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Combining Correlated Streams of Nonrandom Traffic

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The Equivalent Random method is used for engineering many of the telephone overflow-networks in the Bell System. But since this method is not directly applicable to the analysis of graded-multiple trunk-groups which carry overflow traffic, we extend the method to cover such arrangements. The key to this extension is a technique for taking correlation into account when combining dependent streams of traffic which are themselves more variable than Poisson. In principle, the technique is applicable wherever a stream of overflow traffic is divided, submitted to independent trunk groups, and then recombined.

The extended Equivalent Random method provides adequate estimates of load-service relations for graded multiples which carry overflow traffic, provided the grading capacity is not substantially influenced by the network that precedes the grading.

I. INTRODUCTION

The Equivalent Random method[1,2] is used for engineering many of the telephone overflow-networks in the Bell System. However, the method is not directly applicable to the analysis of graded-multiple trunk-groups† which carry overflow traffic; e.g., gradings used as alternate routes in step-by-step switching systems having common control and alternate-routing capability. Apparently, Lotze[5,6] is the only author with results for estimating load-loss relations for gradings

† The reader should have some knowledge of graded multiples and the methods associated with the engineering of telephone overflow-networks. Some familiarity with the step-by-step switching system would also be helpful. Those not acquainted with these concepts may find it worthwhile to consult Ref. 1 for a discussion of telephone overflow-networks. An introduction to graded multiples is given in Ref. 3. Reference 4 contains a description of the pertinent aspects of the step-by-step system.

which carry overflow traffic. Unfortunately, his method requires data which cannot be obtained in a step-by-step system at reasonable cost.

We extend the Equivalent Random method to cover such applications. The key to the extension is a technique for taking correlation into account when combining dependent streams of overflow traffic. (In principle, the technique is applicable wherever a stream of overflow traffic is divided, submitted to independent trunk groups, and then recombined.) In Section II, we derive the appropriate covariance function. In order to describe how the covariance function is used to extend the Equivalent Random method,† we begin with an example of an application of the extended method for the analysis of a step-by-step graded multiple.

Figure 1 represents schematically a step-by-step grading which might be used as a final route. Each horizontal bar denotes one trunk (server). The traffic offered to the grading‡ is an overflow stream from a subordinate network. The traffic is represented by the mean α and variance v of the number of simultaneous calls that would be in progress if this traffic were carried on a full-access group without blocking. The diagram is designed to indicate that an arriving call is first directed at random to one of the four first-choice subgroups; i.e., an arrival is directed to the ith subgroup with probability p_i. After reaching a particular subgroup, the call hunts vertically upward for an idle trunk. If all three trunks in the subgroup are busy, the call overflows into the second major level of the grading. The call then seizes the lowest idle trunk in the second-level subgroup. If both trunks in the second level are busy, the call overflows to the third major level containing five trunks (usually called finals). If all five finals are busy, the call leaves the system and does not return; i.e., the call is blocked and cleared. The symbol α_0 and v_0 denote respectively the mean and variance of the overflow from the grading.

In a step-by-step system, the arrangement of line-finders and selectors through which calls reach a grading causes an inherent load-balancing[4] over the first-choice subgroups; there is positive correlation between the numbers of occupied trunks in the individual subgroups. Attempting to introduce the correlation into our model, we assume

† The Equivalent Random method is known to be adequate for estimating load-service relation for overflow-networks having Poisson input.[7,8]

‡ For the present study, we assume that all offered loads are constant. Hence, we are estimating single-hour capacities of gradings. Utilization of our results for normal engineering involving average busy-hour loads, would require peripheral operations to adjust the load-service relationship to reflect the effects of low, medium, or high day-to-day variations in the load.[9]

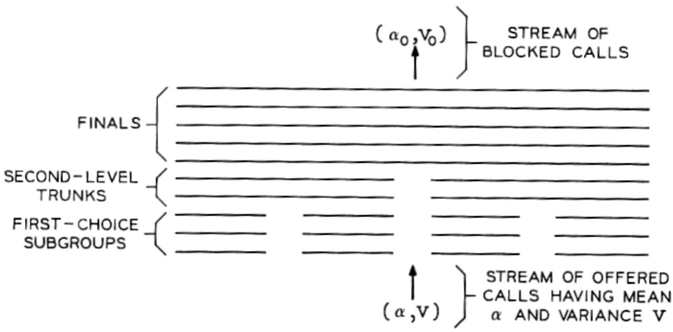

Fig. 1—Schematic representation of a step-by-step graded multiple containing 21 trunks.

the configuration given in Fig. 2. The four arrows above the full-access group denote (correlated) overflow streams caused by the corresponding input streams. The intensity of the ith input stream is $a_i = p_i a$.

At this stage, we have modeled an arrival process having mean α and variance v. The individual substreams to the first-choice subgroups of the grading are certainly correlated. How well the correla-

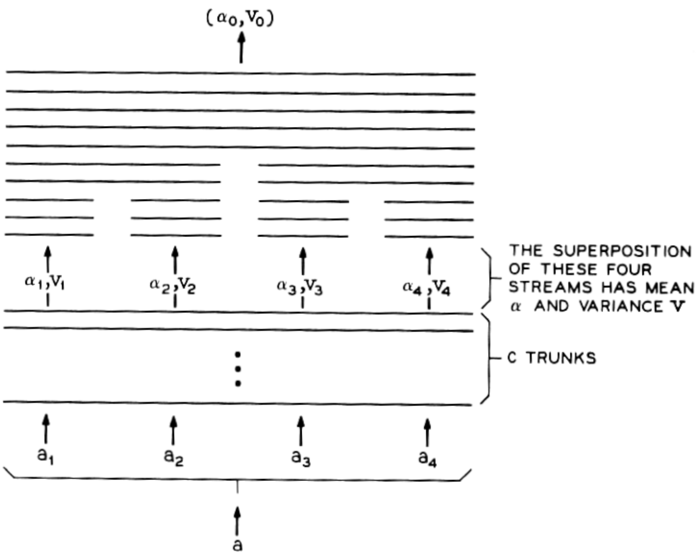

Fig. 2—A model for correlated input streams.

tion approximates the correlation that exists in an actual step-by-step system is a point which must be tested. In Section III, we show that the approximation works quite well when a large group of selectors is connected to the grading. In any event, we can view the above figure as a first approximation for the step-by-step problem. Moreover, the general features of the above configuration are not restricted to step-by-step systems.

Figure 2 also illustrates the two basic features of the Equivalent Random method. First, the method assumes that, for engineering purposes, only the first two moments of an overflow process are required. Second, the method assumes that any overflow process having mean α and variance v is adequately approximated by the overflow from a unique "equivalent system" consisting of a full-access trunk-group with Poisson input. Whenever α and v are known, standard techniques are available[1] to obtain the equivalent system of c trunks and intensity $a = a_1 + a_2 + a_3 + a_4$ of the Poisson input.

