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Phase and Amplitude Variations in Multipath Fading of Microwave Signals

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Experimental data on the duration of individual fades in a 4-GHz radio signal are used to derive the rates of change in amplitude and phase of a signal propagated through a 30-mile multipath medium. A by-product is a relation between the rate of change in phase and the effective vertical velocity of the atmospheric irregularities that can be considered to be the cause of the multipath phenomena. The derived distribution of velocities is then used to predict the variations in fade duration over a wide range of frequency and path length.

I. INTRODUCTION

Multipath fading on line-of-sight paths causes substantial variations in the amplitude and phase of the received signal. Theoretically, if the response to an impulse function were known *with sufficient accuracy,* all magnitudes and rates of change could be computed. In practice, this possibility is somewhat of an illusion because many different impulse responses occur on any given path, with significant diurnal and seasonal variations. Moreover, the desired accuracy is generally not achieved, because either the required impulse function needs to be only one cycle of the radio frequency or extensive correlation techniques are needed to achieve a comparable result.

The quantitative estimates given below are based on elementary theory and on the measured distribution of fade durations. The results agree with (or at least do not contradict) the available experimental data.

II. GENERAL CONSIDERATIONS

The signal received through a multipath medium can be represented by the sum of the principal wave plus the delayed components ($a_m e^{i\Delta t_m}$):

$$E = Ae^{i(\omega+\Delta\omega)t}\left(1 + \sum_{m=1}^{M} a_m e^{i(\omega+\Delta\omega)\Delta t_m}\right) \tag{1}$$
$$= Ave^{i(\omega+\Delta\omega)t}$$

where a_m and Δt_m represent the magnitude and relative time delay of the mth "echo."

The effect of frequency deviation, $\Delta\omega/2\pi$, is discussed in a later section on the variation in net loss within the modulation bandwidth. For the single-frequency case ($\Delta\omega = 0$), it is convenient to define R, L, ϕ, and γ as follows:

$$v = 1 + Re^{i[\phi+(n-1)\pi]} = Le^{-i\gamma}, \tag{2}$$
$$\tan\gamma = \frac{R\sin\phi}{1 - R\cos\phi}$$

where R and ϕ, respectively, are the instantaneous amplitude and residual phase of the "composite echo" given by the summation term in (1) with $\Delta\omega = 0$. For the deep fades $L \ll 1$, $|1 - R| \leqq L$, $|\phi| \leqq L$, and n is an even integer.

It is shown in Appendix A that

$$\frac{dL}{dt} = \frac{d\phi}{dt}\left[\sin\gamma - (\cos\gamma - L)\frac{dR}{R\,d\phi}\right], \tag{3}$$

$$\frac{L\,d\gamma}{dt} = \frac{d\phi}{dt}\left[(\cos\gamma - L) + \sin\gamma\frac{dR}{R\,d\phi}\right], \tag{4}$$

and

$$\frac{dL}{L\,d\gamma} = \tan(\alpha - \beta)$$

where

$$\alpha = \tan^{-1}\left(\frac{\sin\gamma}{\cos\gamma - L}\right) = \tan^{-1}\left(\frac{\sin\phi}{\cos\phi - R}\right)$$
$$= \tan^{-1}\left(\frac{\tan\gamma}{1 - L/\cos\gamma}\right) \approx \gamma,$$
$$\beta = \tan^{-1}\frac{dR}{R\,d\phi}.$$

At the bottom of a fade $dL/dt = 0$ and $L\,d\gamma/dt$ has a maximum value of

$$\frac{d\phi}{dt}\frac{R^2}{(\cos\gamma - L)};$$

conversely, dL/dt has a maximum value of

$$\frac{d\phi}{dt}\frac{R^2}{\sin\gamma} \quad \text{when} \quad d\gamma = 0.$$

Frequency selective fading requires that $d\phi/dt \gg dR/dt(\beta \ll 1)$, but dR/dt may not be entirely negligible at the very deep fade levels. This observation is consistent with physical models of "reflecting" layers that are slowly rising or falling with time. The physical model suggests that during any particular fade $d\phi/dt$ and dR/dt can be constant, or at least they vary much more slowly with time than the "non-linear" parameters L, γ, dL/dt, and $d\gamma/dt$. This information is useful in examining the detailed wave interference characteristics during a few seconds or minutes in time. On the other hand, the process is nonstationary and the *average* fading characteristics taken over many fades depend on the long term *average* values of $d\phi/dt$ and dR/dt, and hence on meteorological conditions.

