is,

$$-2.5 \leqq 20 \log_{10} |F_2(f)| \leqq 1.5, \qquad 400 \leqq f \leqq 2800.$$

Notice that $|F_2(f)|$ can vary in any manner within these bounds. We simply use unity as the estimate of $|F_2(f)|$. Then from (112), $[\eta(f)]_{max} = 1.414$ and $[\eta(f)]_{min} = 0.562$. Since $|1 - [\eta(f)]_{min}| > |1 - [\eta(f)]_{max}|$, we use $|1 - [\eta(f)]_{min}|$ as the upper bound of $|\hat{\lambda}_i|_{max}$, that is, $|\hat{\lambda}_i|_{max} \leqq 0.438$. Using $|\hat{\lambda}_i|_{max} = 0.438$, we obtain from (110)

$$\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k \leqq \left[\frac{1}{5.2}\right]^k \mathbf{e}_0'\mathbf{PAP}'\mathbf{e}_0 .$$

Thus, $\mathbf{e}_k'\mathbf{PAP}'\mathbf{e}_k$ reduces at least 5.2 times after each adjustment (a 7.16 dB reduction per adjustment). Notice that since the actual value of $|\hat{\lambda}_i|_{max}$ may be much less than 0.438, the actual convergence rate can be much faster.

7.3 *Computer Simulation of the Hypothetical System*

The analytical methods in the preceding subsections yield bounds of the convergence rate. In order to demonstrate the actual convergence rate, we simulate the hypothetical system on the computer, using a number of private-line characteristics obtained from field measurement. In order to conserve space we describe here only the most important results. Fig. 6 shows the convergence of the mean-square error for a private line which has severe amplitude and phase distortions (see Table I). Other lines simulated have less severe distortions and hence faster convergence rates. A 30-dB signal-to-noise ratio is assumed at the equalizer input. The bottom curve shows the convergence of ϵ when the new equalizer is used (with a prefixed $\mathbf{P}$ matrix computed from (119) and (89)). It can be seen that, when the

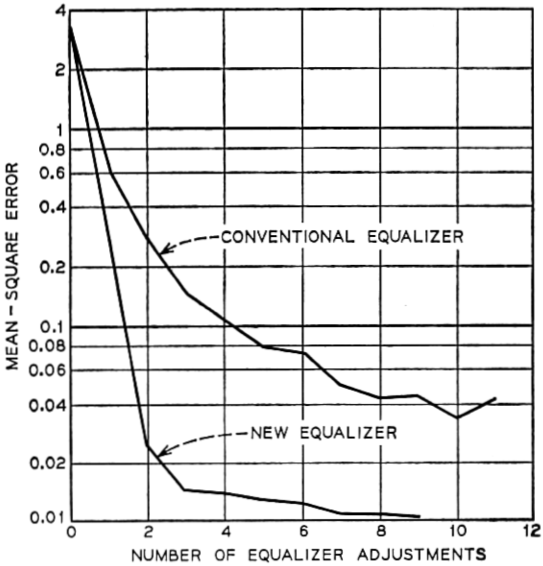

Fig. 6—Convergence of the mean-square error.

Table I—Relative Envelope Delay and Relative Loss Characteristics of Private Line used in Section 7.3.

Frequency (Hz)	Relative Envelope Delay (μs)	Relative Loss (dB)
300	5500	6.3
500	2830	2.4
600	2060	1.9
800	1040	0.6
1000	590	0
1200	390	−0.9
1400	280	−1.3
1600	150	−0.8
1800	0	0
2000	−100	0.7
2200	− 80	1.0
2400	15	2.2
2600	270	2.7
2800	260	4.5
3000	1500	7.1

new equalizer is used, ϵ converges rapidly to its minimum value, and the training period may be terminated after the third adjustment (three adjustments require approximately 12 milliseconds). These simulation results are in full agreement with the theoretical results in the preceding subsections.

The top curve in Fig. 6 shows the convergence of ϵ when the conventional equalizer in Fig. 1 is used (also with $N = 13$). It can be seen that the convergence of ϵ is rather slow, and that it takes more than 11 adjustments (approximately 44 milliseconds) to reduce ϵ to close to its minimum value.