The next step consists of using results obtained independently by Descloux[10] and Lotze[6] to determine the mean α_i and the variance v_i of the overflow (due to a_i) which is submitted to the ith first-choice subgroup of our grading. This "splitting" is determined by[10]

$$\alpha_i = p_i \alpha \tag{1}$$

and

$$\frac{v_i}{\alpha_i} - 1 = p_i\left(\frac{v}{\alpha} - 1\right). \tag{2}$$

The covariance c_{ij} between the ith and jth "split" streams is given by[10]

$$c_{ij} = p_i p_j (v - \alpha). \tag{3}$$

After splitting, our system is represented as

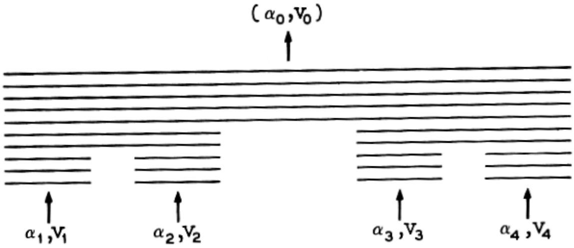

Fig. 3—Graded multiple with correlated input streams.

where it is understood that the parameters α_i and v_i of the individual substreams satisfy equations (1) through (3). Using the Equivalent Random method, we now approximate the individual substreams as overflows from full-access trunk groups. Moreover, to determine the total overflow which goes to the second major level of the grading, we view the entire system in the following manner:

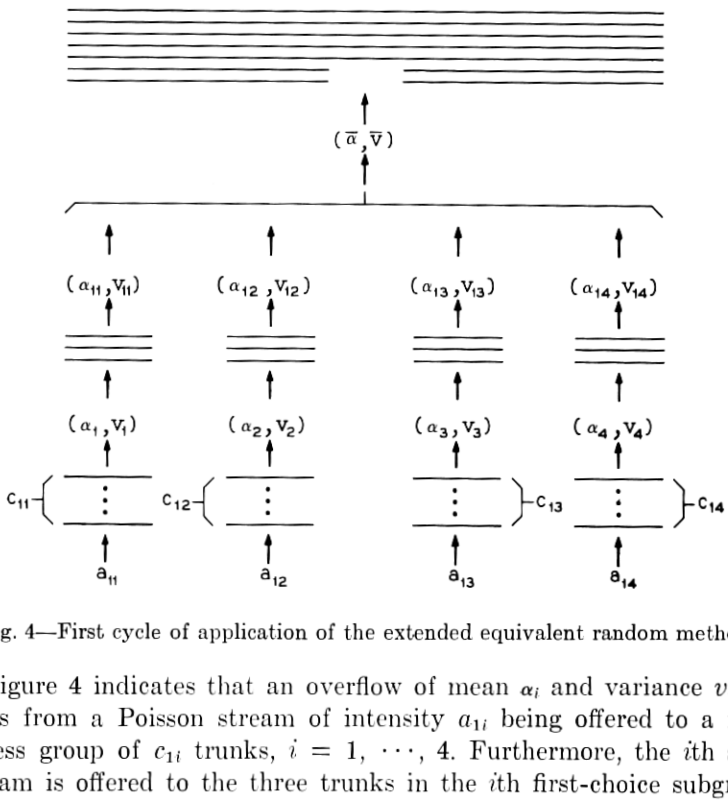

Fig. 4—First cycle of application of the extended equivalent random method.

Figure 4 indicates that an overflow of mean α_i and variance v_i results from a Poisson stream of intensity a_{1i} being offered to a full-access group of c_{1i} trunks, $i = 1, \cdots, 4$. Furthermore, the ith substream is offered to the three trunks in the ith first-choice subgroup of the grading, and causes an overflow of mean α_{1i} and variance v_{1i} to be submitted to the second major level of our grading.

One can see the main reason for looking at the grading in the manner described above: the parameters α_{1i} and v_{1i} are easily computed by[1]

$$\alpha_{1i} = a_{1i}E_{1,c_{1i}+3}(a_{1i})$$

and

$$v_{1i} = \alpha_{1i}\left[1 - \alpha_{1i} + \frac{a_{1i}}{(c_{1i} + 3) + \alpha_{1i} - a_{1i} + 1}\right],$$

where $E_{1,s}(a)$ denotes the first Erlang loss-function (Erlang-B blocking probability).

To complete the first cycle of computation, we need to obtain the mean $\bar{\alpha}$ and variance $\bar{v}$ of the total overflow submitted to the second major level of our grading. The mean $\bar{\alpha}$ is given by $\bar{\alpha} = \sum_{i=1}^{4} \alpha_{1i}$. Unfortunately, the variance $\bar{v}$ is more difficult to obtain, since the individual substreams are correlated.

To obtain $\bar{v}$ we need the covariance, cov (i, j), between the ith and jth overflow streams which are offered to the second level. A reasonable method for computing cov (i, j) has not been available in the past. Our extension of the Equivalent Random method consists of an algorithm for computing cov (i, j) in a fairly efficient fashion for many configurations of interest. A derivation of the algorithm is given in Section II.

After cov (i, j) has been determined, $\bar{v}$ is obtained from

$$\bar{v} = \sum_{i=1}^{4} v_{1i} + \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \text{cov}\,(i, j). \tag{4}$$

Having $\bar{\alpha}$ and $\bar{v}$, we reduce the system configuration to that shown in Fig. 5.

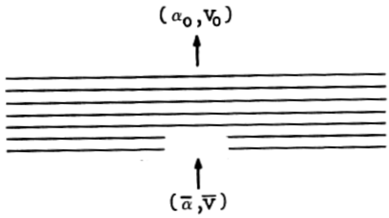

Fig. 5—Starting point for the second cycle of the extended equivalent random method.

The proportion of the traffic $(\bar{\alpha}, \bar{v})$ offered to the first subgroup of the second level is $\bar{p}_1 = (\alpha_{11} + \alpha_{12})/\bar{\alpha}$, and $\bar{p}_2 = (\alpha_{13} + \alpha_{14})/\bar{\alpha}$ is the proportion offered to the second subgroup. Consequently, one cycle of computation is completed.

Repetition of the logic described above will yield estimates of the overflow mean α_0 and variance v_0 . Moreover, the cyclic nature of the procedure allows the logic to be programmed on a digital computer so that load-service tables and other relevant information can be generated in a straightforward manner.

II. MATHEMATICAL MODEL

A service system S is composed of a collection $\mathfrak{M}$ of c first-choice servers, two groups $\mathfrak{N}_1$, $\mathfrak{N}_2$ containing d_1, d_2 second-choice servers respectively, and two last-choice groups $\mathfrak{L}_1$, $\mathfrak{L}_2$ each containing an infinite number of servers (see Fig. 6). The arrivals into the system are generated by ν independent groups of customers $G_1, \cdots, G_\nu$. The arrivals from group G_i occur according to a Poisson process with intensity a_i. All service times throughout the system are independent and have a negative-exponential distribution with unit mean.