Since the variations in signal level are caused by variable metero-logical conditions along the path, experimental data are needed to obtain quantitative results. In principle, it should be possible to predict the radio results from meteorological theory, if the meterological parameters were known with sufficient accuracy. In practice, this indirect approach is not feasible and direct radio measurements are required. Data on the number and durations of deep fades will be used to estimate the probability distribution of $\cos\gamma$ and $d\phi/dt$ for use in (3) and (4).

III. NUMBER OF FADES VS DEPTH OF FADE

Experimental data show that 20-dB fades occur about ten times more often than 40-dB fades and on the average last about ten times longer. In other words, both the number of fades and their average duration are proportional to L.[1] Of all the fades reaching a given level L_1, 30 percent will "bottom out" at or before $0.7L_1$, 50 percent will not go deeper than $0.5L_1$, and only 10 percent will reach or go deeper than $0.1L_1$. Thus the probability can be written as

$$\text{Prob}\left(\frac{L_{\min}}{L_1} \leqq X\right) = X$$

where X varies uniformly from $0 \leqq X \leqq 1$.

It can be seen from (2) that

$$L = (1 - R\cos\phi)\sqrt{1 + \tan^2\gamma} = \frac{1 - R\cos\phi}{\cos\gamma} = \frac{R\sin\phi}{\sin\gamma}$$

and $\phi \leqq L$. At the bottom of a fade, $\phi \to 0$, $dL = 0$, $\gamma = 0$, and

$$L_{\text{min}} \approx 1 - R_0 .$$

It follows that at any level $L_1 < 0.1$ the ratio

$$\frac{L_{\text{min}}}{L_1} = \left(\frac{1 - R_0}{1 - R_1 \cos \phi_1}\right) \cos \gamma_1$$

$$\approx \cos \gamma_1 \quad \text{for} \quad R_1 \approx R_0 . \tag{5}$$

Thus the experimentally determined cumulative distribution for L_{min}/L_1 also applies to $\cos \gamma$, when $dR/dt \ll d\phi/dt$; that is

$$\text{Prob} (\cos \gamma \leqq X) = X$$

and the average values are

$$\overline{\frac{L_{\text{min}}}{L_1}} \approx \overline{\cos \gamma} = \int_0^1 X \, dX = \tfrac{1}{2}.$$

The corresponding average value of $\overline{\sin \gamma}$, needed in a later section, is given by

$$\int_0^1 \sqrt{1 - X^2} \, dX = \frac{\pi}{4}.$$

IV. DURATION OF FADES

The average duration of a 40-dB fade ($L = 0.01$) at 4 GHz on a 30-mile path is about 3 or 4 seconds, and the average duration of a 20-dB fade is about 10 times longer. The results of an extensive propagation experiment have shown that the distribution of fade duration τ follows a log normal distribution with a standard deviation of about $10 \log(\tau_{16}/\tau_{50}) = 5.6$ or $\ln (\tau_{16}/\tau_{50}) = 1.3$. Quantitatively, these results are summarized as follows for 4 GHz on a 30-mile line-of-sight path:

99 percent of the fades are shorter than 4000 L seconds,
70 percent of the fades are shorter than 400 L (average duration),
50 percent of the fades are shorter than 200 L (median duration).

These data are plotted on Fig. 1 and it is assumed that fade durations shorter than the median are given by the extension of this log normal distribution with the same standard deviation.*

* Recent unpublished data on the duration of rapid fades indicate that the standard deviation may be somewhat larger than the "5.6 dB" used in this paper, which means that the lines on Fig. 1 might be more nearly vertical than shown.

