THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 50, No. 7, September, 1971
Printed in U.S.A.

Crosstalk in Outside Plant Cable Systems

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(Manuscript received February 4, 1971)

A transmission model of outside plant wire pair cable systems is given. The model includes direct crosstalk within cable units and has been adapted for computer manipulation in the form of a voice-frequency simulation program. The program can simulate the terminal behavior of any coterminous system composed of a cascade of mixed gauges of pulp and PIC cables, load coils, and bridged taps. There is evidence that the simulation capability can provide useful estimates of direct, within unit, FEXT at least up to the high kilohertz frequency region. Factory data on capacitance unbalance are the essential data source for the crosstalk portion of the simulation program.

For many systems studies a nominally specified cable should be represented stochastically. In this development a Brownian motion process is used to model the stochastic behavior of the capacitance unbalance. The diffusion constants for the various pair-to-pair combinations of capacitance unbalance can be expressed in terms of cable geometry. The development of this expression using capacitance unbalance data forms a basis for selecting pair twist lengths.

I. INTRODUCTION

A transmission model of outside plant wire pair cable systems which includes direct crosstalk is developed here. We begin by presenting a deterministic model of outside plant systems by means of matrix representations of the various elements. The perturbation technique for handling pair-to-pair coupling is developed in detail. An adaptation of the model in the form of a simulation program is discussed and comparisons between simulations and experiments are made. A stochastic treatment of pair-to-pair capacitive coupling is given which is shown to lead to a method of optimal twist length selection.

We stress that the theory we present here has its origin in the work of others (see, for example, Ref. 1 through 4). The ideas of direct crosstalk and of treating capacitance unbalance as random functions with a delta function covariance are familiar to those who have worked on crosstalk. We organize and extend available techniques in the direction of furnishing a suitable systems engineering model of crosstalk. The transformation of our model into a practical engineering tool extends the use of routine data on pair-to-pair capacitance unbalance that are taken at Western Electric's cable manufacturing locations on a daily basis.

II. OUTSIDE PLANT SYSTEMS—GENERAL

For simplicity we limit the discussion to an idealized version of the outside plant. We assume that a plant system is comprised of a cascade of passive n-port networks as depicted in Fig. 1.

These networks are primarily wire pair cables, although we include load coils, splices, and open circuited bridged taps. The objective is to be able to simulate the response of a broad class of outside plant systems to single-frequency excitations. We seek a programmed simulation capability that will enable the evaluation of crosstalk performance. The crosstalk performance (near end or far end) is usually expressed in the form of the cumulative distribution function of pair-to-pair crosstalk values. These cumulative distributions form the basis for answering a large class of systems engineering questions concerning crosstalk.

At this point in the development of this work only the stochastic nature of pair-to-pair capacitance unbalance is included. The effects of moisture, temperature, aging, externally generated noise, or manufacturing variability of other cable parameters are not considered. Active devices are also not included. For many problems involving active devices it is important to characterize the behavior of the passive portions of the plant that interconnect these devices. Furthermore, the technique we present for handling crosstalk appears to be extend-

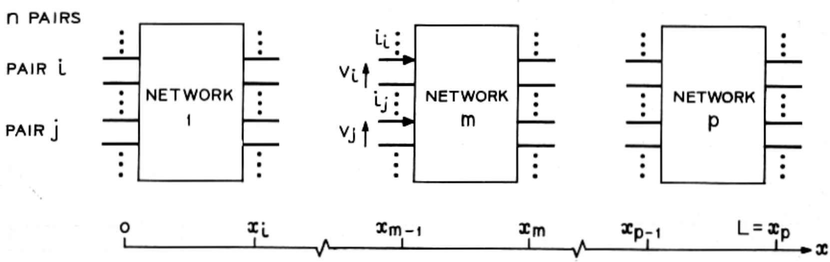

Fig. 1—A cascade of passive n-port networks.

ible to the problem of modeling variations in other cable parameters as will be indicated in the next section. However, the available data on the variability of other cable parameters is sparse and it is therefore premature to attempt to include these at this time in a numerical model.

The mathematical representation of the terminal behavior of these systems is generated from the solutions to the differential equations that characterize the physical laws that the constituent sybsystems obey, and the boundary relations that characterize the interaction between contiguous subsystems. In our model the system is characterized at each point in space (x) and time (t) by the vector of pair voltages $\mathbf{v}(x, t)$ and pair currents $\boldsymbol{\imath}(x, t)$. Thus longitudinal symmetry is implicitly assumed, that is, metallic mode propagation is not affected by any series or shunt primary constant unbalance. Each of the networks is described by a mathematically well behaved first-order homogeneous system of linear differential equations. We make the standard simplification of suppressing the time variable by restricting the excitations to be of the same frequency and in phase. The solution matrices to these differential equations with initial value equal to the identity matrix are ideally suited to cascaded structures. Following the terminology in 2-port network theory, we call this solution the $ABCD$ chain matrix of the subsystem where

$$\begin{bmatrix} \mathbf{v}(x_m) \\ \boldsymbol{\imath}(x_m) \end{bmatrix} = \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} \begin{bmatrix} \mathbf{v}(x_{m-1}) \\ \boldsymbol{\imath}(x_{m-1}) \end{bmatrix}.$$

It then follows that

$$\begin{bmatrix} \mathbf{v}(L) \\ \boldsymbol{\imath}(L) \end{bmatrix} = \prod_{k=1}^{p} \begin{bmatrix} A_{p+1-k} & B_{p+1-k} \\ C_{p+1-k} & D_{p+1-k} \end{bmatrix} \begin{bmatrix} \mathbf{v}(0) \\ \boldsymbol{\imath}(0) \end{bmatrix}$$

where the order of multiplication is important.

Generally when a system is specified there is some degree of uncertainty concerning its behavior. This uncertainty is modeled by assuming any chain matrix to be of the form $\mathsf{D} + \epsilon\mathsf{S}(\omega)$* where D is deterministic and $\mathsf{S}(\omega)$ is a matrix random process with zero mean. The parameter ϵ, which is called the perturbation coefficient, will be needed to facilitate our analysis. Since different system elements have different physical constants we unify the theory by treating these constants as parameters in the chain matrices. Thus the chain matrices are indexed over a set of parameter values and so they are stochastic processes.

* Here ω is used to indicate a random sample point not an angular frequency.

First we deal with the deterministic features of our systems. That is, we discuss features that are common to all realizations of these stochastic systems. Therefore, in the next section the fact that we are really dealing with stochastic subsystems is unimportant and is neglected.