A customer, arriving to find an idle first-choice server, selects an idle server from $\mathfrak{M}$, and service commences immediately. If an arrival from group G_i, $i = 1, 2$, occurs when all c of the first-choice servers are busy, but at least one of the d_i servers from the second-choice group $\mathfrak{N}_i$ is idle, a server is selected from $\mathfrak{N}_i$, and service commences at once. If a customer from group G_i arrives to find all the servers busy in both $\mathfrak{M}$ and $\mathfrak{N}_i$, then he is served by one of the servers from the group $\mathfrak{L}_i$. If $k \neq 1$ and $k \neq 2$, requests for service from group G_k, which occur when all c servers in $\mathfrak{M}$ are busy, are dismissed and do not return.

We assume that the system is in statistical equilibrium and define M, N_i, L_i to be the number of busy servers in $\mathfrak{M}$, $\mathfrak{N}_i$ and $\mathfrak{L}_i$ respectively (at a random instant of time) for $i = 1, 2$. Define the state of the system to be (M, N_1, N_2, L_1, L_2) with joint probability density function $f(m, n_1, n_2, l_1, l_2) =$

$$P\{M = m, N_1 = n_1, N_2 = n_2, L_1 = l_1, L_2 = l_2\}.$$

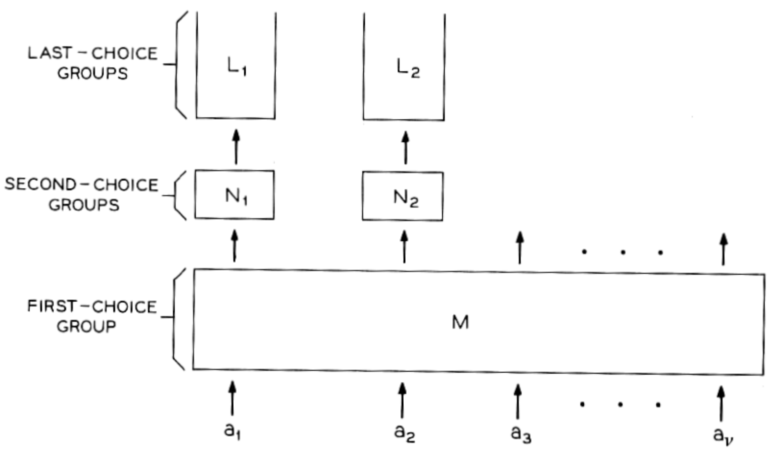

Fig. 6—System configuration.

Setting $a = a_1 + a_2 + \cdots + a_\nu$, it follows that f must satisfy relations of the following form:

For $0 \leqq m \leqq c - 1, 0 \leqq n_i \leqq d_i$, and $l_i \geqq 0$,

$$\begin{aligned}(a + m + n_1 + n_2 + l_1 + l_2)f(m, n_1, n_2, l_1, l_2) \\ = af(m - 1, n_1, n_2, l_1, l_2) \\ + (m + 1)f(m + 1, n_1, n_2, l_1, l_2) \\ + (n_1 + 1)f(m, n_1 + 1, n_2, l_1, l_2) \\ + (n_2 + 1)f(m, n_1, n_2 + 1, l_1, l_2) \\ + (l_1 + 1)f(m, n_1, n_2, l_1 + 1, l_2) \\ + (l_2 + 1)f(m, n_1, n_2, l_1, l_2 + 1). \end{aligned} \tag{5}$$

Similar relations hold on the boundary of the state space $\{(m, n_1, n_2, l_1, l_2) : 0 \leqq m \leqq c, 0 \leqq n_i \leqq d_i, l_i \geqq 0\}$. We define f to be zero at all points not in the state space.

The preceding infinite set of equations is quite difficult to solve. However, when it suffices to know the various moments of the random variables L_1 and L_2, the problem can be simplified by introducing a two-dimensional binomial-moment generating-function. This function is defined by

$$B(m, n_1, n_2; x_1, x_2) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} f(m, n_1, n_2, l_1, l_2)(1 + x_1)^{l_1}(1 + x_2)^{l_2} \tag{6}$$

for $-1 \leqq x_i \leqq 0$, $0 \leqq m \leqq c$, and $0 \leqq n_i \leqq d_i$. Assuming that the binomial moments

$$B_{l_1,l_2}(m, n_1, n_2) = \sum_{k_1=l_1}^{\infty} \sum_{k_2=l_2}^{\infty} \binom{k_1}{l_1} \binom{k_2}{l_2} f(m, n_1, n_2, k_1, k_2) \tag{7}$$

exist, it follows that†

$$B(m, n_1, n_2; x_1, x_2) = \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} B_{l_1,l_2}(m, n_1, n_2) x_1^{l_1} x_2^{l_2}.$$

Of course, the binomial moments $B_{l_1,l_2}(m, n_1, n_2)$ are the entities of interest since

† Various manipulations of these double series will be carried out in the sequel. The mathematical justification for the validity of the manipulations can be obtained from Ref. 11, Sections 5.3 through 5.5.

$$B_{(l_1,l_2)} \overset{\mathrm{d}}{=} \sum_{m=0}^{c} \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} B_{l_1,l_2}(m, n_1, n_2) = E\left[\binom{L_1}{l_1}\binom{L_2}{l_2}\right]. \tag{8}$$

In particular, $B_{(1,0)} = E(L_1)$, $B_{(0,1)} = E(L_2)$, and $B_{(1,1)} = E(L_1L_2)$ so that $\operatorname{cov}(L_1, L_2) = B_{(1,1)} - B_{(1,0)}\, B_{(0,1)}$.

Relations for the binomial moments are obtained by multiplying both sides of Equation (5) (and the boundary equations which were not given) by $(1 + x_1)^{l_1}(1 + x_2)^{l_2}$ and summing on l_1 and l_2. Equating like powers of x_1 and x_2 yields the following finite system of equations:

If $0 \leqq m \leqq c - 1$ and $0 \leqq n_i \leqq d_i$,

$$\begin{aligned}(a + m + n_1 + n_2 + l_1 + l_2)&B_{l_1,l_2}(m, n_1, n_2)\\ = aB_{l_1,l_2}&(m - 1, n_1, n_2) + (m + 1)B_{l_1,l_2}(m + 1, n_1, n_2)\\ &+ (n_1 + 1)B_{l_1,l_2}(m, n_1 + 1, n_2)\\ &+ (n_2 + 1)B_{l_1,l_2}(m, n_1, n_2 + 1).\end{aligned} \tag{9}$$