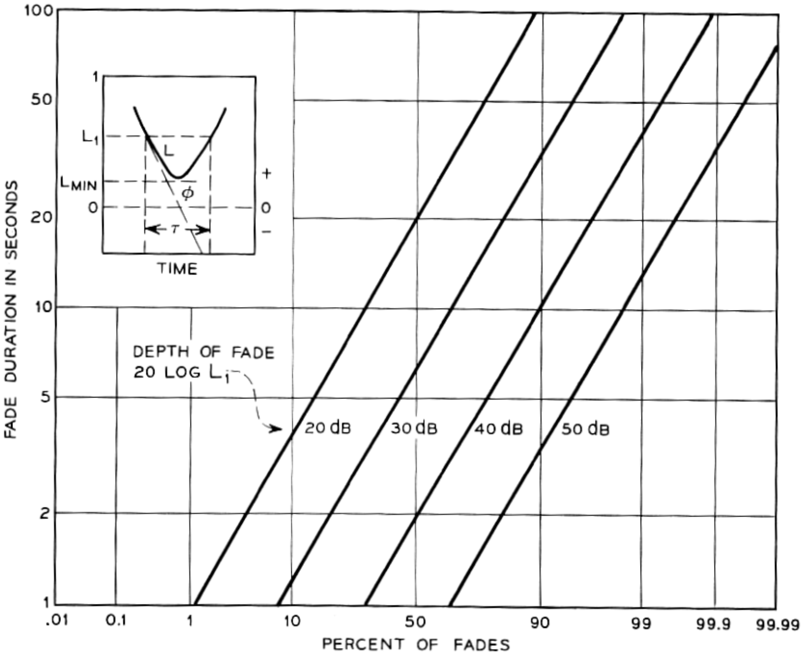

Fig. 1—Duration of fading at 4 GHz on a 30-mile path.

The voltage-time diagram shown on Fig. 1 defines L, $L_{\min}$, ϕ, and the fade duration τ for a fixed frequency. From the diagram it can be seen that for $dR/dt = 0$ the fade duration τ is given by

$$\tau = \frac{2}{\frac{d\phi}{dt}} \int_0^{\sin^{-1}(L_1 \sin \gamma_1/R)} d\phi \approx \frac{2}{\frac{d\phi}{dt}} \frac{L_1 \sin \gamma_1}{R} \tag{6a}$$

and since the average value

$$\overline{\sin \gamma_1} = \frac{\pi}{4},$$

$$\overline{\frac{d\phi}{dt}} = \frac{\pi L_1}{2\tau R}. \tag{6b}$$

Substituting into (3) and (4), the average values are

$$\overline{\frac{dL}{L\,dt}} = \overline{\frac{d(\ln L)}{dt}} = \frac{\pi^2}{8\tau R}, \tag{7a}$$

$$\overline{\frac{d\gamma}{dt}} = \frac{\pi}{2\tau R} \frac{(1 - 2L)}{2} \approx \frac{\pi}{4\tau R} \quad \text{for} \quad L \ll 1. \tag{7b}$$

It follows that, since 20 log L = 8.7 ln L dB,

$$\overline{\frac{d\gamma}{dt}} \approx \frac{2}{\pi} \overline{\frac{d(\ln L)}{dt}} = \frac{2\ \overline{\text{dB/s}}}{8.7\pi} = \frac{\overline{\text{dB/s}}}{13.6}. \tag{8}$$

Figure 2 shows the distribution of rate of change of phase $d\gamma/dt$ and the rate of change of L in dB/second, based on (7a) and (7b) and on the distribution of fade durations shown on Fig. 1.

It will be noted that if γ were uniformly distributed ($\overline{\sin \gamma} = \overline{\cos \gamma}$) it would be expected that

$$\overline{\text{dB/s}} = 8.7 \overline{\frac{d\gamma}{dt}},$$

but this assumption is not consistent with the experimental result that the number of fades is proportional to L.