III. THE PERTURBATION COEFFICIENT—ϵ

We express here the known fact that unwanted coupling between wire pairs in a cable system produces low-level interference on all pairs in proximity to an energized pair. The parameter $\epsilon > 0$ can be thought of as a scale to measure just how close a system is to an ideal one ($\epsilon = 0$) in which the pairs are uncoupled. We shall develop some simplifications that occur as ϵ tends to zero.

We begin by showing the central role of the perturbation coefficient in reducing the size of a cable crosstalk computation. An n-pair cable of length L is a linear passive bilateral $2n$-port network, and at any frequency it has a chain matrix ($ABCD$) characterization which we shall denote by $T(L)$

$$\begin{bmatrix} \mathbf{v}(L) \\ \boldsymbol{\imath}(L) \end{bmatrix} = T(L) \begin{bmatrix} \mathbf{v}(0) \\ \boldsymbol{\imath}(0) \end{bmatrix}, \tag{1}$$

where the v's and the $\imath$'s are the voltages and currents at the $2n$ ports of the network. We assume that $T(x)$ is defined by the matrix initial value problem

$$\frac{dT(x)}{dx} = -\begin{bmatrix} 0 & \mathsf{Z}(x) \\ \mathsf{Y}(x) & 0 \end{bmatrix} T(x) \quad \text{and} \quad T(0) = I, \tag{2}$$

where I is the identity matrix. The matrix $\mathsf{Z}(x)$ is a symmetrical $n \times n$ impedance matrix with entries that are continuous complex functions of the real variable x and whose diagonal terms are all equal and are denoted by z, which is a constant. Similarly for the admittance matrix $\mathsf{Y}(x)$. We write $\mathsf{Z}(x) = zI + \epsilon\boldsymbol{\zeta}(\mathbf{x})$ and $\mathsf{Y}(x) = yI + \epsilon\boldsymbol{\eta}(\mathbf{x})$ where $\boldsymbol{\zeta}(\mathbf{x})$ and $\boldsymbol{\eta}(\mathbf{x})$ have zero diagonals and z and y are constants with positive real and imaginary parts. Recall that ϵ is the perturbation coefficient.

We shall be interested in making crosstalk computations of the type where one pair in a cable is energized and the near-end voltages and far-end voltages on the other pairs are to be solved for. Using $T(L)$, the chain matrix characterization of the cable, and the boundary conditions, we obtain $2n$ equations with $2n$ unknowns. If there are 100 pairs in the cable we must invert a 200×200 complex matrix to solve for the crosstalk voltages. Practical considerations in inverting a matrix of this size places the problem out of reach of the computer. We shall develop an approximation whereby the computation is decomposed into the inversion of 99, 4×4 matrices. This enables the calculation to be handled quite easily by the computer.

For simplicity in what follows we shall assume that at both ends the cable pairs are terminated in their "characteristic impedance" $Z_0 = \sqrt{z/y}$. Also we shall normalize by assuming that the energized pair is impressed with a voltage generator of voltage V ($\approx$ 2 volts) and an internal impedance Z_0 so that one volt is established across the terminals of the energized pair. We shall follow the convention that the near-end crosstalk $NEXT(i, j) = [v_j(0)/v_i(0)]$ and the far-end crosstalk $FEXT(i, j) = [v_j(L)/v_i(L)]$ denote the ratios of the voltage on pair j due to the disturbing voltage on pair i at the cable terminals.

Using the boundary relations and substituting into equation (1) we can obtain the equations for $NEXT(i, j)$ and $FEXT(i, j)$.

Consider the hypothetical cable containing only two pairs whose defining equation is

$$\frac{d\hat{T}(x)}{dx} = -\begin{bmatrix} 0 & 0 & z & \epsilon\zeta_{ij}(x) \\ 0 & 0 & \epsilon\zeta_{ij}(x) & z \\ y & \epsilon\eta_{ij}(x) & 0 & 0 \\ \epsilon\eta_{ij}(x) & y & 0 & 0 \end{bmatrix} \hat{T}(x) \quad \text{and} \quad \hat{T}(0) = I.$$

If pair i is energized the equation for the crosstalk on pair j becomes

$$v_i(L)\begin{bmatrix} 1 \\ F\hat{E}XT(i, j) \\ 1/Z_0 \\ F\hat{E}XT(i, j)/Z_0 \end{bmatrix} = \hat{T}(L)\begin{bmatrix} 1 \\ N\hat{E}XT(i, j) \\ (V-1)/Z_0 \\ -N\hat{E}XT(i, j)/Z_0 \end{bmatrix}. \tag{3}$$

The hatted variables $N\hat{E}XT(i, j)$ and $F\hat{E}XT(i, j)$ are called direct crosstalks. The direct crosstalk which considers only the existence of pairs i and j in the cable and the crosstalk which considers the existence of all n-pairs in the cable are asymptotically equivalent as is shown in the appendix. That is,

$$\lim_{\epsilon \to 0} \frac{N\hat{E}XT(i, j)}{NEXT(i, j)} = 1.$$

There is strong experimental evidence that in the systems of interest ϵ is so small that for most practical purposes asymptotic equivalence can be interpreted as equality.

Next we decompose the problem of determining a 4×4 matrix T into a problem of determining two 2×2 matrices. To accomplish this we introduce some artificial voltages and currents $V_1(x)$, $V_2(x)$, $I_1(x)$, and $I_2(x)$, via the following transformation, which we shall call P

$$\begin{bmatrix} V_1(x) \\ V_2(x) \\ I_1(x) \\ I_2(x) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1(x) \\ v_2(x) \\ i_1(x) \\ i_2(x) \end{bmatrix}.$$