For $0 \leqq n_i \leqq d_i - 1$,

$$\begin{aligned}(a_1 + a_2 + c + n_1 + n_2 + l_1 + l_2)&B_{l_1,l_2}(c, n_1, n_2)\\ = aB_{l_1,l_2}&(c - 1, n_1, n_2)\\ &+ a_1B_{l_1,l_2}(c, n_1 - 1, n_2) + a_2B_{l_1,l_2}(c, n_1, n_2 - 1)\\ &+ (n_1 + 1)B_{l_1,l_2}(c, n_1 + 1, n_2)\\ &+ (n_2 + 1)B_{l_1,l_2}(c, n_1, n_2 + 1).\end{aligned} \tag{10}$$

Whenever $0 \leqq n_1 \leqq d_1 - 1$,

$$\begin{aligned}(a_1 + c + n_1 + d_2 + l_1 + l_2)&B_{l_1,l_2}(c, n_1, d_2)\\ = aB_{l_1,l_2}&(c - 1, n_1, d_2) + a_1B_{l_1,l_2}(c, n_1 - 1, d_2)\\ &+ a_2B_{l_1,l_2}(c, n_1, d_2 - 1)\\ &+ (n_1 + 1)B_{l_1,l_2}(c, n_1 + 1, d_2) + a_2B_{l_1,l_2-1}(c, n_1, d_2).\end{aligned} \tag{11}$$

A similar result holds for $0 \leqq n_2 \leqq d_2 - 1$. At the extreme boundary point (c, d_1, d_2),

$$\begin{aligned}(c + d_1 + d_2 + l_1 + l_2)&B_{l_1,l_2}(c, d_1, d_2)\\ = aB_{l_1,l_2}&(c - 1, d_1, d_2) + a_1B_{l_1,l_2}(c, d_1 - 1, d_2)\\ &+ a_2B_{l_1,l_2}(c, d_1, d_2 - 1)\\ &+ a_1B_{l_1-1,l_2}(c, d_1, d_2) + a_2B_{l_1,l_2-1}(c, d_1, d_2).\end{aligned} \tag{12}$$

Since $B_{0,0}(m, n_1, n_2) = P\{M = m, N_1 = n_1, N_2 = n_2\}$, it follows that

$$\sum_{m=0}^{c} \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} B_{0,0}(m, n_1, n_2) = 1. \tag{13}$$

We set $B_{l_1,l_2}(m, n_1, n_2) = 0$ for any point (l_1, l_2, m, n_1, n_2) not in the set

$$\{(l_1, l_2, m, n_1, n_2):\ l_i \geqq 0,\ \ 0 \leqq m \leqq c,\ \ 0 \leqq n_i \leqq d_i\}.$$

A very useful relation can be obtained by summing all of equations (9) through (12) to obtain

$$(l_1 + l_2)B_{(l_1,l_2)} = a_1 \sum_{n_2=0}^{d_2} B_{l_1-1,l_2}(c, d_1, n_2) + a_2 \sum_{n_1=0}^{d_1} B_{l_1,l_2-1}(c, n_1, d_2). \tag{14}$$

Consequently, $B_{(1,1)} = E\{L_1L_2\}$ can be obtained from

$$q_1(m, n_2) \triangleq \sum_{n_1=0}^{d_1} B_{1,0}(m, n_1, n_2) \tag{15}$$

and

$$q_2(m, n_1) \triangleq \sum_{n_2=0}^{d_2} B_{0,1}(m, n_1, n_2). \tag{16}$$

That is,

$$2B_{(1,1)} = a_1q_2(c, d_1) + a_2q_1(c, d_2). \tag{17}$$

Relations for q_1 and q_2 are obtained directly from equations (9) through (12) (see Appendix B). However, the relations require $B_{0,0}(c, n_1, d_2)$ and $B_{0,0}(c, d_1, n_2)$ for $0 \leqq n_i \leqq d_i$. (See Refs. 12 and 13 for a related problem.) Setting $l_1 = l_2 = 0$ in equations (9) through (13) yields, with equation (14), a system of $(c + 1)$ $(d_1 + 1)$ $(d_2 + 1)$ independent linear equations for $B_{0,0}$ which in principle can be solved numerically. Unfortunately, for the step-by-step applications described in the introduction, c can be quite large (20 or more) although d_1 and d_2 normally do not exceed 5. Consequently, systems of 500 and more equations would not be uncommon. Since a solution might be required for several sets of parameters in any particular network, a direct numerical solution is not attractive.

In Appendix A, we obtain a closed-form expression for $B_{0,0}(m, n_1, n_2)$ in terms of the $(d_1 + 1)$ $(d_2 + 1)$ constants

$$\beta(n_1, n_2) = E\left[\binom{M}{c}\binom{N_1}{n_1}\binom{N_2}{n_2}\right], \qquad 0 \leqq n_i \leqq d_i . \tag{18}$$

Furthermore, these constants satisfy $(d_1 + 1)(d_2 + 1)$ independent linear relations, so that a numerical solution is quite practical. In fact, the *maximum* size of the system of equations requiring solution for step-by-step systems is reduced from more than 500 to 36.

A closed-form expression for $q_1(c, d_2)$ and $q_2(c, d_1)$ is derived in Appendix B. These results combine into the following computational algorithm for $\operatorname{cov}(L_1, L_2)$: First of all,

$$\beta(0, 0) = E_{1,c}(a) \tag{19}$$

(the Erlang-B blocking probability for the first-choice group), and for $0 \leqq n_i \leqq d_i$, $n_1 + n_2 > 0$,

$$\begin{aligned}(n_1 + n_2)\nu_{n_1+n_2}\beta(n_1, n_2) &= a_1\beta(n_1 - 1, n_2) + a_2\beta(n_1, n_2 - 1) \\ &\quad - a_1\binom{d_1}{n_1 - 1}\beta(d_1, n_2) - a_2\binom{d_2}{n_2 - 1}\beta(n_1, d_2).\end{aligned} \tag{20}$$

By definition, $\beta(n_1, -1) = \beta(-1, n_2) = 0$ for $0 \leqq n_i \leqq d_i$. The numbers ν_n are intimately related to various aspects of overflow systems[1] and satisfy the following recurrence relation:

$$\frac{1}{\nu_0} = E_{1,c}(a) \tag{21}$$

and

$$\nu_n = \frac{a}{n\nu_{n-1}} + 1 + \frac{c - a}{n} \quad \text{for} \quad n \geqq 1. \tag{22}$$

Then, (from Appendix B)

$$q_1(c, d_2) = \frac{\dfrac{a_1}{a_2}\displaystyle\sum_{j=0}^{d_2}\left[\beta(d_1, j)\prod_{k=j+1}^{d_2+1}\frac{a_2}{k\nu_k}\right]}{1 + \displaystyle\sum_{j=1}^{d_2}\left[\binom{d_2}{j-1}\prod_{k=j+1}^{d_2+1}\frac{a_2}{k\nu_k}\right]} \tag{23}$$

and

$$q_2(c, d_1) = \frac{\dfrac{a_2}{a_1}\displaystyle\sum_{j=0}^{d_1}\left[\beta(j, d_2)\prod_{k=j+1}^{d_1+1}\frac{a_1}{k\nu_k}\right]}{1 + \displaystyle\sum_{j=1}^{d_1}\left[\binom{d_1}{j-1}\prod_{k=j+1}^{d_1+1}\frac{a_1}{k\nu_k}\right]}. \tag{24}$$

From Appendix A,

$$E\{L_1\} = a_1\beta(d_1, 0) \quad \text{and} \quad E\{L_2\} = a_2\beta(0, d_2), \tag{25}$$

so that

$$2 \operatorname{cov}(L_1, L_2) = a_1q_2(c, d_1) + a_2q_1(c, d_2) - 2a_1a_2\beta(d_1, 0)\beta(0, d_2). \tag{26}$$

Equations (19) through (26) constitute an algorithm for the computation of $\operatorname{cov}(L_1, L_2)$.