In a similar manner, when $d\phi/dt = 0$ ($|\phi|$ is a constant $< L_{\min}$) the fade duration might be represented by

$$\tau = \frac{2}{\overline{\frac{dR}{dt}}} \int_{R_0}^{R_1} dR = \frac{2}{\overline{\frac{dR}{dt}}} (R_1 - R_0)$$

or

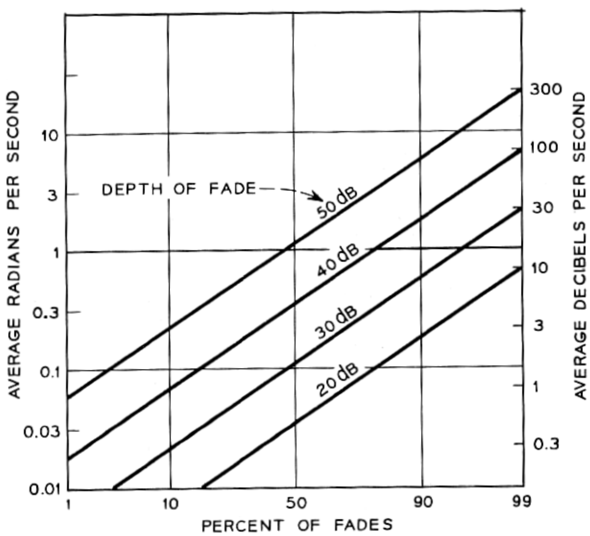

Fig. 2—Rates of change in amplitude and phase during fading at 4 GHz on a 30-mile path.

$$\frac{dR}{dt} = \frac{2L}{\tau}\left[\frac{R_1 - R_0}{L\cos\phi}\right]. \tag{9}$$

There is no obvious way to relate a change in R to a change in meteorological conditions and there is doubt that it is important to do so. The frequency selective nature of fading requires that $d\phi/dt$ be controlling most of the time. In addition, essentially all fast "flat" fading can be explained as a phase interference effect with a delay of one or three half-cycles as well as by a supposed change in R. Moreover, the experimental results that the number of fades and their average durations are both proportional to L are characteristic of a Rayleigh type distribution, which is usually explained on the basis of near-random phase. Consequently, it is assumed in the remainder of the paper that

$$\overline{\frac{d\phi}{dt}} \gg \overline{\frac{dR}{dt}}.$$

V. VERTICAL VELOCITY OF ATMOSPHERIC IRREGULARITIES

Although dL/dt and $d\gamma/dt$ are important parameters in the design of communication systems, the search for a better physical understanding of the fading mechanism needs to begin with the less complicated parameters R, ϕ, and $d\phi/dt$ of the composite echo given by the bracketed term in (1). It may be seen from (4) that for $dR/dt = 0$,

$$\frac{d\phi}{dt} = \frac{L\,d\gamma}{dt}\,\frac{1}{(\cos\gamma - L)}$$

which on the average is

$$\overline{\frac{d\phi}{dt}} \approx \overline{\frac{2L\,d\gamma}{dt}}. \tag{10}$$

In principle, the rate of change in phase $d\phi/dt$ might also be obtained by either: (*i*) direct measurement of phase, or (*ii*) computation from known or assumed values for vertical motion in the atmosphere. For example, it is reasonable to expect that

$$\overline{\frac{d\phi}{dt}} = \frac{\pi\bar{v}}{H_n - H_{n-1}} \tag{11}$$

or conversely,

$$\bar{v} = \left(\frac{H_n - H_{n-1}}{\pi}\right)\overline{\frac{d\phi}{dt}}$$

where $d\phi/dt$ is the rate of change in phase caused by an echo source moving vertically through a thickness $(H_n - H_{n-1})$ at a velocity v. H_n is the height of the nth Fresnel zone and $H_n - H_{n-1}$ is the thickness of the nth Fresnel zone. The relation between $d\phi/dt$ and the effective velocity v is shown on Fig. 3 and the probability of exceeding specified values of $d\phi/dt$ has been obtained from (10) and from Fig. 2. It appears that a vertical motion of only a few centimeters per second (a few feet per minute) is sufficient to explain the fade duration measurements and the values of $d\gamma/dt$ and dB/s derived therefrom. An alternative derivation of (11) is given in Appendix B.