The above transformation maps the problem of two uniform coupled pairs to that of determining the chain matrices for two nonuniform but uncoupled pairs. The dependence of T on ϵ will be important now. So, since x is understood to be fixed at $L > 0$, we shall write $T(\epsilon)$ instead of $T(L)$. Letting $\tilde{T}(\epsilon)$ denote the chain matrix of the system of two uncoupled pairs we have $T(\epsilon) = P^{-1}\tilde{T}(\epsilon)P$. As a consequence $T(\epsilon)$ has the following form:

$$T(\epsilon) = \frac{1}{2}\begin{bmatrix} \mathsf{A}_1(\epsilon) + \mathsf{A}_2(\epsilon) & \mathsf{A}_1(\epsilon) - \mathsf{A}_2(\epsilon) & \mathsf{B}_1(\epsilon) + \mathsf{B}_2(\epsilon) & \mathsf{B}_1(\epsilon) - \mathsf{B}_2(\epsilon) \\ \mathsf{A}_1(\epsilon) - \mathsf{A}_2(\epsilon) & \mathsf{A}_1(\epsilon) + \mathsf{A}_2(\epsilon) & \mathsf{B}_1(\epsilon) - \mathsf{B}_2(\epsilon) & \mathsf{B}_1(\epsilon) + \mathsf{B}_2(\epsilon) \\ \mathsf{C}_1(\epsilon) + \mathsf{C}_2(\epsilon) & \mathsf{C}_1(\epsilon) - \mathsf{C}_2(\epsilon) & \mathsf{D}_1(\epsilon) + \mathsf{D}_2(\epsilon) & \mathsf{D}_1(\epsilon) - \mathsf{D}_2(\epsilon) \\ \mathsf{C}_1(\epsilon) - \mathsf{C}_2(\epsilon) & \mathsf{C}_1(\epsilon) + \mathsf{C}_2(\epsilon) & \mathsf{D}_1(\epsilon) - \mathsf{D}_2(\epsilon) & \mathsf{D}_1(\epsilon) + \mathsf{D}_2(\epsilon) \end{bmatrix},$$

where the matrices

$$\begin{bmatrix} \mathsf{A}_i(\epsilon) & \mathsf{B}_i(\epsilon) \\ \mathsf{C}_i(\epsilon) & \mathsf{D}_i(\epsilon) \end{bmatrix} \qquad i = 1, 2$$

are the $ABCD$ matrices for the nonuniform lines without coupling. Figures 2a and 2b portray the transformation in detail. So $\mathsf{A}_2(\epsilon) = \mathsf{A}_1(-\epsilon)$, $\mathsf{B}_2(\epsilon) = \mathsf{B}_1(-\epsilon)$, $\mathsf{C}_2(\epsilon) = \mathsf{C}_1(-\epsilon)$, and $\mathsf{D}_2(\epsilon) = \mathsf{D}_1(-\epsilon)$. Using the definition of derivative we obtain that for ϵ sufficiently small

$$T(\epsilon) \approx \begin{bmatrix} \mathsf{A}_1(0)I & \mathsf{B}_1(0)I \\ \mathsf{C}_1(0)I & \mathsf{D}_1(0)I \end{bmatrix} + \epsilon \begin{bmatrix} \mathsf{A}_1'(0)J & \mathsf{B}_1'(0)J \\ \mathsf{C}_1'(0)J & \mathsf{D}_1'(0)J \end{bmatrix},$$

where

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

This formula is particularly useful for generating the linear approximation to the $T(\epsilon)$ matrix for the case of constant coupling, that is, the case when $\eta(x)$ and $\zeta(x)$ are constant functions. The matrix is as follows, with $\Gamma = \sqrt{zy}$,

$$
\begin{bmatrix}
\cosh \Gamma L & -\epsilon \dfrac{(Z_0\eta + Z_0^{-1}\zeta)L}{2}\sinh \Gamma L & -Z_0 \sinh \Gamma L & \epsilon Z_0\left(\dfrac{(Z_0^{-1}\zeta + Z_0\eta)L}{2}\cosh \Gamma L - \dfrac{(\eta Z_0 - Z_0^{-1}\zeta)L}{2\Gamma L}\sinh \Gamma L\right) \\
-\epsilon \dfrac{(Z_0\eta + Z_0^{-1}\zeta)L}{2}\sinh \Gamma L & \cosh \Gamma L & \epsilon Z_0\left(\dfrac{(Z_0^{-1}\zeta + Z_0\eta)L}{2}\cosh \Gamma L - \dfrac{(\eta Z_0 - Z_0^{-1}\zeta)L}{2\Gamma L}\sinh \Gamma L\right) & -Z_0 \sinh \Gamma L \\
-\dfrac{1}{Z_0}\sinh \Gamma L & \epsilon \dfrac{1}{Z_0}\left(\dfrac{(Z_0\eta + Z_0^{-1}\zeta)L}{2}\cosh \Gamma L + \dfrac{(\eta Z_0 - Z_0^{-1}\zeta)L}{2\Gamma L}\sinh \Gamma L\right) & \cosh \Gamma L & -\epsilon \dfrac{(Z_0\eta + Z_0^{-1}\zeta)L}{2}\sinh \Gamma L \\
\epsilon \dfrac{1}{Z_0}\left(\dfrac{(Z_0\eta + Z_0^{-1}\zeta)L}{2}\cosh \Gamma L + \dfrac{(\eta Z_0 - Z_0^{-1}\zeta)L}{2\Gamma L}\sinh \Gamma L\right) & -\dfrac{1}{Z_0}\sinh \Gamma L & -\epsilon \dfrac{(Z_0\eta + Z_0^{-1}\zeta)L}{2}\sinh \Gamma L & \cosh \Gamma L
\end{bmatrix}. \tag{4}
$$

For the case when $\eta(x)$ and $\zeta(x)$ are arbitrary continuous functions we obtain the following linear approximation to $T(\epsilon)$

$$\begin{bmatrix} \cosh \Gamma L & 0 & -Z_0 \sinh \Gamma L & 0 \\ 0 & \cosh \Gamma L & 0 & -Z_0 \sinh \Gamma L \\ -\dfrac{\sinh \Gamma L}{Z_0} & 0 & \cosh \Gamma L & 0 \\ 0 & -\dfrac{\sinh \Gamma L}{Z_0} & 0 & \cosh \Gamma L \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & -\epsilon \displaystyle\int_0^L \frac{(\eta(x)Z_0 - \zeta(x)Z_0^{-1})}{2} \sinh 2\Gamma x \, dx & 0 & \epsilon Z_0 \displaystyle\int_0^L (Z_0\eta(x) \sinh^2 \Gamma x - Z_0^{-1}\zeta(x) \cosh^2 \Gamma x) \, dx \\ -\epsilon \displaystyle\int_0^L \frac{(\eta(x)Z_0 - \zeta(x)Z_0^{-1})}{2} \sinh 2\Gamma x \, dx & 1 & \epsilon Z_0 \displaystyle\int_0^L (Z_0\eta(x) \sinh^2 \Gamma x - Z_0^{-1}\zeta(x) \cosh^2 \Gamma x) \, dx & 0 \\ 0 & -\dfrac{\epsilon}{Z_0} \displaystyle\int_0^L (Z_0\eta(x) \cosh^2 \Gamma x - Z_0^{-1}\zeta(x) \sinh^2 \Gamma x) \, dx & 1 & \epsilon \displaystyle\int_0^L \frac{(\eta(x)Z_0 - \zeta(x)Z_0^{-1})}{2} \sinh 2\Gamma x \, dx \\ -\dfrac{\epsilon}{Z_0} \displaystyle\int_0^L (Z_0\eta(x) \cosh^2 \Gamma x - Z_0^{-1}\zeta(x) \sinh^2 \Gamma x) \, dx & 0 & \epsilon \displaystyle\int_0^L \frac{(\eta(x)Z_0 - \zeta(x)Z_0^{-1})}{2} \sinh 2\Gamma x \, dx & 1 \end{bmatrix}. \tag{5}$$