Whenever

$$a_1 = a_2 \quad \text{and} \quad d_1 = d_2$$

the symmetry of the problem (see Fig. 1 and equation (18)) implies that

$$\beta(n_1, n_2) = \beta(n_2, n_1). \tag{27}$$

The symmetry required by (27) would prevail for most step-by-step graded multiples. In such cases (27) can be used to reduce the number of equations to $\frac{1}{2}(d_1 + 1)(d_1 + 2)$. Consequently, the dimensions of the systems of equations needed for an analysis of most step-by-step graded multiples would not exceed 21. Such systems can be solved very efficiently by numerical matrix inversion.

III. NUMERICAL RESULTS

In order to establish a base for comparison, we used a simulation to obtain load-service relations for a 25-trunk and a 45-trunk step-by-step graded multiple. We obtained results for several values of variance-to-mean ratio $z = v/\alpha$ and several selector configurations. We found that the extended Equivalent Random method furnished adequate estimates of blocking probability in each case where the maximum number of selectors was used. However, as the number of selectors was reduced for a particular grading, the inherent load-balancing[4] caused actual grading capacity to be higher than indicated by the extended Equivalent Random method. Consequently, we conclude that *the extended Equivalent Random method provides adequate estimates of load-service relations for graded multiples which carry overflow traffic, provided that the network through which calls reach the grading does not significantly influence grading capacity.*

In view of the effort required to obtain the covariance $\operatorname{cov}(L_i, L_j)$, one naturally questions the necessity of accounting for the correlation which occurs in our problem. In fact, on several occasions, we encountered the following question: What sort of results would be obtained if both mean and variance were split by proportion (i.e., $\alpha_i = p_i\alpha$ and $v_i = p_i v$) when splitting is required, and $\text{cov}(L_i, L_j)$ were assumed to be zero whenever a variance recombination is required?

In order to consider the preceding question, as well as to obtain a better understanding of the behavior of $\text{cov}(L_i, L_j)$, three computer programs were written to generate load-loss relations for step-by-step graded multiples. These programs were based respectively on the following assumptions:

(*i*) The input traffic is completely random, i.e., Poisson.†

(*ii*) The input traffic is nonrandom. Mean and variance are split by proportion when required and cov (L_i, L_j) is assumed to be zero.‡

(*iii*) The input traffic is nonrandom, and the gradings are analyzed by using the extended Equivalent Random method.

Throughout the study, the offered traffic was assumed to be balanced over the subgroups of any grading under consideration (i.e., $p_i = p_j$ for all i, j).

For comparison, we used each of the three assumptions to compute load-service relations for a 25-trunk and a 45-trunk graded multiple. The results are displayed in Fig. 7. For these examples, the variance-to-mean ratio of the nonrandom offered traffic was held constant at 2.25.

The different results arising from assumptions (*ii*) and (*iii*) were surprising. It was originally felt by some that assumption (*ii*) would cause over-trunking, but not by the amounts indicated. For example, notice that the 45-trunk grading yields a B.01 blocking probability for an offered load of 330 ccs (9.17 erlangs) with a variance-to-mean ratio of 2.25 when the correlation is neglected as outlined in assumption (*ii*). However, Fig. 7 indicates that the same traffic can actually be handled at B.01 on the 25-trunk grading when the correlation is taken into account. Hence, assumption (*ii*) leads to at least 80 percent overtrunking for the example. An examination of other portions of the curves yields similar results. Consequently assumption (*ii*) must be discarded.

Lower bounds to the load-loss relations for nonrandom traffic

† Assumption (*i*) is known to cause an underprovision of trunks.[1]

‡ Assumption (*ii*) was considered by many to be the most natural approach to improving on assumption (*i*).

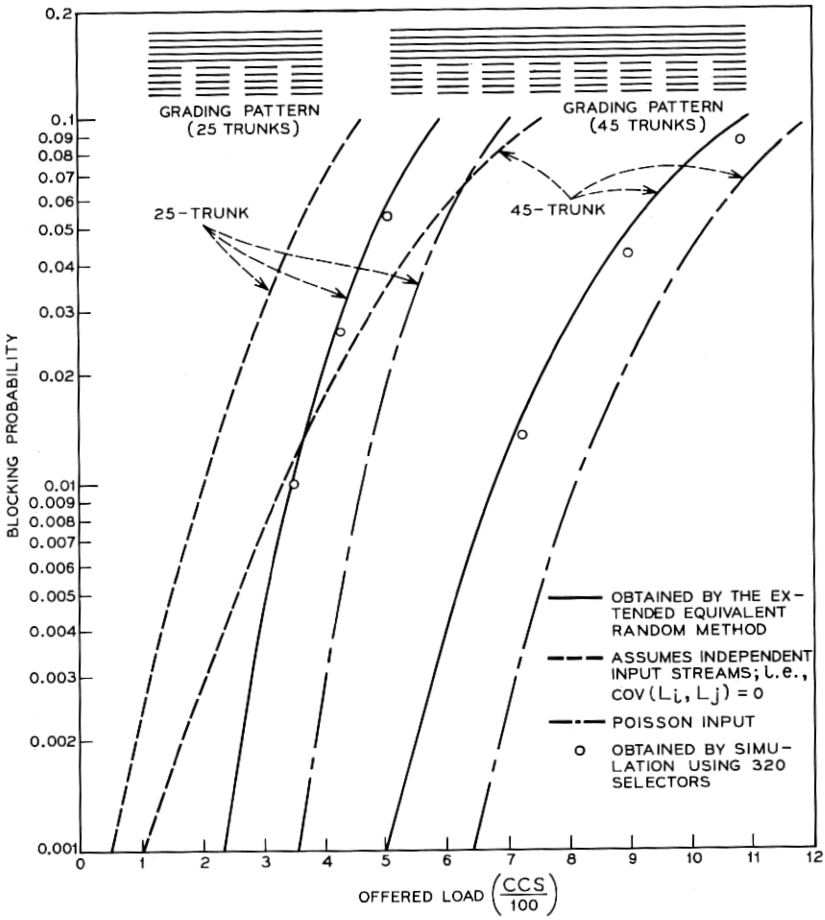

Fig. 7—Load-loss relations for the 25-trunk and the 45-trunk graded multiples. Variance-to-mean ratio of the offered load is $z = 2.25$.

result from assumption (*i*) as illustrated in Fig. 7. For these examples, undertrunking by 18 to 25 percent results from approximation (*i*). Since the disparity will increase for larger variance-to-mean ratios, assumption (*i*) does not seem applicable either.