VI. FADE DURATION AT OTHER FREQUENCIES AND PATH LENGTHS

The distribution of velocities obtained in the preceding paragraph is expected to be a meteorological phenomena independent of radio frequency. It should be possible to use this information to predict the fade duration at other frequencies and path lengths.

Substituting (6b) into (11), and using equation (17), it follows that the

$$\text{Average Fade Duration} = \frac{1}{2}\frac{L}{\bar{v}}(H_n - H_{n-1}) = \frac{L}{8\bar{v}}\sqrt{\frac{\lambda D}{n}} \tag{12}$$

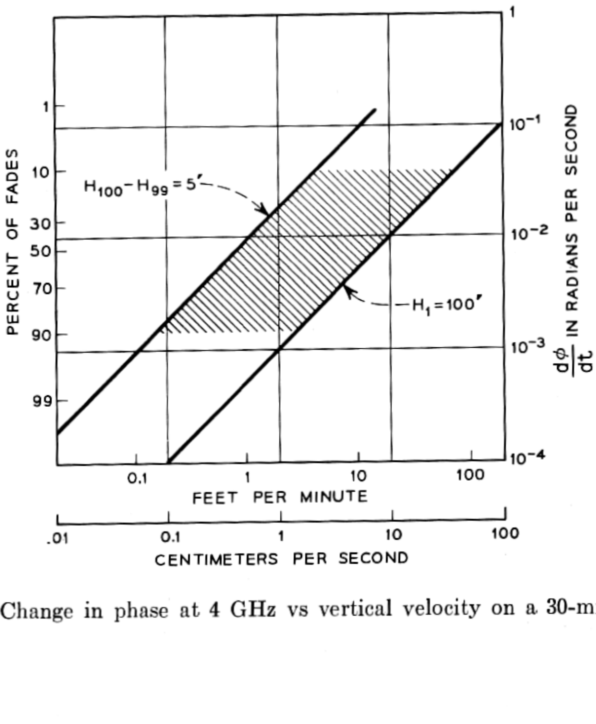

Fig. 3—Change in phase at 4 GHz vs vertical velocity on a 30-mile path.

where D is the path length and λ is the wavelength. Equation (12) indicates that the fade durations vary as the $\sqrt{\lambda D}$, *if* the effective value of n is essentially independent of frequency for the average fade.

The preceding information and assumptions lead to the following estimates for the average fade duration at various frequencies, when the path length D is measured in km.

Frequency	Average Fade Duration in Seconds
160 MHz	$2000\,L\sqrt{\frac{D}{50}}$
4 GHz	$400\,L\sqrt{\frac{D}{50}}$
6 GHz	$320\,L\sqrt{\frac{D}{50}}$
6×10^5 GHz (visible light)	$L\sqrt{\frac{D}{50}}$

The experimental results at 4 and 6 GHz are not accurate enough to establish the $\lambda^{1/2}$ variation over this narrow range, but at lower frequencies the fades, when they occur, last longer than at 4 and 6 GHz so the fading rate is not independent of λ.

An extrapolation of the radio data to optical frequencies indicates that the *average* fading rate on a 5-km path through the atmosphere would be of the order of one or two cycles per second with much more rapid fluctuations for a small percentage of the time. In principle, scintillations are rapid multipath phenomena. Although differences, such as the effect of water vapor, antenna beamwidth, etc., need to be included, the fading rate must vary with the wavelength as λ^p, where p is a positive fraction less than one. A variation of as much as λ^1 is unrealistic because it would predict that the twinkling of the stars would ordinarily be above the flicker rate acceptable by the human eye. A consideration of both radio and optical phenomena requires a variation in fade duration with frequency close to $\lambda^{1/2}$ (at least in the range between $\lambda^{1/3}$ and $\lambda^{2/3}$), so the elementary result shown in (12) is a reasonable working hypothesis. At optical frequencies the Fresnel zones are more nearly comparable in size to the atmospheric irregularities and both horizontal and vertical motion of atmospheric irregularities produce significant scintillations.