Substituting this evaluation of $T(\epsilon)$ into equation (3) and solving for the linear approximations to the crosstalks yields the classical formula

$$N\hat{E}XT(i,j) \approx -\epsilon \int_0^L \frac{(Z_0\eta(x) - Z_0^{-1}\zeta(x))}{2} e^{-2\Gamma x}\, dx$$

$$F\hat{E}XT(i,j) \approx -\epsilon \int_0^L \frac{(Z_0\eta(x) + Z_0^{-1}\zeta(x))}{2}\, dx.$$

In this section we have achieved a canonical representation of coupled pairs in a cable. Our representation is complete enough to permit pair-to-pair crosstalk calculations to be made, but our representations are approximate. We have not developed any analytical relationships between the magnitude of errors involved and ϵ. Pragmatically we have not had any need for such relationships as we have been able to rely on experimental verification of our assumptions.

The structure of the cable chain matrix that we have developed expedites the determination of the chain matrix experimentally using

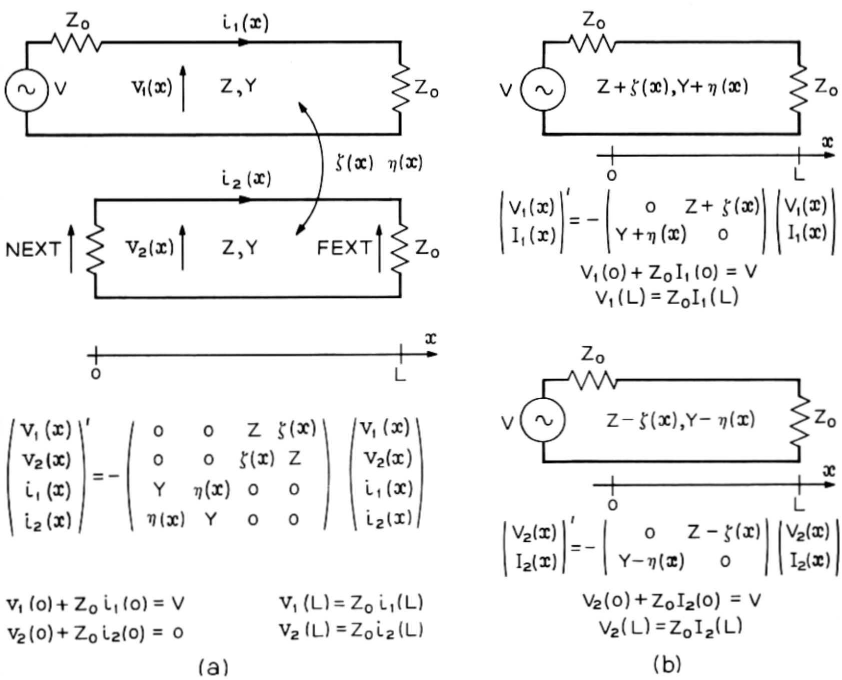

Fig. 2a—Two identical coupled transmission lines and their characterizing equations.
Fig. 2b—Two nonuniform uncoupled transmission lines and their characterizing equations.

a generalized open circuit–short circuit technique. Specifically each of the A, B, C, and D matrices has the form of a perturbation of some nonzero multiple of the identity. A straightforward generalization of the classical open circuit–short circuit technique, capitalizing on the approximation that can be made in inverting a matrix of the aforementioned form, makes for a simple determination of the A, B, C, and D matrices. Since the experimental technique is straightforward we shall not discuss it further.

In this section we have discussed crosstalk computations for a single cable. The extension to a cascade of cables is straightforward. We need only modify (3) by replacing $T(L)$ by the ordered product $\prod_{k=1}^{p} T_{p-k+1}$ and Z_0 can be replaced by possibly distinct impedances at $x = 0$ and $x = L$. In the Appendix we have proven asymptotic equivalence in a special case of a single cable. (We spare the reader the general proof for cable systems.) We shall use asymptotic equivalence of crosstalk and direct crosstalk for cable systems in the sections that follow. We note that if bridged taps are excluded the extension of the asymptotic equivalence to cable systems is immediate.

IV. SIMULATING THE PLANT AT VOICE FREQUENCIES

The first phase of the computer simulation of the outside plant has been limited to voice frequencies. The program generates the subsystem chain matrices and multiplies them together to get a realization of the $ABCD$ matrix for the system. The program can simulate an extensive class of coterminous systems; specifically, the loop can be comprised of any cascade of PIC cables, pulp cables, load coils, and bridge taps. Thus, given the termination impedances and excitations we can solve for any of the terminal voltages and currents. The important feature of the program is that it provides the capability to make crosstalk computations.

There were a number of reasons why we began the simulation effort with the voice plant. First the very vastness of the overall project of outside plant cable simulation dictates that one should begin by handling a somewhat self-contained portion of the project that could form the basis for future effort. The simplification that arises in dealing with voice plant makes it an attractive starting place as we now show.