Figure 6 is obtained with a specific setting of system timing and demodulating carrier phase. The computer program has also been executed for a sufficiently large number of other timing and carrier phase settings. Curves similar to those in Fig. 6 have been obtained. The results may be summarized in Fig. 7. The horizontal axis in Fig. 7

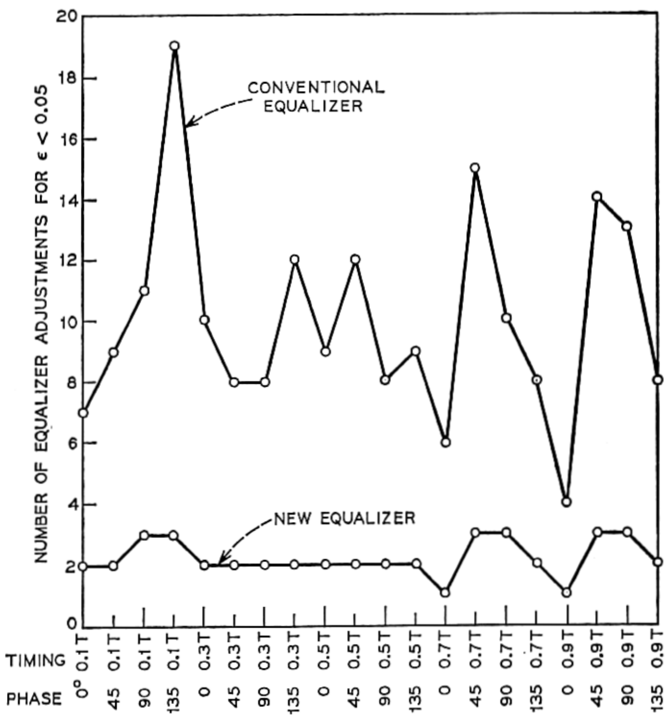

Fig. 7—Number of equalizer adjustments required to reduce the mean-square error to less than 0.05 for different timing and carrier-phase settings.

shows the setting of the timing and the carrier phase, while the vertical axis shows the number of adjustments required to reduce the mean-square error ϵ to less than 0.05. For example, when timing is set to $0.1T$ and the carrier phase is set to 90 degrees, three adjustments are required to reduce ϵ to less than 0.05 when the new equalizer is used, while 11 adjustments are required when the conventional equalizer is used. It can be seen from the top curve in Fig. 7 that when the conventional equalizer is used, the number of required adjustments varies a great deal with timing and carrier phase, and as many as 19 adjustments can be required. On the other hand, when the new equalizer is used (bottom curve), the number of required adjustments remains small for all time and carrier-phase settings. (This is due to the first convergence of the new equalizer.) Thus, when the new equalizer is used, it is not necessary to make fine adjustments of timing and carrier phase at the beginning of the training period. We may simply set these parameters to some reasonable values, and then use the new equalizer to quickly reduce the mean-square error. It is possible that the whole process of adjusting the timing, the carrier phase, and the equalizer can be completed in a brief training period.

VIII. SUMMARY AND CONCLUSIONS

Section II considers amplitude modulation data communication systems that use transversal filters for automatic equalization. The gain controls of the transversal filters are adjusted by the gradient method [equation (15)] to minimize the mean-square error between the received and the desired pulses. After the fundamentals are reviewed, the mean-square error is decomposed into N error-components (N is the number of taps on the transversal filter). The error-components depend on the eigenvalues $\lambda_1, \cdots, \lambda_N$ of the correlation matrix $\mathbf{A}$. These eigenvalues and the convergence of the error-components depend on the signaling scheme and the channel characteristics. Two important classes of digital communication systems are examined: the class IV partial-response system and the SSB Nyquist system. It is shown that for both systems, the engenvalues $\lambda_1, \cdots, \lambda_N$ are independent of the demodulation carrier phase, the system timing, the phase characteristic of the transmission medium, and the phase characteristics of the transmitting and receiving filters. Consequently, the eigenvalues depend on only the amplitude characteristics of the transmission medium and the filters. Since amplitude characteristics of the filters depend on the signaling scheme, it might be said that the eigenvalues depend on only the signaling scheme and the amplitude characteristic $|F_2(f)|$ of the transmission medium. For applications where $|F_2(f)|$ does not vary considerably from channel to channel (such as private-line systems), the eigenvalues depend primarily on the signaling scheme. For the SSB Nyquist system, the eigenvalues $\lambda_1, \cdots, \lambda_N$ are nearly equal. Consequently, as shown in Section 3.2, each adjustment reduces each of the error-components by a large factor, and the mean-square error can be minimized after only a few adjustments (for example, two adjustments). For the class IV partial-response system, the eigenvalues $\lambda_1, \cdots, \lambda_N$ are very different in magnitude. (The ratio of the maximum eigenvalue to the minimum eigenvalue is very large; e.g., 40 or larger.) Therefore, each adjustment cannot reduce each error-component by a large factor, and, as can be seen from Figs. 6 and 7, the convergence rate may not be satisfactory for fast start-up purposes.