VII. VARIATIONS IN NET LOSS WITH FREQUENCY

The principal interest thus far has been in the characteristics of fading at a single fixed frequency. Even if an automatic equalizer could be built to compensate perfectly for the effects of fading at any reference frequency, the resulting equalization would be less than perfect at frequencies other than the reference frequency.[2–4] This type of distortion results in variations in net loss (deviations from a constant output level) over the modulation bandwidth. By the use of the concept of Fresnel zones and some simplifying assumptions, it is shown in Appendix C that the variations in net loss relative to a fixed level at the carrier or pilot frequency is given approximately by

$$20 \log \left| 1 - i \frac{\delta_{\text{eff}}}{L} \right| , \tag{13}$$

$$10 \log \left(1 + \left(\frac{\delta_{\text{eff}}}{L} \right)^2 \right) , \tag{14}$$

where

$$\delta_{\text{eff}} = \frac{\Delta f}{f} \pi n_{\text{eff}} .$$

$20 \log L$ is the depth of fade in decibels ($L \ll 1$) and n_{eff} is the effective number of half-wavelengths in the relative delay of the strongest delayed component (or composite "echo") that is causing the fade at the reference frequency. The corresponding variations in net loss are shown in Table I and on Fig. 4. Table I indicates that most of the deep fades are caused by "echo" delays of 1, 3, or 5 half-wavelengths ($n_{\text{eff}} \leqq 5$), because experience has shown that good diversity requires a frequency separation greater than 1 percent and that acceptable transmission quality can almost always be obtained with bandwidths of 0.1 percent or less. The corresponding values of $\Delta f/f$ for a 20-dB fade are ten times larger than shown in Table I.

A comparison between theory and experiment is shown on Fig. 4 for the case of 40-dB fades. This figure also suggests that the median 40-dB fade corresponds to $1 \leqq n_{\text{eff}} \leqq 5$, but that 10 percent of the 40-dB fades correspond to delays of many half-wavelengths. The curve for $n = 100$ represents the approximate upper boundary set by 40-dB antenna patterns at 4 GHz on a 30-mile path. The corresponding theoretical curves for a 20-dB fade would be shifted one decade to the right. The crossover of the experimental 40-dB fade data and the $n = 1$ curve shown on the figure is not possible with a single echo,

TABLE I—VARIATIONS IN NET LOSS

$\frac{\delta_{eff}}{L}$	Net Loss Variation (dB)	Quality	$\frac{\Delta f}{f}$ for 40-dB fade		
			$n_{eff} = 1$ (%)	$n_{eff} = 3$ (%)	$n_{eff} = 5$ (%)
0.1	0.04	Excellent transmission	0.03	0.01	0.006
0.3	0.4	Acceptable transmission	0.1	0.03	0.02
1.0	3.0	Poor transmission	0.3	0.1	0.06
3.0	10.0	Fair diversity	1.0	0.3	0.2
10.0	20.0	Good diversity	3.0	1.0	0.6

and is an indication that more than two components are present much of the time. The "experimental 40-dB fade" frequently represents the combination of two or more echoes whose effects add to 40 dB; in this case, the limiting theoretical curve for $n = 1$ is shifted to the right because the principal component of the composite echo, taken by itself, represents a fade of less than 40 dB.

The agreement between theory and experiment is considered satisfactory and as evidence that most of the fades are caused by strong components delayed by only a few half-wavelengths. It has been suggested that a better fit with the shape of the experimental data exceeded 10 percent and 50 percent of the time can be obtained if the $i = \sqrt{-1}$ in (13) could be omitted; such an assumption is rejected

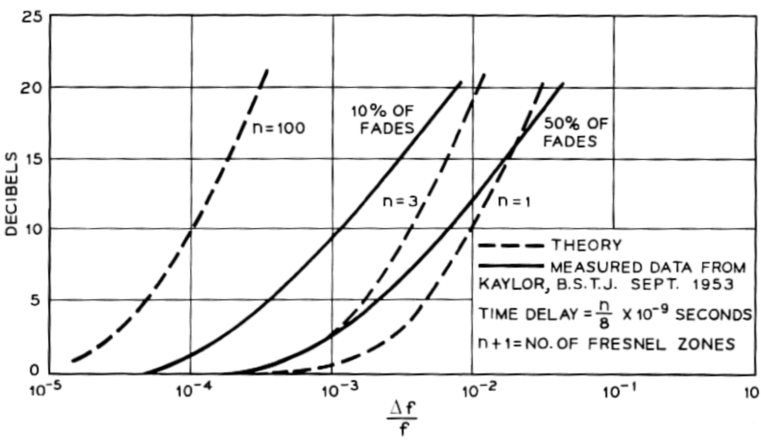

Fig. 4—Variations in net loss during 40-dB fades at 4 GHz on a 30-mile path.