Two other approximations were also tried but the results were very poor. The first used the correct splitting equations (1) through (3) but assumed $\text{cov}(L_i, L_j) = 0$. As a result, some of the load-service curves actually intersected each other. The second also used the correct splitting formulas but assumed the correlation coefficient.

$$\rho(L_i\ ,\ L_j) = \frac{\text{cov}\ (L_i\ ,\ L_j)}{[\text{var}\ (L_i)\ \text{var}\ (L_j)]^{\frac{1}{2}}}$$

to be constant at recombination points and equal to the correlation coefficient for the split (nonrandom) offered traffic. The predicted blocking resulting from the last approximation actually *decreased* as the intensity of the offered traffic increased.

IV. CONCLUSIONS

We have presented a technique for taking correlation into account when combining certain dependent streams of overflow traffic. The result was used to define an extension of the Equivalent Random method; an engineering approximation for estimating the capacities of overflow networks.

The extended Equivalent Random method yields good estimates of load-service relations for graded multiples which carry overflow traffic provided the network through which calls reach the grading does not significantly affect grading capacity. Consequently, for application to step-by-step gradings, the extended method is restricted to gradings which are connected to large groups of selectors. We are currently investigating techniques which would allow us to consider systems which do influence grading capacity.

V. ACKNOWLEDGMENTS

I am grateful to Mrs. H. J. Tauson for writing the computer programs which were necessary for this study. My understanding of basic aspects of step-by-step switching systems is primarily due to the patience of M. F. Morse and R. I. Wilkinson. Discussions with P. J. Burke were very helpful in establishing a proper mathematical model. Burke also pointed out the original work of Kosten[14] which was the key to the solutions presented in Appendixes A and B.

APPENDIX A

System State Probabilities

In this section, we obtain the solution for the system of equations (9) through (13). We use a generalization of a technique employed by Kosten.[14]

For notational simplicity, let $p(m, n_1, n_2) = B_{0,0}(m, n_1, n_2)$, and let $p_1(m, n_1, n_2)$ denote any function which belongs to the family $\mathcal{A}$ of functions satisfying equation (9) for $0 \leqq m < \infty$ and $0 \leqq n_i \leqq d_i$. Define

$$P(t, q_1, q_2) = \sum_{m=0}^{\infty} \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} p_1(m, n_1, n_2) t^m q_1^{n_1} q_2^{n_2}. \tag{28}$$

It follows from equation (9) that

$$(1 - t)\frac{\partial P}{\partial t} + (1 - q_1)\frac{\partial P}{\partial q_1} + (1 - q_2)\frac{\partial P}{\partial q_2} = a(1 - t)P. \tag{29}$$

This linear first-order partial differential equation can be solved by (Lagrange's) method of characteristics (see Ref. 15, Chap. 2). The characteristic equations are

$$\frac{dt}{1 - t} = \frac{dq_1}{1 - q_1} = \frac{dq_2}{1 - q_2} = \frac{dP}{a(1 - t)P}$$

with solutions

$$\frac{1 - q_i}{1 - t} = k_i \quad \text{and} \quad k = e^{-at}P$$

where k and k_i are arbitrary constants. Hence, the solution to equation (29) is given by

$$P(t, q_1, q_2) = e^{at} H\left(\frac{1 - q_1}{1 - t}, \frac{1 - q_2}{1 - t}\right) \tag{30}$$

where $H(z_1, z_2)$ denotes any analytic function of the arguments z_1, z_2. From (28), it follows that the Taylor series for H must be finite, and so

$$P(t, q_1, q_2) = e^{at} \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \alpha(n_1, n_2)\left(\frac{1 - q_1}{1 - t}\right)^{n_1}\left(\frac{1 - q_2}{1 - t}\right)^{n_2}. \tag{31}$$

Following the notation of Riordan (Ref. 1, p. 89), let $\{\sigma_k(m): m = 0, 1, \cdots\}$ be the sequence with generating function $(1 - t)^{-k} \exp(at)$, that is,

$$\sum_{m=0}^{\infty} \sigma_k(m) t^m = \frac{e^{at}}{(1 - t)^k}. \tag{32}$$

The variables $\sigma_k(m)$ satisfy the following recurrence relations.[1,14] For $m \geqq 0$ and $k \geqq 0$,

$$m\sigma_k(m) = a\sigma_k(m - 1) + k\sigma_{k+1}(m - 1), \tag{33}$$

$$k\sigma_{k+1}(m) = (k + m - a)\sigma_k(m) + a\sigma_{k-1}(m), \tag{34}$$

and

$$\sum_{m=0}^{j} \sigma_k(m) = \sigma_{k+1}(j). \tag{35}$$

It is convenient to define

$$\sigma_{-1}(m) = \sigma_k(-1) = 0. \tag{36}$$

Hence, from Equations (31) and (32),

$$P(t, q_1, q_2)$$

$$= \sum_{m=0}^{\infty} \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \alpha(n_1, n_2)\sigma_{n_1+n_2}(m)t^m(1 - q_1)^{n_1}(1 - q_2)^{n_2} \tag{37}$$

$$= \sum_{m=0}^{\infty} \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \left[(-1)^{n_1+n_2} \sum_{k_1=n_1}^{d_1} \sum_{k_2=n_2}^{d_2} \binom{k_1}{n_1}\binom{k_2}{n_2}\alpha(k_1, k_2) \cdot \sigma_{k_1+k_2}(m)\right] t^m q_1^{n_1} q_2^{n_2}. \tag{38}$$

Comparing equations (29) and (38), one can see that the functions p_1 in $\mathcal{A}$ are of the form

$$p_1(m, n_1, n_2) = (-1)^{n_1+n_2} \sum_{k_1=n_1}^{d_1} \sum_{k_2=n_2}^{d_2} \binom{k_1}{n_1}\binom{k_2}{n_2}\alpha(k_1, k_2)\sigma_{k_1+k_2}(m), \tag{39}$$

and are determined up to the $(d_1 + 1)(d_2 + 1)$ constants $\{\alpha(n_1, n_2): 0 \leqq n_i \leqq d_i\}$. There are $(d_1 + 1)(d_2 + 1)$ independent linear equations among equations (10) through (13), and so it follows that there is exactly one function, p^*, in $\mathcal{A}$ which satisfies equations (9) through (13). The appropriate restriction of p^* must be p. The remainder of this section is devoted to a derivation of the relations which the constants $\{\alpha(n_1, n_2)\}$ must satisfy in order to obtain the solution.