The results in Section III suggest that in order to improve the convergence rate an attempt should be made to reduce the differences between the eignevalues. As emphasized in Section IV, differences in eigenvalues are caused by the correlation between the inputs to the gain controls. When the inputs to the gain controls are orthonormal, the eigenvalues are all equal and the mean-square error can be minimized in only one adjustment. The inputs to the gain controls can be made orthonormal by using the generalized equalizer structure in Fig. 3. The generalized equalizer can be realized with digital or analog filters. Because of the press toward digitalization and integrated circuits, digital filters are used. A realization using nonrecursive digital filters is depicted in Fig. 4. This equalizer structure, referred to as the new equalizer structure, is analyzed in detail. It is shown that for any given channel the coefficients of the digital filters can be set according to (69) or (74) to make the inputs to the gain controls orthonormal. Then each adjustment of the gain controls can reduce the mean-square error ϵ to its minimum value, regardless of the magnitude of ϵ prior to that adjustment. In Section V it is shown that such a fast convergence (one-step convergence) is obtained without changing the residual noise power, the minimum mean-square error, and the convexity of the adjustments, and without complicating the gain-control adjustment loops.

The theory in Sections IV and V is general in that it applies regardless of the type of modulation (SSB, VSB, or DSB) or the signaling scheme (partial-response or Nyquist). Sections VI and VII consider the application of the theory to a Class IV partial-response system. Most importantly, it is shown that the coefficients P_{ij} of the digital filters in the new equalizer can be prefixed. It is first observed that for Class IV partial-response systems the coefficients P_{ij} in the new equalizer depend only on the amplitude characteristic $|F_2(f)|$ of the transmission medium. Consequently, for private-line systems and systems where $|F_2(f)|$ does not vary considerably from channel to channel (delay distortion can vary arbitrarily), we may compute the coefficients P_{ij} from an estimated (typical) $|F_2(f)|$ and use these prefixed P_{ij} for all channels. One-step convergence is obtained when actual $|F_2(f)|$ agrees with the estimated $|F_2(f)|$. When actual $|F_2(f)|$ differs from the estimated $|F_2(f)|$, the convergence rate of the mean-square error is reduced. In order to determine if the reduced convergence rate is satisfactory, two analytical methods are developed to relate the convergence rate to $|F_2(f)|$. These methods are first presented in Section VI, and are used in Section VII for a hypothetical data communication system. This hypothetical system uses a Class IV partial-response signal and transmits over private voice lines at a data rate of 4800 bauds. For private lines, $|F_2(f)|$ does not vary considerably in the passband 400 to 2800 Hz; therefore, a constant is used as the estimate of $|F_2(f)|$ and the prefixed P_{ij} are computed from (89) and (119). It is shown analytically in Section 7.1 that the use of the prefixed P_{ij} yields a satisfactory convergence rate even when the line has severe amplitude distortion. To further demonstrate the convergence rate, the hypothetical system is simulated on a digital computer, using the prefixed P_{ij} and a number of private-line characteristics obtained from field measurements. A 30-dB signal-to-noise ratio at the equalizer input is assumed. Figs. 6 and 7 illustrate the convergence of the mean-square error for a private line that has severe amplitude and phase distortion. (The convergence is faster for other lines.) It can be seen from these figures that, when the new equalizer is used, the equalizer settles after three adjustments (approximately 12 milliseconds). Furthermore, because of the fast convergence, the settling time remains small for all settings of timing and demodulating carrier phase. Thus, when the new equalizer is used, it is not necessary to make fine adjustments of timing and demodulating carrier phase in the start-up period. These parameters can simply be set to some reasonable values at the beginning of the start-up period. This can further reduce the overall start-up time of the system.

IX. ACKNOWLEDGMENT

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APPENDIX A

A.1 *Determination of Eigenvalues and Eigenvectors*

In this appendix we determine the eigenvalues and eigenvectors of the $N \times N$ matrix $\mathbf{A}$ in equation (40). The two cases of odd N and even N are considered separately.

Case 1. Odd N.