because it seems contrary to logic and because it would create difficulties in explaining the data exceeded by, say, 90 percent of the fades.

VIII. CONCLUSION

Experimental data on the number and duration of fading have been used to estimate the rate of change in phase and the rate of change in amplitude of an electromagnetic wave received through a multipath medium. The elementary theory used herein achieves quantitative results, primarily because it explicitly includes the first-order effects of phase associated with the use of Fresnel zones. Alternative methods starting from the more fundamental wave equations are more sophisticated and may be more exact in a mathematical sense, but their inherent generality does not ordinarily lead to results that can be compared with experiment, unless somewhat arbitrary "constants" or correlation functions are assumed.

APPENDIX A

The relative magnitude and phase of a multipath signal can, by definition, be represented as

$$v = 1 + Re^{i[\phi + (n-1)\pi]} = Le^{-i\gamma}$$

where n is an even integer and R and ϕ are, respectively, the instantaneous magnitude and residual phase of the sum of all the delayed components. It follows that

$$\begin{aligned} L &= \sqrt{(1 - R\cos\phi)^2 + (R\sin\phi)^2} \\ &= (1 - R\cos\phi)\sqrt{1 + \tan^2\gamma} \\ &= \sqrt{(1 - R)^2 + 2R(1 - \cos\phi)} \\ &= \frac{1 - R\cos\phi}{\cos\gamma} = \frac{R\sin\phi}{\sin\gamma}. \end{aligned}$$

By differentiation,

$$\begin{aligned} dL &= \frac{R\sin\phi\, d\phi - (\cos\phi - R)\, dR}{L} \\ &= \sin\gamma\, d\phi - \frac{(R\cos\phi - R^2 + 1 - 2R\cos\phi - 1 + 2R\cos\phi)}{L}\frac{dR}{R} \\ &= \sin\gamma\, d\phi - \left(\frac{1 - R\cos\phi - L^2}{L}\right)\frac{dR}{R} \end{aligned}$$

$$= \sin\gamma \, d\phi - (\cos\gamma - L)\frac{dR}{R}.$$

In a similar manner,

$$L \, d\gamma = (\cos\gamma - L) \, d\phi + \sin\gamma \frac{dR}{R}.$$

APPENDIX B

As shown in several textbooks, the height of the nth Fresnel zone at the middle of path length D is given by

$$H_n = y_n = \frac{\sqrt{\lambda \, Dn}}{2}. \tag{15}$$

The corresponding phase delay is $\theta_n = n\pi$, by definition of Fresnel zones.

In order to restrict the parameter n to integral values, we can generalize as follows:

$$\theta = n\pi - \phi; \qquad \frac{d\theta}{dt} = -\frac{d\phi}{dt}$$

and

$$y = \frac{1}{2}\sqrt{\frac{\lambda \, D(\theta + \phi)}{\pi}}.$$

It follows that

$$v = \frac{dy}{dt} = \frac{1}{4\pi}\sqrt{\frac{\lambda \, D}{n}} \frac{d\phi}{dt} \tag{16}$$

where v is the effective velocity in a plane perpendicular to the direction of propagation.

From (15) we have

$$H_n - H_{n-1} = \tfrac{1}{2}\sqrt{\lambda \, Dn}\left(1 - \sqrt{\frac{n-1}{n}}\right)$$

$$\approx \frac{1}{4}\sqrt{\frac{\lambda \, D}{n}} \quad \text{for} \quad n > 3. \tag{17}$$

By substituting (17) into (16), we have

$$\frac{d\phi}{dt} = \frac{\pi v}{H_n - H_{n-1}},$$

which is the same as equation (11).