Ostensibly the basis for the simulation of individual cables is equation (5). The chain matrices for load coils and bridge taps can be obtained from the matrix for the cable. We note that the entries of the

ABCD matrix depicted in equation (5), that involve pair-to-pair unbalances have the following form

$$\int_0^L [H_1(f, x)Z_0\eta(x) + H_2(f, x)Z_0^{-1}\zeta(x)]\, dx,$$

where the H_i are continuous complex hyperbolic functions. For any common set of cable parameters the $\zeta(x)$ term is negligible at voice frequencies ($f < 4000$ Hz). To help see why inductance unbalance can be neglected at voice frequencies we first note that for any of the cable pairs simulated, the absolute value of the characteristic impedance decreases by an order of magnitude in changing from its dc value to its high-frequency asymptote. Then by looking at the manner in which Z_0 appears in (5), or (4), we see that the capacitive coupling effect is accentuated while the inductive coupling effect is de-emphasized at voice frequencies. Hence in the voice band we need only consider

$$\int_0^L H_1(f, x)\eta(x)\, dx.$$

Replace the continuous function $H_1(f, x)$ by its average value $L^{-1} \int_0^L H_1(f, x)\, dx$ in the integral. By continuity we shall lose negligible accuracy so long as L is sufficiently small. For values of f in the voice band the actual cable section lengths between splices are sufficiently small. Hence the integral becomes

$$\int_0^L \left[\frac{1}{L}\int_0^L H_1(f, y)\, dy\right]\eta(x)\, dx = \left[\frac{1}{L}\int_0^L \eta(x)\, dx\right]\int_0^L H_1(f, y)\, dy.$$

So in effect we have replaced $\eta(x)$ by its average value. That is to say we can treat $\eta(x)$ as if it were a constant function. So equation (4) with $\zeta(x) = 0$ becomes the basic representation of cables in the voice band.

The function $\eta(x)$ has the form

$$\eta(x) = \sqrt{-1}\, 2\pi f c(x),$$

where $c(x)$ is the distributed capacitance unbalance. In an idealized situation where only two pairs are present, let $1'$, $1''$ and $2'$, $2''$ denote the wires in the first and second pairs, respectively; then we have from Ref. 1,

$$c(x) \approx \tfrac{1}{4}[c_{1'2'}(x) - c_{2'1''}(x) + c_{1''2''}(x) - c_{2''1'}(x)].$$

At voice frequencies the numerical values of a cable chain matrix can be obtained if the values of Γ, Z_0, and $\int_0^L c(x)\, dx$ are known. For quality assurance, Western Electric undertakes a bridge measurement of $| \int_0^L c(x)\, dx |$ for various pair combinations in the cables that it manufactures. Thus we have the data necessary to compute these chain matrices at voice frequencies.

Another important reason for beginning with the simulation of voice plant is that there are several systems engineering studies for which voice plant crosstalk data is an essential ingredient. For example, the comparison of unigauge plant with existing plant, determining gain limitations for station sets, and evaluating the performance of PBX trunk circuits are some of the topics of interest where crosstalk performance, particularly near-end crosstalk performance, is important.

To test the program two 3000-foot sections of cable and a bank of load coils were measured for unbalance of capacitance and inductance respectively. The cables were spliced together first with and then without the load coils at the splice. The near-end crosstalk distributions were measured in both cases. The unbalance data along with the nominal values of primary constants for the cables and coils were fed to the program. Figures 3 and 4 depict the success of the simulation.

In most systems engineering studies, a cable's behavior is evaluated

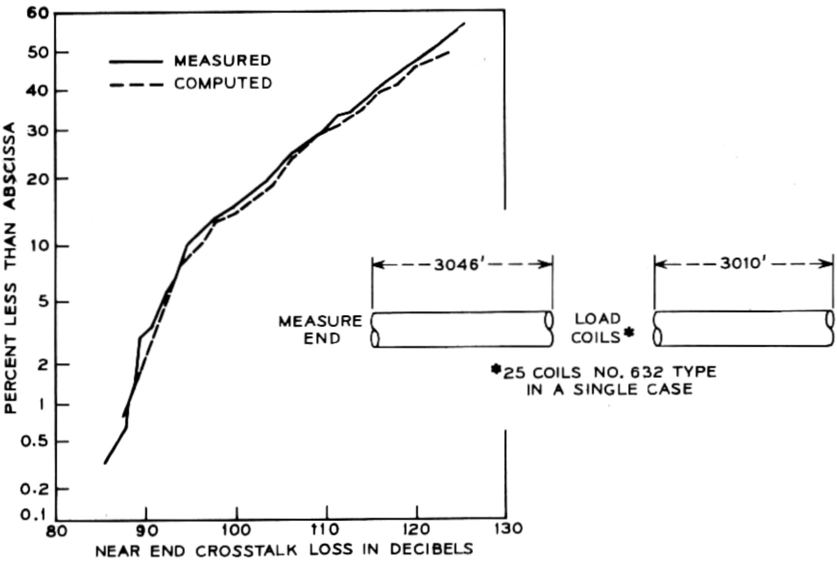

Fig. 3—Comparison of measured and computed distributions of 1-kHz within unit (25 pair) near-end crosstalk on 6065 feet of 26-gauge polyethylene cable (loaded, 600 Ω termination). (On normal probability paper.)

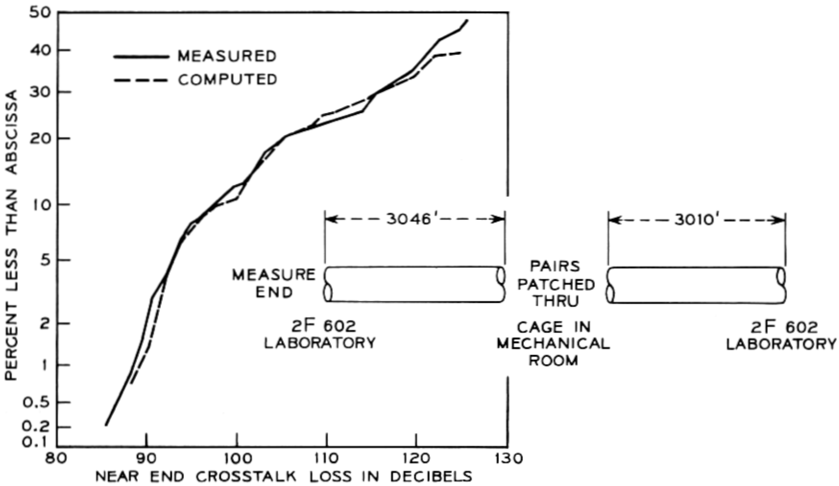

Fig. 4—Comparison of measured and computed distributions of 1-kHz within unit (25 pair) near-end crosstalk on 6065 feet of 26-gauge polyethylene cable (non-loaded, 600 Ω termination). (On normal probability paper.)

from a combination of nominally specified transmission constants and empirically derived crosstalk estimates. We are striving to utilize more of a cable's inherent communication capacity and so a more detailed description of cable crosstalk behavior is needed. By modeling the stochastic nature of cable crosstalk we get a pertinent refinement of the crosstalk evaluation capability. In this section we indicate the reason why crosstalk should be treated stochastically. We give a heuristic technique for simulating this stochastic behavior at voice frequencies.