Let λ be an eigenvalue of $\mathbf{A}$, and let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \tag{122}$$

be the corresponding eigenvector. We can rewrite the equation $\mathbf{A}\,\mathbf{u} = \lambda\,\mathbf{u}$ as

$$\sum_{j=1}^{N} a_{ij} u_j = \lambda u_i \,, \qquad i = 1 \quad \text{to} \quad N. \tag{123}$$

Equation (123) can be split into the following two equations:

$$\sum_{j=1}^{N} a_{ij} u_j = \lambda u_i \,, \qquad i = \text{odd} \tag{124}$$

and

$$\sum_{j=1}^{N} a_{ij} u_j = \lambda u_i \,, \qquad i = \text{even}. \tag{125}$$

From (39), $a_{ij} = 0$ when $i - j$ is odd. Thus (124) can be reduced to

$$\sum_{j=1,3,\cdots,N} a_{ij} u_j = \lambda u_i \,, \qquad i = 1, 3, \cdots, N.$$

Using (39), we can write the above equation as

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ \vdots \\ u_N \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \\ \vdots \\ u_N \end{bmatrix}. \qquad (126)$$

The first matrix on the left-hand side of (126) is an $(N+1)/2 \times (N+1)/2$ tridiagonal matrix. In addition to the trivial solution

$$u_1 = u_3 = u_5 = \cdots = u_N = 0,$$

there are $(N+1)/2$ nonzero solutions to (126). The kth such solution, $k = 1, 2, \cdots, (N+1)/2$, is[14]

$$\lambda = 1 - \cos \frac{k\pi}{\frac{N+1}{2} + 1} \qquad (127)$$

and

$$\begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_N \end{bmatrix} = \left[\frac{2}{\frac{N+1}{2} + 1} \right]^{\frac{1}{2}} \begin{bmatrix} \sin \frac{k\pi}{\frac{N+1}{2} + 1} \\ \sin \frac{k2\pi}{\frac{N+1}{2} + 1} \\ \vdots \\ \sin \frac{k \frac{N+1}{2} \pi}{\frac{N+1}{2} + 1} \end{bmatrix}. \qquad (128)$$

From (39), (125) can be written as

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & \cdots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \cdots & 0 \\ 0 & -\frac{1}{2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_4 \\ u_6 \\ \vdots \\ u_{(N-1)} \end{bmatrix} = \lambda \begin{bmatrix} u_2 \\ u_4 \\ u_6 \\ \vdots \\ u_{(N-1)} \end{bmatrix}. \qquad (129)$$

where the first matrix on the left-hand side is an $(N - 1)/2 \times (N - 1)/2$ tridiagonal matrix. In addition to the trivial solution

$$u_2 = u_4 = u_6 = \cdots = u_{(N-1)} = 0,$$

there are $(N - 1)/2$ nonzero solutions to (129). The kth such solution, $k = 1, 2, \cdots, (N - 1)/2$, is

$$\lambda = 1 - \cos \frac{k\pi}{\frac{N-1}{2} + 1} \tag{130}$$

and

$$\begin{bmatrix} u_2 \\ u_4 \\ \vdots \\ u_{(N-1)} \end{bmatrix} = \left(\frac{2}{\frac{N-1}{2} + 1} \right)^{\frac{1}{2}} \begin{bmatrix} \sin \frac{k\pi}{\frac{N-1}{2} + 1} \\ \sin \frac{k2\pi}{\frac{N-1}{2} + 1} \\ \vdots \\ \sin \frac{k\frac{N-1}{2}\pi}{\frac{N-1}{2} + 1} \end{bmatrix}. \tag{131}$$

It can be shown that the λ given by (127) is not equal to that given by (130). Therefore, (126) and (129) do not have the same nonzero solution.

We have split (123) into (124) and (125), and rewritten (124) and (125), respectively, as (126) and (129). Thus, λ and $\mathbf{u}$ satisfy (123) if and only if they satisfy both (126) and (129). Since (126) and (129) do not have the same nonzero solution, λ and $\mathbf{u}$ are given either by the nonzero solution of (126) plus the trivial solution of (129), or by the nonzero solution of (129) plus the trivial solution of (126).

