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Time-Varying Spectra and Linear Transformation

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One of the most important prerequisites for defining the spectrum of a nonstationary process is that the spectrum should transform simply and reasonably when the process is transformed linearly, and should lead to information about the important response statistics. Presented in this paper are some useful transform relationships for linear causal systems in terms of C. H. Page's time-varying spectra. Expressions suitable for direct analysis or numerical computation of the time-history of the response process and its bounds, the response power spectrum, the total energy of the system, and the upper bounds on the response shock spectra are given.

I. INTRODUCTION

R. M. Loynes[1] recently established a list of desirable properties for the spectrum of nonstationary processes. The elementary properties include: (*i*) a nonstationary spectrum should be rigorously defined, (*ii*) it should describe in some sense the energy distribution over frequency and time, and (*iii*) it reduces to the ordinary spectrum when the process is stationary (Loynes' properties A1, A2, and A5). Both Page's instantaneous power spectrum[2] and M. B. Priestley's evolutionary spectrum[3] satisfy these basic requirements. Another spectrum definition based on the notions of two-dimensional spectra arising in the consideration of harmonizable processes[4,5] also satisfy these basic requirements with some qualifications. However, from the practical point of view, especially when the filtering and convolution of a random process is involved the most important requirement is the existence of simple transform relationships for linear systems. A spectrum should transform simply and reasonably when the process is transformed linearly (property A3, Loynes). In other words, input-output relationships are required so that a knowledge of the spectrum of a process determines the spectrum of the transformed process. In addition, these relationships must be suitable for numerical computations. This is because most vibration data are available in the discrete form and the analytical solution of most nonstationary problems is difficult to obtain. Presented in this paper are some useful transform relationships for nonstationary processes using Page's time-varying spectra.

II. BACKGROUND

Consider a real function $x(t)$ in $(-\infty, \infty)$, which may be either a sample function of a random process, or a shock function which is zero outside the range $t \in [a, b]$ for finite a and b and is Riemann integrable in this same range. The function $x(t)$ is assumed to have finite energy. The running spectrum of $x(t)$ is defined as[2]

$$X(t, \omega) = \int_{-\infty}^{t} x(\tau) \exp(-i\omega\tau)\, d\tau, \tag{1}$$

where "i" is the imaginary unit. The instantaneous power spectrum is defined as

$$\begin{aligned} \rho_x(t, \omega) &= \frac{\partial}{\partial t} \mid X(t, \omega) \mid^2 \\ &= 2x(t) \operatorname{Re} [\exp(i\omega t) X(t, \omega)] \end{aligned} \tag{2}$$

which is even in ω. If $x(t)$ is a random process having a time-dependent autocorrelation function $R_x(t, \tau) = E[x(t)x(t - \tau)]$ where E denotes expectation, the instantaneous power spectrum of the process is understood to be the average of the spectra of all its sample functions. Let $S_x(t, \omega) = E[\rho_x(t, \omega)]$ and take the expectation of both sides of (2), we obtain

$$S_x(t, \omega) = 2 \int_0^{\infty} R_x(t, \tau) \cos \omega\tau \, d\tau. \tag{3}$$

The above relation shows that $S_x(t, \omega)$ is completely defined by $R_x(t, \tau)$. When $x(t)$ is stationary, (3) reduces to the ordinary relation as $R_x(t, \tau) = R_x(\tau)$ and $S_x(t, \omega) = S(\omega)$, both independent of t.

Some important properties of $X(t, \omega)$ and $\rho(t, \omega)$ as evident from their definitions are in order.

(*i*) $X(t, \omega)$ is Hermitian, i.e., $X(t, -\omega) = X^*(t, \omega)$. The symbol * denotes the complex conjugate.

(*ii*) Let $E(t)$ be the energy of $x(t)$ up to time t. The function $\rho_x(t, \omega)$ can be regarded as the energy density in the (t, ω) plane as seen by

$$E(t) = \int_{-\infty}^{t} | x(\tau) |^2 \, d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} | X(t, \omega) |^2 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \rho_x(\tau, \omega) \, d\omega \, d\tau. \tag{4}$$

(*iii*) Both $X(t, \omega)$ and $\rho_x(t, \omega)$ depend on the past history of $x(t)$, but not on the future.