For convenience we order the array of pair-to-pair capacitance unbalances $\{\int_0^L c_{ij}(x)\,dx;\, i = 2, 3, \cdots, n,\, i < j\}$ to form a vector $\mathbf{C}(L)$. We have limited $i < j$ since $c_{ij}(x) = c_{ji}(x)$. Ample Western Electric quality assurance data on capacitance unbalance are on file. Data are available on all types of cables from all Western cable manufacturing plants going back many years.

To provide a voice plant crosstalk simulation capability we need data on $\mathbf{C}(L)$ for all the kinds of cable involved in the outside plant. The information concerning a cable that is usually available to the systems engineer is the pair size, gauge, mutual capacitance, insulation, and length. A review of quality assurance capacitance unbalance data shows that the values of $\mathbf{C}(L)$ for cables meeting the same nominal specification are not identical. Thus we see the need to treat $\mathbf{C}(L)$ as a stochastic process $\mathbf{C}(L, \omega)$ where ω indexes the sample space Ω.

Next we discuss how representative quality assurance data were selected for simulation purposes.

The vector $\mathbf{C}(L, \omega)$ has signs associated with its components as will be explained in the next section. The quality assurance samples of $\mathbf{C}(L, \omega)$ are unsigned. Some special measurements on signed values of $\mathbf{C}(L, \omega)$ were made and it was concluded that $\mathbf{C}(L, \omega)$ could be assumed to have mean $\mathbf{0}$. For simulation purposes the quality assurance data were randomly signed with equal probability of being positive or negative. (Thus we are implicitly assuming that the components of $\mathbf{C}(L, \omega)$ are symmetrically distributed. This assumption is consistent with the detailed model of capacitance unbalance in the sequel. Of course the accuracy of the simulation ultimately tests our assumptions.)

Another shortcoming of the quality assurance data is that for cable units in excess of 25 pairs a biased sampling of the capacitance unbalance values is taken. Only certain pair combinations that are prone to high capacitance unbalance are measured. So the data needed for simulation are incomplete. Techniques can be developed for determining the parameters of the true distributions from the biased samples. Because of the difficulty involved with such techniques it was decided to approximate units of 50 and 100 pairs as combinations of 25-pair units. Because of this approximation the pair combinations prone to high crosstalk loss are not correctly modeled. This is of no consequence since it is the distribution of the values of low crosstalk loss that we seek. The degree of success of a simulation of one of the field experiments on in-place cables in Scranton, Pennsylvania, is depicted by Fig. 5. Finally, to use the data, the individual capacitance values are adjusted for length according to a square-root correction factor. The square-root correction factor is well known; it has been verified experimentally and will be discussed further in the next section.

For each type of cable several capacitance unbalance matrices are necessary since several sections of a single type of cable may be used in one system. Capacitance unbalance data on 19-, 22-, 24-, and 26-gauge pulp insulated cable and 22-, 24-, and 26-gauge PIC cable were obtained. The data spanned a 1-year period at the Western Electric Plant in Hawthorne, Illinois, and at least 25 reels of each type of cable were represented in the data. The annual interval is particularly significant for pulp data since the capacitance unbalance is sensitive to the humidity at the time of manufacture. An analysis of the data substantiated the view that the pulp capacitance unbalance is not a function of gauge.

To use a square-root length correction on the data means that, given

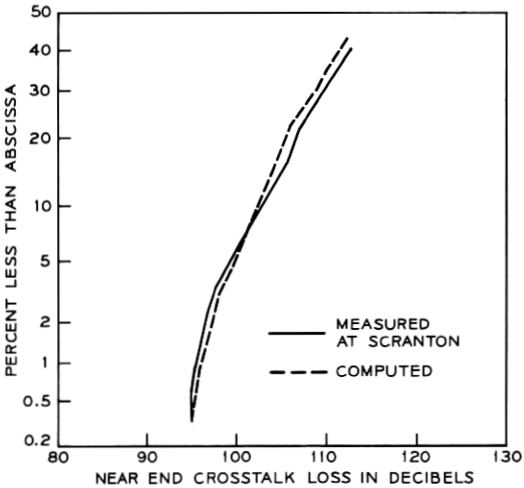

Fig. 5—Comparison of measured and computed 1-kHz within unit (50 pair) near-end crosstalk on 2800 feet of 22-gauge pulp cable (600 Ω termination). Two 19-gauge, 25-pair pulp units were used in the computer simulation of the 50-pair unit. (On normal probability paper.)

a sample $\mathbf{C}(L, \omega)$, to get a sample of $\mathbf{C}(l, \omega)$ we multiply $\mathbf{C}(L, \omega)$ by $\sqrt{l/L}$. As we mentioned, the data was randomly signed and a pulp, PIC distinction was maintained while a gauge distinction within pulp was not acknowledged. The problem remained of providing separate sequences of capacitance tables for pulp and for PIC cables. The problem was resolved differently for pulp than for PIC. In both cases a distribution, called a universal distribution, was constructed by pooling accumulated data from over 25 cables. For pulp a typical cable was modeled by using a set of 300 pair-to-pair capacitance unbalance values whose distribution matched the universal distribution. Numerous shuffled copies of the list of 300 values were prepared to enable modeling of randomly spliced pulp sections. Since PIC is spliced color to color the procedure used to represent pulp was not valid for PIC. For each PIC gauge the tables were sequenced to give the most representative distributions priority in selecting tables for a simulation study. In comparing the individual distributions to the universal distribution particular emphasis was given to the fit of the high capacitance tail. The simulation scheme provides for placing the most representative distributions at the near end in order to assure the most representative estimates of near-end crosstalk.