The above may be summarized in the form of a lemma:

Lemma A-1: For odd N, the eigenvalues and eigenvectors of $\mathbf{A}$ *may be divided into two groups. The kth eigenvalue and eigenvector,* $k = 1, 2, \cdots, (N + 1)/2$, *in the first group are given by* (127), (128), *and*

$$u_2 = u_4 = u_6 = \cdots = u_{(N-1)} = 0.$$

The kth eigenvalue and eigenvector, $k = 1, 2, \cdots, (N-1)/2$, *in the second group are given by* (130), (131), *and*

$$u_1 = u_3 = u_5 = \cdots = u_N = 0.$$

It can be shown that the eigenvectors specified in Lemma A-1 form a set of N orthonormal eigenvectors.

Case 2. Even N

Using the same method as in Case 1, one can verify the following lemma:

Lemma A-2: For even N, the eigenvalues and eigenvectors of **A** *may be divided into two groups. The kth eigenvalue and eigenvector in the first group,* $k = 1, 2, \cdots, N/2$, *are given by*

$$\lambda = 1 - \cos \frac{k\pi}{\frac{N}{2} + 1} \tag{132}$$

and

$$\begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{(N-1)} \end{bmatrix} = \left(\frac{2}{\frac{N}{2} + 1} \right)^{\frac{1}{2}} \begin{bmatrix} \sin \frac{k\pi}{\frac{N}{2}+1} \\ \sin \frac{k2\pi}{\frac{N}{2}+1} \\ \vdots \\ \sin \frac{k\frac{N}{2}\pi}{\frac{N}{2}+1} \end{bmatrix} \tag{133}$$

and

$$u_2 = u_4 = u_6 = \cdots = u_N = 0. \tag{134}$$

The kth eigenvalue and eigenvector in the second group, $k = 1, 2, \cdots, N/2$, *are given by*

$$\lambda = 1 - \cos \frac{k\pi}{\frac{N}{2} + 1} \tag{135}$$

and

$$\begin{bmatrix} u_2 \\ \\ u_4 \\ \\ \vdots \\ \\ u_N \end{bmatrix} = \left(\frac{2}{\frac{N}{2}+1}\right)^{\frac{1}{2}} \begin{bmatrix} \sin \dfrac{k\pi}{\frac{N}{2}+1} \\ \sin \dfrac{k2\pi}{\frac{N}{2}+1} \\ \vdots \\ \sin \dfrac{k\frac{N}{2}\pi}{\frac{N}{2}+1} \end{bmatrix} \tag{136}$$

and

$$u_1 = u_3 = u_5 = \cdots = u_{(N-1)} = 0. \tag{137}$$

The eigenvectors given by (133) and (134) together with the eigenvectors given by (136) and (137) form a set of N orthonormal eigenvectors.

APPENDIX B

B.1 *Bounds on Eigenvalues*

It has been defined in Section VI that $\hat{\lambda}_i$, $i = 1$ to N, are the eigenvalues of $\mathbf{RS}^{-1}$. In this appendix we study the relation between $\hat{\lambda}_i$ and the amplitude characteristic $|F_2(f)|$ of the transmission medium. It is easier to first study the relation between $|F_2(f)|$ and the eigenvalues of $\mathbf{AS}^{-1}$. Let $\tilde{\lambda}$ denote the eigenvalue of $\mathbf{AS}^{-1}$. Since $\mathbf{AS}^{-1}$ and $\mathbf{S}^{-1}\mathbf{A}$ have the same eigenvalues, $\tilde{\lambda}$ is also the eigenvalue of $\mathbf{S}^{-1}\mathbf{A}$. Thus

$$\mathbf{S}^{-1}\mathbf{A}\tilde{\mathbf{x}} = \tilde{\lambda}\tilde{\mathbf{x}}, \tag{138}$$

where $\tilde{\mathbf{x}}$ is the eigenvector of $\mathbf{S}^{-1}\mathbf{A}$. Premultiplying both sides of (138) with $\tilde{\mathbf{x}}'\mathbf{S}$ gives

$$\tilde{\mathbf{x}}'\mathbf{A}\tilde{\mathbf{x}} = \tilde{\lambda}\tilde{\mathbf{x}}'\mathbf{S}\tilde{\mathbf{x}}. \tag{139}$$