(*iv*) The function $\rho_x(t, \omega)$ may take negative values (see Ref. 2, p. 106) but the integral $\int_{-\infty}^{t} \rho_x(\tau, \omega) \, d\tau$, the energy spectral density of frequency ω, is always positive.

(*v*) The time derivative of $X(t, \omega)$ relates to the signal $x(t)$ itself by

$$x(t) = \exp (i\omega t) \, \partial X(t, \omega)/\partial t$$

$$= \exp (-i\omega t) \, \partial X^*(t, \omega)/\partial t. \tag{5}$$

We will now proceed to derive from some input-output relations for a simple causal linear system in terms of $X(t, \omega)$ and $\rho_x(t, \omega)$.

III. INPUT-OUTPUT RELATIONS

Consider a simple, second-order causal linear system whose equation of motion is given by

$$\ddot{y} + 2\lambda\omega_o\dot{y} + \omega_o^2 y = x(t) \tag{6}$$

with homogeneous initial conditions $y(0) = \dot{y}(0) = 0$, where $\lambda < 1.0$ and ω_o are positive constants representing respectively the damping and natural frequency of the system. This system has a transfer function between the output y and input x given by $H(\omega) = (\omega_0^2 - \omega^2 + 2i\lambda\omega\omega_o)^{-1}$ and a corresponding impulse response $h(t) = [\exp (-\lambda\omega_o t) \sin pt]/p$ for $t > 0$, where $p = (1 - \lambda^2)^{\frac{1}{2}}\omega_o$. We obtain the following results for different response parameters.

3.1 *Time-Varying Spectra of Response Process*

Using the convolution integral and the casual property of the system, i.e., $h(t) = 0$ for $t < 0$, and assuming the excitation begins at $t = 0$, the response $y(t)$ in (6) is given in terms of $X(t, \omega)$ by

$$y(t) = \int_0^t x(\tau)h(t - \tau) \, d\tau$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)X(t, \omega) \exp (i\omega t) \, d\omega. \tag{7}$$

Following the definition of (1), the running spectrum of $y(t)$ is

$$Y(t, \omega) = X(t, \omega)H(\omega) - e_Y(t, \omega), \tag{8}$$

where

$$e_Y(t, \omega) = \int_0^t x(\tau_1) \exp(-i\omega\tau_1) \int_{t-\tau_1}^{\infty} h(\tau_2) \exp(-i\omega\tau_2)\, d\tau_2\, d\tau_1 \tag{9}$$

is the error involved in the approximation $Y(t, \omega) \simeq X(t, \omega)H(\omega)$. Notice from (8) and (9) that $Y(t, \omega)$ is given by the double integral in the $\tau_1 - \tau_2$ plane of a function $f(\tau_1, \tau_2) = x(\tau_1)h(\tau_2) \exp(-i\omega\tau_1 - i\omega\tau_2)$ over an area of an equal-sided triangle formed by the line $\tau_1 + \tau_2 = t$ in the first quadrant. The value of $X(t, \omega)H(\omega)$ is given by the same integral over an extended area from 0 to t along the τ_1 axis and from 0 to ∞ along the τ_2 axis. The error e_Y is given by the difference of these two integrals.

Let $F\{x_t\}$ be the Fourier transform of x_t, the part of $x(t)$ from t on, and $F\{y_t\}$ be the Fourier transform of y_t, the part of $y(t)$ from t on. It can be shown that the error $e_Y(t, \omega)$ in (9) is also given by

$$e_Y(t, \omega) = H(\omega)F\{x_t\} - F\{y_t\}. \tag{10}$$

It follows from (8) that the instantaneous power spectrum of $y(t)$ is

$$\begin{aligned} \rho_y(t, \omega) &= \left(\frac{\partial X}{\partial t} H - \frac{\partial e_Y}{\partial t}\right)(X^*H^* - e_Y^*) + (XH - e_Y)\left(\frac{\partial X^*}{\partial t} H^* - \frac{\partial e_Y^*}{\partial t}\right) \\ &= \rho_x(t, \omega) \mid H(\omega) \mid^2 + e_\rho(t, \omega), \end{aligned} \tag{11}$$