An equivalent but less complex set of boundary conditions is obtained by putting $m = c$ in (9) and subtracting equations (2) through (13) respectively. Hence, if $0 \leqq n_i \leqq d_i - 1$,

$$(a - a_1 - a_2)p(c, n_1, n_2) + a_1p(c, n_1 - 1, n_2) + a_2p(c, n_1, n_2 - 1) = (c + 1)p(c + 1, n_1, n_2), \tag{40}$$

for $0 \leqq n_1 \leqq d_1 - 1$,

$$(a - a_1)p(c, n_1, d_2) + a_1 p(c, n_1 - 1, d_2) + a_2 p(c, n_1, d_2 - 1) = (c + 1)p(c + 1, n_1, d_2), \tag{41}$$

and a similar relation holds for $0 \leqq n_2 \leqq d_2 - 1$. Finally,

$$ap(c, d_1, d_2) + a_1 p(c, d_1 - 1, d_2) + a_2 p(c, d_1, d_2 - 1) = (c + 1)p(c + 1, d_1, d_2). \tag{42}$$

Now, define

$$P_m(q_1, q_2) = \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} p(m, n_1, n_2) q_1^{n_1} q_2^{n_2}, \tag{43}$$

$$G_{m,n_1}(q_2) = \sum_{n_2=0}^{d_2} p(m, n_1, n_2) q_2^{n_2}, \tag{44}$$

and

$$H_{m,n_2}(q_1) = \sum_{n_1=0}^{d_1} p(m, n_1, n_2) q_1^{n_1}. \tag{45}$$

It follows from equations (40) through (42) that

$$[a - a_1(1 - q_1) - a_2(1 - q_2)]P_c(q_1, q_2) + a_1(1 - q_1)q_1^{d_1} G_{c,d_1}(q_2) + a_2(1 - q_2)q_2^{d_2} H_{c,d_2}(q_1) = (c + 1)P_{c+1}(q_1, q_2). \tag{46}$$

Equations (39) and (43) through (45) imply that

$$P_m(q_1, q_2) = \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \alpha(n_1, n_2)\sigma_{n_1+n_2}(m)(1 - q_1)^{n_1}(1 - q_2)^{n_2}, \tag{47}$$

$$G_{c,d_1}(q_2) = (-1)^{d_1} \sum_{n_2=0}^{d_2} \alpha(d_1, n_2)\sigma_{d_1+n_2}(c)(1 - q_2)^{n_2}, \tag{48}$$

and

$$H_{c,d_2}(q_1) = (-1)^{d_2} \sum_{n_1=0}^{d_1} \alpha(n_1, d_2)\sigma_{n_1+d_2}(c)(1 - q_1)^{n_1}. \tag{49}$$

Using the identity $q_i = 1 - (1 - q_i)$ and the relations (47) through (49) in (46) obtains

$$a \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \alpha(n_1, n_2)\sigma_{n_1+n_2}(c)(1 - q_1)^{n_1}(1 - q_2)^{n_2} - a_1 \sum_{n_1=1}^{d_1+1} \sum_{n_2=0}^{d_2} \alpha(n_1 - 1, n_2)\sigma_{n_1+n_2-1}(c)(1 - q_1)^{n_1}(1 - q_2)^{n_2}$$

$$- a_2 \sum_{n_1=0}^{d_1} \sum_{n_2=1}^{d_2+1} \alpha(n_1\,, n_2 - 1)\sigma_{n_1+n_2-1}(c)(1 - q_1)^{n_1}(1 - q_2)^{n_2}$$

$$- (-1)^{d_1} a_1 \sum_{n_1=1}^{d_1+1} \sum_{n_2=0}^{d_2} (-1)^{n_1} \begin{bmatrix} d_1 \\ n_1 - 1 \end{bmatrix}$$

$$\cdot \alpha(d_1\,, n_2)\sigma_{d_1+n_2}(c)(1 - q_1)^{n_1}(1 - q_2)^{n_2}$$

$$- (-1)^{d_2} a_2 \sum_{n_1=0}^{d_1} \sum_{n_2=1}^{d_2+1} (-1)^{n_2} \begin{bmatrix} d_2 \\ n_1 - 1 \end{bmatrix}$$

$$\cdot \alpha(n_1\,, d_2)\sigma_{n_1+d_2}(c)(1 - q_1)^{n_1}(1 - q_2)^{n_2}$$

$$= (c + 1) \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \alpha(n_1\,, n_2)\sigma_{n_1+n_2}(c + 1)(1 - q_1)^{n_1}(1 - q_2)^{n_2}.$$

Equating the coefficients of $(1 - q_1)^{n_1}(1 - q_2)^{n_2}$ yields

$$\begin{aligned} &[a\sigma_{n_1+n_2}(c) - (c + 1)\sigma_{n_1+n_2}(c + 1)]\alpha(n_1\,, n_2) \\ &\quad = [a_1\alpha(n_1 - 1, n_2) + a_2\alpha(n_1\,, n_2 - 1) + a_2\alpha(n_1\,, n_2 - 1]\sigma_{n_1+n_2-1}(c) \\ &\qquad + (-1)^{d_1+n_1} a_1 \begin{bmatrix} d_1 \\ n_1 - 1 \end{bmatrix} \alpha(d_1\,, n_2)\sigma_{d_1+n_2}(c) \\ &\qquad + (-1)^{n_2+d_2} a_2 \begin{bmatrix} d_2 \\ n_2 - 1 \end{bmatrix} \alpha(n_1\,, d_2)\sigma_{n_1+d_2}(c). \end{aligned} \tag{50}$$

Using (33), we see that

$$a\sigma_{n_1+n_2}(c) - (c + 1)\sigma_{n_1+n_2}(c + 1) = -(n_1 + n_2)\sigma_{n_1+n_2+1}(c). \tag{51}$$

It is worthwhile to define

$$\beta(n_1\,, n_2) = (-1)^{n_1+n_2}\alpha(n_1\,, n_2)\sigma_{n_1+n_2}(c), \quad \text{and} \quad n = n_1 + n_2\,. \tag{52}$$

Now, substitute (51) and (52) into (50) to obtain

$$\begin{aligned} n\,\frac{\sigma_{n+1}(c)}{\sigma_n(c)}\,\beta(n_1\,, n_2) &= a_1\beta(n_1 - 1, n_2) + a_2\beta(n_1\,, n_2 - 1) \\ &\quad - a_1 \begin{bmatrix} d_1 \\ n_1 - 1 \end{bmatrix} \beta(d_1\,, n_2) \\ &\quad - a_2 \begin{bmatrix} d_2 \\ n_2 - 1 \end{bmatrix} \beta(n_1\,, d_2) \end{aligned} \tag{53}$$

for $0 \leqq n_i \leqq d_i$ and $n > 0$. (Both sides vanish for $n_1 = n_2 = 0$.)