APPENDIX C

As noted in (1) the signal received through a multipath medium can be represented by the sum of the principal wave plus the delayed components ($a_m e^{i\Delta t_m}$):

$$E = Ae^{i(\omega+\Delta\omega)t}\left(1 + \sum_{m=1}^{M} a_m e^{i(\omega+\Delta\omega)\Delta t_m}\right) \tag{18}$$

where a_m and Δt_m represent the magnitude and relative time delay of the mth "echo." The term in brackets can also be resolved into Fresnel zones so that

$$v = 1 + \sum_{n=1}^{N} a_n e^{i(\omega+\Delta\omega)(n\pi/\omega)} \tag{19}$$

where n is an integer and a_n is the magnitude of the signal in the $(n + 1)$th Fresnel zone. The upper limit N is the maximum number of Fresnel zones enclosed by the antenna beamwidth; components with longer delays, if present, are relatively unimportant because their magnitudes are reduced by the off-beam antenna patterns. The expression for v can be separated into the following three terms:

$$v = 1 + \sum_{n=1}^{N} a_n e^{in\pi} + \sum_{n=1}^{N} a_n e^{in\pi}(e^{i\delta} - 1) \tag{20}$$

where $\delta = (\Delta f/f)n\pi$. The first two terms in (20) show the fading characteristic at frequency f, which in principle can be corrected by a perfect automatic equalizer based on a carrier frequency or pilot tone. The third term on the right in (20) shows the relative variation (distortion) at frequency Δf.

By definition, let the first two terms be

$$1 + \sum_{n=1}^{N} a_n e^{in\pi} = Le^{-i\gamma}. \tag{21}$$

Substituting into (20) and introducing an effective value of δ, we have

$$\begin{aligned} v &= Le^{-i\gamma} + i\delta_{\text{eff}} \sum a_n e^{in\pi}\left(\frac{e^{i\delta} - 1}{i\delta_{\text{eff}}}\right) \\ &= Le^{-i\gamma} + i\delta_{\text{eff}}\left[Le^{-i\gamma} - 1 - \sum_{n=1}^{N} a_n e^{in\pi} + \sum_{n=1}^{N} a_n e^{in\pi}\left(\frac{e^{i\delta} - 1}{i\delta_{\text{eff}}}\right)\right] \\ &= Le^{-i\gamma} - i\delta_{\text{eff}}\left[1 - Le^{-i\gamma} - \sum_{n=1}^{N} a_n e^{in\pi}\left(\frac{e^{i\delta} - 1}{i\delta_{\text{eff}}} - 1\right)\right]. \end{aligned} \tag{22}$$

By this series approximation and a suitable choice for the effective value of δ (δ_{eff}), the last term on the right can be made negligible during fading conditions ($L \ll 1$),* so that

$$v \approx Le^{-i\gamma}\left(1 - \frac{i\delta_{eff}e^{i\gamma}}{L}\right). \tag{23}$$

A "perfect" equalizer (at any given fixed frequency) would leave a residual distortion of

$$v' = 1 - \frac{i\delta_{eff}e^{i\gamma}}{L}. \tag{24}$$

The phase angle γ is zero at the bottom of the fade, is ± 45 degrees at 3 dB above the bottom, and varies over the range $-\pi/2 < \gamma < \pi/2$. In the region of good transmission ($\delta_{eff}/L \ll 1$) it is sufficiently accurate to approximate the net loss variations as

$$20 \log v' = 20 \log \left| 1 - \frac{i\delta_{eff}}{L} \right| = 10 \log \left[1 + \left(\frac{\delta_{eff}}{L}\right)^2\right]. \tag{25}$$

While both δ_{eff} and L are individually less than unity (and usually <0.1) their ratio can be greater than unity, and in that case the net loss variations calculated from (25) need to be interpreted as the median value of a distribution of net loss variations.

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* An associated assumption is that the maximum value of δ is restricted to

$$\delta_{max} = \frac{\Delta f}{f} N\pi < 1.$$