where

$$e_\rho(t, \omega) = \frac{\partial}{\partial t} \mid e_Y \mid^2 - He_Y^* \frac{\partial X}{\partial t} - H^*e_Y \frac{\partial X^*}{\partial t} \tag{12}$$

is the error involved in the approximation $\rho_y(t, \omega) \cong \rho_x(t, \omega) \mid H(\omega) \mid^2$.

Some interesting remarks can now be made about e_Y, and subsequently about e_ρ which is closely related to e_Y through (12). The energy bound in ω for $e_Y(t, \omega)$ can be considered as follows. Let the norm of an arbitrary function $Q(\omega)$ be defined as $\| Q \|_\omega = [\int_{-\infty}^{\infty} \mid Q(\omega) \mid^2 d\omega]^{\frac{1}{2}}$, then from Parseval's Theorem $\| F\{x_t\} \|_\omega = \| x_t \|_\tau = [\int_t^\infty \mid x(\tau) \mid^2 d\tau]^{\frac{1}{2}}$, and similarly $\| F\{y_t\} \|_\omega = \| y_t \|_\tau = [\int_t^\infty |y(\tau)|^2 d\tau]^{\frac{1}{2}}$. Suppose $h(t)$ is integrable so that $\mid H(\omega) \mid \leqq A$ is bounded, then it follows from (10) that

$$\| e_Y(t, \omega) \|_\omega \leqq A\left[\int_t^\infty \mid x(\tau) \mid^2 d\tau\right]^{\frac{1}{2}} + \left[\int_t^\infty \mid y(\tau) \mid^2 d\tau\right]^{\frac{1}{2}}$$

which $\rightarrow 0$ as $t \rightarrow \infty$. If furthermore $x(t)$ is integrable, from which $\mid F\{x_t\} \mid \leqq \int_t^\infty \mid x(\tau) \mid d\tau$ which $\rightarrow 0$ with t and consequently $\mid F\{y_t\} \mid$

also $\rightarrow 0$ with t, then (10) indicates $| e_Y(t, \omega) | \rightarrow 0$ in t, uniformly in ω. This result can also be concluded from (9) because in which the function $f(\tau_1, \tau_2) = x(\tau_1)h(\tau_2) \exp(-i\omega\tau_1 - i\omega\tau_2)$ is integrable in two dimensions.

For a more practical purpose, we now consider the familiar first- and second-order simple linear systems. The running spectrum of the response $y(t)$ and other pertinent quantities are given in Table I, in which $a^{-1} = \tau_o =$ decay time of the impulse response $h(t)$ of the first-order system and $\tau_o = (\lambda\omega_o)^{-1}$ decay time of $h(t)$ of the second-order system. From this table, it can be seen that for large t/τ_o ratio the error $e_Y(t, \omega)$ for both systems [given by the second term in the expression for $Y(t, \omega)$] will be small. In general at the same time t, the error for a system having short decay time (broad bandwidth) is smaller than that having long decay time (narrow bandwidth). Therefore $e_Y(t, \omega)$ for systems with a flat spectrum (high damping) is smaller compared with that for high resonant systems (low damping). This observation is also evident from the damping term appearing in the expression of $Y(t, \omega)$ for the second-order system in Table I.

In many cases the impulse responses and the input are both bounded functions defined on positive t axis such that $|h(t)| \leqq A_1 \exp(-a_1 t)$ and $|x(t)| \leqq A_2 \exp(-a_2 t)$ for some positive A_1, A_2, and $a_1 > a_2$. Then from (9) $|e_Y(t, \omega)| \leqq A_1 A_2[\exp(-a_2 t) - \exp(-a_1 t)]/a_1(a_1 - a_2)$. Therefore the error $|e_Y(t, \omega)|$ approaches zero as the time t increases; and for a certain time $t > t_o$ the error can be regarded as negligibly small.