Since $\mathbf{S}$ is a correlation matrix like $\mathbf{A}$, $\mathbf{S}$ is positive definite. Consequently $\tilde{\mathbf{x}}'\mathbf{S}\tilde{\mathbf{x}} > 0$ for every $\tilde{\mathbf{x}}$ except $\tilde{\mathbf{x}} = \mathbf{0}$. Since $\tilde{\mathbf{x}}$ is an eigenvector, $\tilde{\mathbf{x}} \neq \mathbf{0}$. Therefore

$$\tilde{\mathbf{x}}'\mathbf{S}\tilde{\mathbf{x}} > 0.$$

From (139), we have

$$\tilde{\lambda} = \frac{\tilde{\mathbf{x}}'\mathbf{A}\tilde{\mathbf{x}}}{\tilde{\mathbf{x}}'\mathbf{S}_{\wedge}}. \tag{140}$$

In this appendix, $\mathbf{c}$ is the vector defined in (4) for the conventional equalizer. Let $[(\mathbf{c}'\mathbf{A}\mathbf{c})/(\mathbf{c}'\mathbf{S}\mathbf{c})]_{\min}$ denote the minimum value of $(\mathbf{c}'\mathbf{A}\mathbf{c})/(\mathbf{c}'\mathbf{S}\mathbf{c})$ over all $\mathbf{c} \neq \mathbf{0}$, and let $[(\mathbf{c}'\mathbf{A}\mathbf{c})/(\mathbf{c}'\mathbf{S}\mathbf{c})]_{\max}$ denote the maximum value of $(\mathbf{c}'\mathbf{A}\mathbf{c})/(\mathbf{c}'\mathbf{S}\mathbf{c})$ over all $\mathbf{c} \neq \mathbf{0}$. Then since the eigenvectors $\tilde{\mathbf{x}}$ are elements of the set $\{\mathbf{c} \mid \mathbf{c} \neq \mathbf{0}\}$, we have

$$\left[\frac{\mathbf{c}'\mathbf{A}\mathbf{c}}{\mathbf{c}'\mathbf{S}\mathbf{c}}\right]_{\min} \leqq \frac{\tilde{\mathbf{x}}\ \mathbf{A}\tilde{\mathbf{x}}'}{\tilde{\mathbf{x}}'\mathbf{S}\tilde{\mathbf{x}}} \leqq \left[\frac{\mathbf{c}'\mathbf{A}\mathbf{c}}{\mathbf{c}'\mathbf{S}\mathbf{c}}\right]_{\max}. \tag{141}$$

Now we evaluate the two bounds in (141). Using the notations in Section II, we can easily show that

$$\mathbf{c}'\mathbf{A}\mathbf{c} = \sum_{i=-\infty}^{\infty} y_i^2 . \tag{142}$$

Since y_i are time samples of $y(t)$ taken at the Nyquist rate, we have by sampling theorem

$$\sum_{i=-\infty}^{\infty} y_i^2 = 2(f_2 - f_1) \int_{-\infty}^{\infty} [y(t)]^2 \, dt. \tag{143}$$

According to Parseval's theorem

$$\int_{-\infty}^{\infty} [y(t)]^2 \, dt = \int_{-\infty}^{\infty} Y(f)\, Y^*(f) \, df, \tag{144}$$

where $Y(f)$ is the Fourier transform of $y(t)$, and $Y^*(f)$ is the complex conjugate of $Y(f)$. From (2), we obtain

$$Y(f) = X(f) e^{J2\pi fT} \sum_{k=1}^{N} c_k e^{-J2\pi fkT}. \tag{145}$$

Substituting (145) into (144) gives

$$\int_{-\infty}^{\infty} [y(t)]^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \left[\sum_{k=1}^{N} c_k e^{-J2\pi fkT}\right]\left[\sum_{k=1}^{N} c_k e^{J2\pi fkT}\right] df. \tag{146}$$

One can rearrange (146) into the following form

$$\int_{-\infty}^{\infty} [y(t)]^2 \, dt = 2 \int_{0}^{\infty} |X(f)|^2 \, q(f) \, df, \tag{147}$$

where

$$q(f) = \left[\sum_{k=1}^{N} c_k \cos 2\pi fkT \right]^2 + \left[\sum_{k=1}^{N} c_k \sin 2\pi fkT \right]^2. \tag{148}$$

Since we are considering a Class IV partial-response system, we can use (30) in Section 3.1 to determine $|X(f)|$, and use (35) for the product $|F_1(f - f_c)F_3(f - f_c)|$ in $|X(f)|$. From these equations, (147) can be reduced to the following form

$$\int_{-\infty}^{\infty} [y(t)]^2 \, dt = \frac{1}{2} \int_0^{f_2 - f_1} |F_2(f - f_c)|^2 \left[\sin \frac{\pi f}{f_2 - f_1} \right]^2 q(f) \, df. \tag{149}$$