One additional relation is required. Using (39),

$$\sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} p(m, n_1, n_2) = \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} \alpha(n_1, n_2)\sigma_{n_1+n_2}(m) \cdot \left[\sum_{k_1=0}^{n_1} \binom{n_1}{k_1}(-1)^{k_1}\right] \left[\sum_{k_2=0}^{n_2} \binom{n_2}{k_2}(-1)^{k_2}\right]$$

$$= \alpha(0, 0)\sigma_0(m).$$

Consequently, noting relation (36), one obtains

$$1 = \sum_{m=0}^{c} \sum_{n_1=0}^{d_1} \sum_{n_2=0}^{d_2} p(m, n_1, n_2) = \alpha(0, 0) \sum_{m=0}^{c} \sigma_0(m)$$

$$= \alpha(0, 0)\sigma_1(c).$$

Thus,

$$\alpha(0, 0) = \frac{1}{\sigma_1(c)}$$

and

$$\beta(0, 0) = \frac{\sigma_0(c)}{\sigma_1(c)} = E_{1,c}(a); \tag{54}$$

i.e., $\beta_{(0,0)}$ is the Erlang-B blocking probability (also known as the first Erlang loss-function) for the first-choice trunk group.

Equations (53) and (54) completely determine $\beta(n_1, n_2)$ for $0 \leqq n_i \leqq d_i$. Using (52) and (39), we see that the state probabilities are given by

$$p(m, n_1, n_2) = \sum_{k_1=n_1}^{d_1} \sum_{k_2=n_2}^{d_2} (-1)^{k_1-n_1}(-1)^{k_2-n_2}\binom{k_1}{n_1}\binom{k_2}{n_2}\beta(k_1, k_2) \frac{\sigma_{k_1+k_2}(m)}{\sigma_{k_1+k_2}(c)}. \tag{55}$$

Using the relation

$$\binom{k}{m}\binom{m}{n} = \binom{k}{n}\binom{k-n}{m-n}$$

it is straightforward to show that if

$$\Lambda_{(m,n_1,n_2)} = \sum_{i=m}^{c} \sum_{k_1=n_1}^{d_1} \sum_{k_2=n_2}^{d_2} \binom{i}{m}\binom{k_1}{n_1}\binom{k_1}{n_1} p(i, k_1, k_2)$$

for $0 \leqq m \leqq c$ and $0 \leqq n_i \leqq d_i$, then an inverse relation is given by

$$p(m, n_1, n_2) = \sum_{i=m}^{c} \sum_{k_1=n_1}^{d_1} \sum_{k_2=n_2}^{d_2} (-1)^{i-m}(-1)^{k_1-n_1}(-1)^{k_2-n_2} \binom{i}{m} \binom{k_1}{n_1} \binom{k_2}{n_2} \cdot \Lambda_{(i,k_1,k_2)} \tag{56}$$

for $0 \leqq m \leqq c$ and $0 \leqq n_i \leqq d_i$.

Consequently, comparing (55) and (56) we see that

$$\beta(n_1, n_2) = \Lambda_{(c,n_1,n_2)} = E\left\{\binom{M}{c}\binom{N_1}{n_1}\binom{N_2}{n_2}\right\}. \tag{57}$$

Equations (14) and (55), imply that

$$E\{L_1\} = a_1\beta(d_1, 0), \tag{58}$$

and

$$E\{L_2\} = a_2\beta(0, d_2). \tag{59}$$

For the computation of $\beta(n_1, n_2)$ using (53), the ratio

$$\nu_n = \frac{\sigma_{n+1}(c)}{\sigma_n(c)}$$

is required for $n \geqq 1$. A recursive relation for the ratio is obtained from (32) via

$$\frac{\sigma_{n+1}(c)}{\sigma_n(c)} = \left[1 + \frac{c-a}{n}\right] + \frac{a}{n}\frac{\sigma_{n-1}(c)}{\sigma_n(c)};$$

that is,

$$\nu_n = \frac{a}{n}\left(\frac{1}{\nu_{n-1}}\right) + \frac{c-a}{n} + 1 \quad \text{for} \quad n \geqq 1. \tag{60}$$

The first-order (nonlinear) algorithm is initiated with

$$\frac{1}{\nu_0} = \frac{\sigma_0(c)}{\sigma_1(c)} = E_{1,c}(a); \tag{61}$$

i.e., the Erlang-B blocking probability for the first-choice trunk group.

APPENDIX B

A Conditional Mean

In this appendix, a formula is obtained for the computation of†

$$q(m, n) = \sum_{n_1=0}^{d_1} B_{1,0}(m, n_1, n)$$
$$= E\{L_1 \mid M = m, N_2 = n\}P\{M = m, N_2 = n\}. \qquad (62)$$

From equations (9) through (12), it follows that for $0 \leqq m \leqq c - 1$ and $0 \leqq n \leqq d_2$,

$$(a + m + n + 1)q(m, n) = aq(m - 1, n) + (m + 1)q(m + 1, n) + (n + 1)q(m, n + 1) \qquad (63)$$

and for $0 \leqq n \leqq d_2 - 1$,

$$(a_2 + c + n + 1)q(c, n) = aq(c - 1, n) + a_2q(c, n - 1) + (n + 1)q(c, n + 1) + a_1B_{0,0}(c, d_1, n). \qquad (64)$$

Also,

$$(c + d_2 + 1)q(c, d_2) = aq(c - 1, d_2) + a_2q(c, d_2 - 1) + a_1B_{0,0}(c, d_1, d_2). \qquad (65)$$

Using the methods and results presented in Appendix A, we can show that

$$q(m, n) = \sum_{k=n}^{d_2} (-1)^{n-k}\omega_k \frac{\sigma_{k+1}(m)}{\sigma_{k+1}(c)} \quad \text{for} \quad 0 \leqq m \leqq c \quad \text{and} \quad 0 \leqq n \leqq d_2, \qquad (66)$$

where $\omega_{-1} = 0$, and

$$(n + 1)\nu_{n+1}\omega_n = a_2\omega_{n-1} - a_2 \begin{bmatrix} d_2 \\ n - 1 \end{bmatrix} \omega_{d_2} + a_1\beta(d_1, n) \quad \text{for} \quad 0 \leqq n \leqq d_2. \qquad (67)$$

In particular, $q(c, d_2) = \omega_{d_2}$, and can be obtained from (67) in the following closed form:

† Throughout this appendix, the subscript 1 on q_1 has been omitted for notational simplicity.

$$q(c, d_2) = \frac{\dfrac{a_1}{a_2} \displaystyle\sum_{j=0}^{d_2} \left[\beta(d_1\,,\, j) \prod_{k=j+1}^{d_2+1} \frac{a_2}{k\nu_k} \right]}{1 + \displaystyle\sum_{j=1}^{d_2} \left[\binom{d_2}{j-1} \prod_{k=j+1}^{d_2+1} \frac{a_2}{k\nu_k} \right]}. \tag{68}$$

A similar expression is valid for $q_2(c, d_1)$.

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