From (8) another useful pair of equations relating to the time derivatives of the input and output running spectra can be obtained:

$$\left.\begin{aligned} \frac{\partial Y(t, \omega)}{\partial t} &= \frac{\partial X(t, \omega)}{\partial t} H(\omega) - \frac{\partial e_Y(t, \omega)}{\partial t} \\ \frac{\partial Y^*(t, \omega)}{\partial t} &= \frac{\partial X^*(t, \omega)}{\partial t} H^*(\omega) - \frac{\partial e_Y^*(t, \omega)}{\partial t} \end{aligned}\right\} \tag{13}$$

which in turn will lead to the following relationship between the input and output time histories.

3.2 *Response Time-History and Some Bounds*

Using (13) and the analogous relation for $y(t)$ as $x(t)$ in (5), we obtain

$$\begin{aligned} y(t) &= x(t)H(\omega) - \frac{\partial e_Y}{\partial t} e^{i\omega t} \\ &= x(t)H^*(\omega) - \frac{\partial e_Y^*}{\partial t} e^{-i\omega t} \end{aligned} \tag{14}$$

TABLE I—RESPONSE RUNNING SPECTRUM AND RUNNING FREQUENCY RESPONSE SPECTRUM

	First-order system	Second-order system†
$h(t),\ t > 0$	$a \exp(-at)$	$\frac{1}{p} \exp(-\lambda\omega_o t) \sin pt$
$H(\omega)$	$\frac{a}{a + i\omega}$	$\frac{1}{\omega_o^2 - \omega^2 + 2i\lambda\omega_o\omega}$
$H(t, \omega)$	$H(\omega)[1 - e^{-(a+i\omega)t}]$	$H(\omega)[1 - g(t)e^{-(\lambda\omega_o+i\omega)t}]$
$Y(t, \omega)$	$H(\omega)\left[X(t, \omega) - e^{-(a+i\omega)t} \int_0^t x(\tau)e^{-(a+2\omega)\tau}\, d\tau\right]$	$H(\omega)\left[X(t, \omega) - e^{-(\lambda\omega_o+i\omega)t} \int_0^t x(\tau)g(t - \tau)e^{-\lambda\omega_o\tau}\, d\tau\right]$

† $g(t) = \cos pt + \frac{\lambda\omega_0 + i\omega}{p} \sin pt.$

which relates the input and output time histories in terms of the transfer function of the system and the error function in (9). The time-history of the response function $y(t)$ can also be related to the time-history of the input function $x(t)$ through the simple expression

$$x(t)y(t) = \frac{1}{\pi} \int_0^\infty \text{Re}\,[H(\omega)]\rho_x(t, \omega)\, d\omega. \tag{15}$$

Notice that the above relation is advantageous when the direct multiplication of input and output functions is involved. We prove (15) in the following. Consider the integral

$$\int_0^t x(\tau)y(\tau)\, d\tau = \frac{1}{2\pi} \int_{-\infty}^\infty H(\omega) \int_0^t x(\tau)X(\tau, \omega) \exp(i\omega\tau)\, d\tau\, d\omega.$$

It follows from the causal property of the system and Parseval's theorem that

$$\int_0^t x(\tau)y(\tau)\, d\tau = \frac{1}{2\pi} \int_{-\infty}^\infty H(\omega) \mid X(t, \omega) \mid^2 d\omega.$$

Differentiating both sides with respect to t, (15) immediately follows.