Combining (142), (143), and (149) gives

$$\mathbf{c'Ac} = (f_2 - f_1) \int_0^{f_2 - f_1} |F_2(f - f_c)|^2 \left[\sin \frac{\pi f}{f_2 - f_1} \right]^2 q(f) \, df. \tag{150}$$

In the following discussion, we have to distinguish between the actual $|F_2(f - f_c)|$ (denoted $|F_2(f - f_c)|_{\text{act}}$) and the estimated $|F_2(f - f_c)|$ (denoted $|F_2(f - f_c)|_{\text{est}}$). In order to emphasize that the $|F_2(f - f_c)|$ in (150) is the actual $|F_2(f - f_c)|$, we rewrite (150) as

$$\mathbf{c'Ac} = (f_2 - f_1) \int_0^{f_2 - f_1} [|F_2(f - f_c)|_{\text{act}}]^2 \left[\sin \frac{\pi f}{f_2 - f_1} \right]^2 q(f) \, df. \tag{151}$$

When the actual $|F_2(f - f_c)|$ is replaced by the estimated $|F_2(f - f_c)|$, $\mathbf{A}$ is replaced by $\mathbf{S}$, and (151) becomes

$$\mathbf{c'Sc} = (f_2 - f_1) \int_0^{f_2 - f_1} [|F_2(f - f_c)|_{\text{est}}]^2 \left[\sin \frac{\pi f}{f_2 - f_1} \right]^2 q(f) \, df. \tag{152}$$

Let $\eta(f)$ be defined as

$$\eta(f) = \frac{[|F_2(f - f_c)|_{\text{act}}]^2}{[|F_2(f - f_c)|_{\text{est}}]^2}, \qquad 0 \leqq f \leqq f_2 - f_1 . \tag{153}$$

From (151), (152), and (153), we obtain

$$\frac{\mathbf{c'Ac}}{\mathbf{c'Sc}} = \frac{\displaystyle\int_0^{f_2 - f_1} \eta(f) \Omega(f) \, df}{\displaystyle\int_0^{f_2 - f_1} \Omega(f) \, df} \tag{154}$$

where

$$\Omega(f) = [|F_2(f - f_c)|_{\text{est}}]^2 \left[\sin \frac{\pi f}{f_2 - f_1} \right]^2 \cdot q(f). \tag{155}$$

It can be easily seen from (155) and (148) that

$$\Omega(f) \geqq 0, \qquad 0 \leqq f \leqq f_2 - f_1 . \tag{156}$$

Furthermore, for $\mathbf{c} \neq \mathbf{0}$, $\Omega(f)$ cannot be identically 0 in the frequency range $0 \leqq f \leqq f_2 - f_1$ (for if so $\mathbf{c}'\mathbf{S}\mathbf{c}$ would be 0, contradicting the fact that $\mathbf{S}$ is positive definite). It can be seen from (153) that

$$\eta(f) \geqq 0, \qquad 0 \leqq f \leqq f_2 - f_1 . \tag{157}$$

From these properties of $\Omega(f)$ and $\eta(f)$, one can see that

$$\frac{\int_0^{f_2-f_1} \eta(f)\Omega(f)\, df}{\int_0^{f_2-f_1} \Omega(f)\, df} \leqq [\eta(f)]_{\max} , \tag{158}$$

where $[\eta(f)]_{\max}$ is the maximum value of $\eta(f)$ in the frequency range $0 \leqq f \leqq f_2 - f_1$. It can alse be seen that

$$\frac{\int_0^{f_2-f_1} \eta(f)\Omega(f)\, df}{\int_0^{f_2-f_1} \Omega(f)\, df} \geqq [\eta(f)]_{\min} , \tag{159}$$

where $[\eta(f)]_{\min}$ is the minimum value of $\eta(f)$ in the frequency range $0 \leqq f \leqq f_2 - f_1$. Now we tie the results together. From (140), (141), (154), and (158), we have

$$\tilde{\lambda} = \frac{\tilde{\mathbf{x}}'\mathbf{A}\tilde{\mathbf{x}}}{\tilde{\mathbf{x}}'\mathbf{S}\tilde{\mathbf{x}}} \leqq \left[\frac{\mathbf{c}'\mathbf{A}\mathbf{c}}{\mathbf{c}'\mathbf{S}\mathbf{c}}\right]_{\max} \leqq [\eta(f)]_{\max} . \tag{160}$$