The formulas used for $NEXT$ and $FEXT$ were obtained as follows. Let $\{T_i\}$ $i = 1, 2, \cdots, k$ represent the chain matrices of k subsystems ordered so that T_1 is at what we shall arbitrarily call the near end and T_k is at the far end. Thus the terminal behavior of the system is described by $T = T_k T_{k-1} \cdots T_1$. Let $\mathbf{n} = (\eta_1, \eta_2, \cdots, \eta_k)^t$ be a vector such that η_i is the coupling in the ith subsystem. Recall that the A, B, C, and D submatrices of T_i contain η_i in the off-diagonal positions only. Neglecting quadratic terms and higher in the η_i variables we obtain

$$T \approx \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_2 & a_1 & b_2 & b_1 \\ c_1 & c_2 & d_1 & d_2 \\ c_2 & c_2 & d_2 & d_1 \end{bmatrix} \triangleq$$

$$\prod_{i=1}^{k} \begin{bmatrix} a_1^{(i)} I & b_1^{(i)} I \\ c_1^{(i)} I & d_1^{(i)} I \end{bmatrix}$$

$$+ \sum_{i=1}^{k} \eta_i \left(\prod_{m=1}^{i-1} \begin{bmatrix} a_1^{(m)} I & b_1^{(m)} I \\ c_1^{(m)} I & d_1^{(m)} I \end{bmatrix} \right) \begin{bmatrix} a_2^{(i)} J & b_2^{(i)} J \\ c_2^{(i)} J & d_2^{(i)} J \end{bmatrix} \left(\prod_{m=i+1}^{k} \begin{bmatrix} a_1^{(m)} I & b_1^{(m)} I \\ c_1^{(m)} I & d_1^{(m)} I \end{bmatrix} \right).$$

Let us assume that the near end is terminated in an impedance z_S and that the far end is terminated in an impedance z_R. Using the boundary conditions we obtain

$$v_1(L) \begin{bmatrix} 1 \\ FEXT \\ \dfrac{1}{z_R} \\ \dfrac{FEXT}{z_R} \end{bmatrix} = T \begin{bmatrix} 1 \\ NEXT \\ \dfrac{V-1}{z_S} \\ \dfrac{-NEXT}{z_S} \end{bmatrix}.$$

Solving and neglecting quadratic terms in the variable with a subscript 2 we obtain

$$NEXT \approx \left[\frac{b_2 - z_R d_2}{b_1 - z_R d_1} - \frac{z_S a_2 - b_2 - c_2 z_S z_R + d_2 z_R}{z_S a_1 - b_1 - c_1 z_S z_R + d_1 z_R} \right]$$

and

$$FEXT \approx \left[(a_1 d_2 - a_2 d_1 - b_2 c_1 - b_1 c_2) - \frac{z_S a_2 - b_2 - c_2 z_S z_R + d_2 z_R}{z_S a_1 - b_1 - c_1 z_S z_R + d_1 z_R} \right].$$

If the system does not contain bridged taps then the parenthetic term in the expression for $FEXT$ can be neglected. Since we have neglected higher order terms both $NEXT$ and $FEXT$ are linear in $\mathbf{n}$. We are led to introduce $\mathbf{N}$ and $\mathbf{F}$ such that

$$NEXT = \langle \mathbf{n}, \mathbf{N} \rangle \quad \text{and}$$

$$FEXT = \langle \mathbf{n}, \mathbf{F} \rangle .$$

In our simulation studies the system description yields the information necessary to determine $\mathbf{N}$ and $\mathbf{F}$ and the sample space of $\mathbf{n}$. We randomly select enough $\mathbf{n}$ vectors and form the scalar products to determine the statistics of $NEXT$ and $FEXT$.

V. THE STOCHASTIC NATURE OF A CABLE AND OPTIMAL TWIST LENGTH SELECTION

In the previous section it was mentioned that the nominal specification of a cable leaves its behavior somewhat uncertain. It followed that the class of cables that meet a nominal specification has different chain matrix representations. Here we present a model for this stochastic behavior of the chain matrices. Specifically we deal only with the coupling terms in these matrices and the other terms are assumed to be constants.*

A Gaussian process $X(t, \omega)$ with zero mean and covariance $k^2 \times \min(t_1, t_2)$ is called a Brownian Motion or Wiener Process. If $X(0, \omega) = 0$ with probability one the process is said to be centered.

The process was first used to model the motion of a particle in a fluid where $X(t, \omega)$ is one of the displacement coordinates after time t and ω indexes over all particles. The number k is called the diffusion constant. This constant is a measure of the tendency of a particle to stray from its initial position.

In what follows we shall concentrate on capacitance but analogous statements are to be applied to inductance as well. Let $c_{ij}(x, \omega)$ represent the capacitance coupling function between pair i and pair j. Let $C_{ij}(x, \omega)$ represent the accumulated capacitance, that is, let

$$C_{ij}(x, \omega) = \int_0^x c_{ij}(\xi, \omega)\, d\xi .$$

* The model we present is intended for standard PIC and pulp cable designs and should not be applied to highly precision engineered cables wherein determinism would play a more significant role than we allow here.

Here ω indexes over all realizations of a cable of a particular type. We shall model the accumulated capacitance unbalance $C_{ij}(x, \omega)$ by treating it as a sample function of a Brownian Motion process. This is an approximation that was reached after considering the manufacturing process and after reviewing quality assurance records on accumulated capacitance unbalance data. The model is in agreement with the empirically established correction of a 10-dB translation of $FEXT$ distributions for a tenfold increase in cable length.

In our application, distance plays the role of time. Therefore, the coupling terms in the cable matrix can be viewed as stochastic integrals of the form $\int_0^L H(x)\, dC(x, \omega)$. When $\int_0^L c(x, \omega)\, dx$ is a Brownian Motion process the stochastic integral* is called a linearly conditioned Brownian Motion process. The matrix of stochastic integrals is a multivariate, centered, linearly conditioned Brownian Motion process.

The capacitance unbalance function $c_{ij}(x, \omega)$ represents an effective distributed capacitance. It is a mathematically simple idealization which attempts to include the effects of all the capacitive coupling between pair i and pair j. The strongest contribution to this coupling function is believed to arise from the capacitances between each wire of pair i and each wire of pair j. The net effect of these four capacitances is expressed as a difference of nearly equal quantities. This difference can be positive or negative at x depending on the geometric relationship of the pairs at x. The pairs are twisted with different twist lengths in an attempt to cause the coupling capacitance to be positive as much as it is negative.

Presently available data is in the form of samples of $| \int_0^L c(x, \omega)\, dx |$ where only one value of L is associated with each ω. This provides no information pertaining to the covariance of $c(x, \omega)$ and so we make an assumption. Pair twist lengths are short (a few inches long) in comparison with the cable lengths of interest (thousands of feet). Thus in the light of our previous statements, it is reasonable to suggest that $c(x, \omega)$ be modeled as having a delta function covariance and this assumption is consistent with the Brownian motion model. From equation (5) we see that the coupling interacts with hyperbolic functions of Γx so that the assumption of a delta function covariance is most reasonable at voice frequencies where for the cables of interest the wavelengths are of the order of at least tens of miles.