We now establish the upper bound for $|y(t)|$ and $|y(t)|^2$ in terms of transfer characteristics of the system and input spectrum. From (7), Schwarz's inequality yields

$$\mid y(t) \mid^2 \leqq E(t)N, \tag{16}$$

$$\mid y(t) \mid \leqq E(t)^{\frac{1}{2}}N^{\frac{1}{2}}, \tag{17}$$

where $E(t)$ is the total energy of the input function $x(t)$ up to the time t and is given by (4), and $N = \int_{-\infty}^{\infty} \mid h(\tau) \mid^2 d\tau > 0$. Equations (16) and (17) are also valid if $x(t)$ is a random process. In this case the left-hand side quantities $\mid y(t) \mid^2$ and $\mid y(t) \mid$ are replaced by $E[\mid y(t) \mid^2]$ and $E[\mid y(t) \mid]$ respectively. For our simple system, $N = (4\lambda\omega_o^3)^{-1}$, and

$$E[\mid y(t) \mid^2] \leqq \frac{1}{8\pi\lambda\omega_o^3} \int_{-\infty}^t \int_{-\infty}^\infty S_x(t, \omega)\, d\omega\, d\tau. \tag{18}$$

3.3 *Total Energy of the System*

The total energy $E_s(t)$ of the system described by (6) at time t is given by

$$E_s(t) = \tfrac{1}{2} \mid Z(t, p) \mid^2 \exp(2\lambda\omega_o t), \tag{19}$$

where $Z(t, p)$ is the running spectrum of $z(t) = x(t) \exp(\lambda\omega_o t)$ at the frequency p. The relation (19) can be proved as follows. Taking, according to (1), the running Fourier transform of both sides of (6) and integrating by parts, we have

$$[\dot{y} + (i\omega + 2\lambda\omega_o)y] \exp(-i\omega t) + [\omega_o^2 + 2i\lambda\omega\omega_o - \omega^2]Y(t, \omega) = X(t, \omega)$$

in which the initial conditions have been used. The second term on the left-hand side of the above equation vanishes when setting $\omega = \theta_1 = i\lambda\omega_o + p$ or $\omega = \theta_2 = i\lambda\omega_o - p$. Thus

$$[\dot{y} + (i\theta_1 + 2\lambda\omega_o)y] \exp(-i\theta_1 t) = X(t, \theta_1) = Z(t, p), \tag{20}$$

$$[\dot{y} + (i\theta_2 + 2\lambda\omega_o)y] \exp(-i\theta_2 t) = X(t, \theta_2) = Z^*(t, p). \tag{21}$$

Multiplying (20) and (21) and dividing the result by 2, we have

$$\tfrac{1}{2}[\dot{y}^2 + 2\lambda\omega_o y\dot{y} + \omega_o^2 y^2] = \tfrac{1}{2} \mid Z(t, p) \mid^2 \exp(2\lambda\omega_o t).$$

But the left-hand side is the total energy of the system and therefore (19) is proved.

Taking expectation of both sides of (19), we obtain the expected energy of the system subjected to the random excitation $x(t)$

$$E[E_s(t)] = \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{t} R_x(\tau_1, \tau_1 - \tau_2) \cdot \exp[\lambda\omega_o(\tau_1 + \tau_2) - ip(\tau_1 - \tau_2)] \, d\tau_1 \, d\tau_2 . \tag{22}$$

For the undamped system, i.e., at the limit when $\lambda \to 0$, (19) reduces to $\frac{1}{2}(\dot{y}^2 + y^2) = \frac{1}{2} \mid X(t, \omega_o) \mid^2$ which agrees with the principle of energy conservation. It is also interesting to note that a simple manipulation of (20) and (21) can lead to the familiar convolution relation between input and output

$$y(t) = \frac{1}{p} \int_{-\infty}^{t} x(\tau) \exp[-\lambda\omega_o(t - \tau)] \sin p(t - \tau) \, d\tau$$

as given by (7).

3.4 *Bounds on the Shock Spectrum*

The shock spectrum of $x(t)$, $t \in [a, b]$ is the maximum absolute response defined as $S_d(\omega_o, \lambda) = \sup_t \mid y \mid$ = the displacement spectrum or $S_v(\omega_0, \lambda) = \sup_t \mid \dot{y} \mid$ = the velocity spectrum. We establish the following upper bounds to them:

$$S_d(\omega_o, \lambda) \leqq \sup_t \left[\frac{1}{p} \mid Z(t, p) \mid \exp(-\lambda\omega_o t)\right] \leqq \sup_t \frac{1}{p} \mid Z(t, p) \mid , \tag{23}$$

$$S_v(\omega_o, \lambda) \leqq \sup_t \left[\frac{1}{(1 - \lambda^2)^{\frac{1}{2}}} \mid Z(t, p) \mid \exp(-\lambda\omega_o t)\right] \leqq \sup_t \mid Z(t, p) \mid . \tag{24}$$