Similarly, from (140), (141), (154), and (159),

$$\tilde{\lambda} = \frac{\tilde{\mathbf{x}}'\mathbf{A}\tilde{\mathbf{x}}}{\tilde{\mathbf{x}}'\mathbf{S}\tilde{\mathbf{x}}} \geqq \left[\frac{\mathbf{c}'\mathbf{A}\mathbf{c}}{\mathbf{c}'\mathbf{S}\mathbf{c}}\right]_{\min} \geqq [\eta(f)]_{\min} . \tag{161}$$

We may combine (160) and (161) as

$$[\eta(f)]_{\min} \leqq \tilde{\lambda} \leqq [\eta(f)]_{\max} . \tag{162}$$

It can be easily shown that the eigenvalue $\hat{\lambda}_i$ of $\mathbf{R}\mathbf{S}^{-1}$ is related to the eigenvalue $\tilde{\lambda}$ of $\mathbf{A}\mathbf{S}^{-1}$ by

$$1 - \hat{\lambda}_i = \tilde{\lambda}. \tag{163}$$

From (163) and (162), we obtain the final result

$$1 - [\eta(f)]_{\max} \leqq \hat{\lambda}_i \leqq 1 - [\eta(f)]_{\min} , \qquad i = 1 \text{ to } N. \tag{164}$$

Equations (164) and (153) are, respectively, (111) and (112) in Section VI.

REFERENCES

1. Lucky, R. W., "Automatic Equalization for Digital Communication," B.S.T.J., *44,* No. 4, part 1 (April 1965), pp. 547–588.
2. DiToro, M. J., "Communication in Time-Frequency Spread Media Using Adaptive Equalization," Proc. IEEE *56,* (October 1968), pp. 1653–1679.
3. Lucky, R. W., and Rudin, H. R., "An Automatic Equalizer for General-Purpose Communication Channels," B.S.T.J., *46,* No. 9, part 2 (November 1967), pp. 2179–2208.
4. Gersho, A., "Adaptive Equalization of Highly Dispersive Channels for Data Transmission," B.S.T.J., *48,* No. 1, part 1 (January 1969), pp. 55–70.
5. Koman, C. W., unpublished work.
6. Chang, R. W., "Joint Optimization of Automatic Equalization and Carrier Acquisition for Digital Communication," B.S.T.J., *49,* No. 6 (July-August 1970), pp. 1069–1104.
7. Lucky, R. W., Salz, J., and Weldon, E. J., Jr., *Principles of Data Communication,* New York: McGraw-Hill Book Company, 1968.
8. Kretzmer, E. R., "Generalization of a Technique for Binary Data Communication," IEEE Trans. Comm. Tech., *COM-14,* (February 1966), pp. 67–68.
9. Becker, F. K., Kretzmer, E. R., and Sheehan, J. R., "A New Signal Format for Efficient Data Transmission," B.S.T.J., *45,* No. 5, part 1 (May–June 1966), pp. 755–758.
10. Gerrish, A. M., and Lawless, W. J., "A New Wideband Partial-Response Data Set," *Conference Record,* IEEE Int. Conf. Commun., San Francisco, June 8–10, 1970.
11. Gunn, J. F., and Weller, D. C., "A Digital Mastergroup Channel for Modern Coaxial Carrier Systems," *Conference Record,* IEEE Int. Conf. Commun., San Francisco, June 8–10, 1970.
12. Lender, A., "Decision-Directed Digital Adaptive Equalization Technique for High-Speed Data Transmission," *Conference Record,* IEEE Int. Conf. Commun., San Francisco, June 8–10, 1970.
13. Forney, G. D., "Error Correction for Partial Response Modems," Int. Symp. Inform. Theory, Noordwijk, the Netherlands, June 1970.
14. Grenander, U., and Szegö, G., *Toeplitz Forms and their Applications,* Berkeley, California: University of California Press, 1958.
15. Saltzberg, B. R., "Intersymbol Interference Error Bounds with Application to Ideal Bandlimited Signaling," IEEE Trans. Inform. Theory, *IT-14,* (July 1968), pp. 563–568.
16. Kuo, F. F., and Kaiser, J. F., *System Analysis by Digital Computer,* New York: John Wiley and Sons, 1966.
17. Fleming, W. H., *Functions of Several Variables,* Reading, Mass.: Addison-Wesley Publishing Company, 1965.