The proof of these relations is straightforward. Write $Z(t, \omega) = |Z(t, \omega)| \exp [-i\phi_z(t, \omega)]$ in which the phase angle $\phi_z(t, \omega) = \tan^{-1} [-S_z(t, \omega)/C_z(t, \omega)]$, S_z and C_z are the running sine and cosine transform of $z(t)$ respectively. It follows from (20) or (21) that

$$\dot{y} + \lambda\omega_o y + ipy = | Z(t, p) | \exp \{-\lambda\omega_o t + i[pt - \phi_z(t, p)]\}.$$

The above relation leads to

$$\dot{y} + \lambda\omega_o y = | Z(t, p) | \exp (-\lambda\omega_o t) \cos [pt - \phi_z(t, p)],$$
$$py = | Z(t, p) | \exp (-\lambda\omega_o t) \sin [pt - \phi_z(t, p)],$$

and (23) and (24) follow directly.

For the undamped system ($\lambda \to 0$), $p = \omega_0$, $Z(t, p) = X(t, \omega_0)$; therefore (23) and (24) reduce to

$$\omega_o S_d(\omega_o , 0) \leqq \sup_t | X(t, \omega_0) | \quad \text{and} \quad S_v(\omega_o , 0) \leqq \sup_t | X(t, \omega_o) |,$$

respectively.

IV. ILLUSTRATIONS

We now consider the following nonstationary processes for their time-varying spectra.

4.1 *Multiplicative Process*

$$x(t) = n(t)\phi(t),$$

where $n(t)$ is a stationary process with an autocorrelation function $R_n(\tau)$ and spectral density $S_n(\omega)$, and $\phi(t)$ is a causal deterministic function. The time dependent correlation function and power spectrum are

$$R_x(t, \tau) = R_n(\tau)\phi(t - \tau),$$

$$S_x(t, \omega) = 2\phi(t) \int_0^t R_n(\tau)\phi(t - \tau) \cos \omega\tau \, d\tau.$$

When $n(t)$ is a white noise with $R_n(\tau) = R_o\delta(\tau)$, $R_o > 0$, then $S_x(t, \omega) = S(t) = 2R_o\phi^2(t)$ becomes frequency independent. When $\phi(t)$ is slowly varying such that $\phi(t - \tau/2) \simeq \phi(t + \tau/2)$, then $R_x(t, \tau) = \phi^2(t)R_n(\tau)$ and $S_x(t, \omega) = \phi^2(t)S_n(\omega)$. This implies that $x(t)$ is locally stationary and its power spectrum changes with respect to time not in the general shape but in its area covered with the ω-axis only.

4.2 *Periodic Nonstationary Process*[6,7]

In this class of processes, the time dependent autocorrelation function is given as $R_x(t, \tau) = \Sigma\psi_k(\tau)\ exp\ (2\pi ikt/T)$ for a constant T. It can be readily shown that

$$S_x(t, \omega) = \frac{1}{2\pi} \sum_k \Psi_k(\omega) \exp (2\pi ikt/T),$$

where $\Psi_k(\omega)$ is the Fourier transform of $\Psi_k(\tau)$ and has the property that $\Psi_o(\omega) \geqq 0$ and $\Psi_k(\omega) = \Psi_k(2\pi k/T - \omega)$. Moreover, it is easy to show that the time-varying power spectral density of the response process $y(t)$ is also periodic with the same period T:

$$S_y(t, \omega) = \frac{1}{2\pi} \sum_k q_k(\omega) \exp (2\pi ikt/T)$$

in which $q_k(\omega) = \Psi_k(\omega)H(\omega)H(2\pi k/T - \omega)$.

The application of the rest of the input-output relations of Section III to these two types of nonstationary processes for either direct analysis or numerical computation is a straightforward exercise.

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