There are many reasons to suspect that the capacitance unbalance function is an extremely irregular function when considered over thou-

* For a treatment of Brownian Motion and the stochastic integral concept see Ref. 5.

sands of feet. There are random disturbances that can change the function drastically over inches. For example, the vibrations of the cable manufacturing machinery, impurities or eccentricities in the insulation, tertiary coupling paths, slipping of the twist, etc. Furthermore the twist lengths are not precisely maintained. Thus, for cable lengths of interest (a thousand feet or so), it is quite reasonable to treat $\int_0^x c_{ij}(\xi, \omega)\, d\xi$ stochastically.

We are assuming that $\operatorname{Var} C_{ij}(x, \omega) = \mathrm{k}_{ij}^2 x$, where k_{ij} is a constant and x is the cable length. If we let $C(x, \omega)$ denote the random variable whose sample space is all pairs in cables of a particular type then $\operatorname{Var} C(x, \omega)$ is simply the average of the individual variances, that is,

$$\operatorname{Var} C(x, \omega) = \frac{1}{(n^2 - n)/2} \sum_{i<j} EC_{ij}^2 = \frac{L}{(n^2 - n)/2} \sum_{i<j} \mathrm{k}_{ij}^2 .$$

So by pooling data on accumulated capacitance for all the different i, j combinations the data should still exhibit a variance which is linear with length.

Following the work reported here W. N. Bell has explored the accuracy and effectiveness of this Brownian motion model utilizing factory data. His conclusions (unpublished) are positive. Extensive data at two different lengths verified the linear dependence of $\operatorname{Var} C(x, \omega)$ with length [$\operatorname{Var} C(0, \omega) = 0$ at the third point]. Thus the use of the $\sqrt{L}$ correction of the data that was mentioned previously is indicated. The $C(x, \omega)$ are shown to be normal in the tails while in the neighborhood of the mean (0) the density is lower than normal and this is compensated for by an excess at zero. The departure from normality may be due to the measurement technique but the departure is unimportant since tail behavior is what concerns us. Furthermore if there really is an excess of zero capitances unbalances then the Brownian motion model is a slightly pessimistic one and hence useful in bounding system performance.

The diffusion constant k_{ij} appears to be an excellent way to express the capacitance unbalance of the pair combination. Furthermore, $\text{maximum}_{i,j}\, \mathrm{k}_{ij}$ is a good measure of the capacitance unbalance of a cable. A statistical analysis of the quality assurance data reveals the relationship between k_{ij} and the twist lengths of pair i and pair j (t_i and t_j). The uncovering of this relationship provides a basis for optimal twist length selection. To demonstrate this idea W. N. Bell used quality assurance data on 145 25-pair PIC cable units to estimate the diffusion constants for the neighboring pair combinations. As suspected, the data revealed a strong relationship between the diffusion constants and cable geometry. An extensive statistical analysis of this relationship was undertaken by A. K. Jain who determined that the formula

$$k_{ij}^2 L = \left[0.090 \times \frac{t_i^{0.62} t_j^{0.62}}{| t_i - t_j |^{0.23}} \right]^2 L \qquad (t_i \neq t_j)$$

(k_{ij} has dimensions of $pf/(ft)^{1/2}$.)

explained 79 percent of the total variation in the data. Besides establishing the relationship between the cable geometry and crosstalk performance, statistical analysis also provides an alternate means for simulating capacitance unbalance distributions for standard PIC and pulp cable designs.

As in the motion of particles in a liquid the mathematical model of Brownian motion is an approximation of the physical situation that must be applied discriminately. We stress this point since the continuity of $c_{ij}(x, \omega)$ was used to solve the differential equations that lead to the chain matrix while the mathematical model of Brownian motion does not account for this continuity. Thus while $c_{ij}(x, \omega)$ is sufficiently irregular so as to model its integral as Brownian Motion, it is smooth enough to be called continuous when expressing the physical laws that a cable must obey.

The inductance unbalance process ($\int_0^L l_{ij}(x, \omega)\, dx$) has not been studied. A small appreciation for how the inductance unbalance relates to the capacitance unbalance enables us to speculate that it is also a Brownian Motion process that is correlated with the capacitance process. Since both forms of unbalance derive from the cable geometry, it is reasonable to attempt to relate these unbalances. In our simulation program we neglect inductance unbalance since this neglect causes negligible error at voice frequencies. The frequency at which inductance unbalance causes an appreciable contribution to crosstalk remains to be determined.

A computer simulation of Scranton, Pennsylvania, field measurements of far-end crosstalk was made. In this case the distribution tails were nearly reproduced up to 775 kHz. At the higher frequencies it appears that the inductance unbalance is the cause of the departure between the measured and computed tails (see Fig. 6). Another simulation was made of three sets of factory far-end crosstalk measurements (3.15 MHz). In each case the computed tails were about 5 dB different from the measured tails due to inductance unbalance (see Fig. 7 where one of these cases is depicted). Figures 6 and 7 do indicate a significant correlation between 1 kHz capacitance unbalance and the crosstalk at

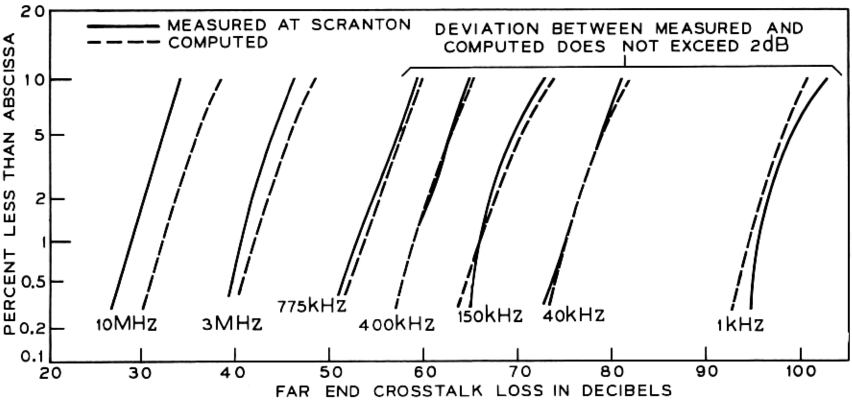

Fig. 6—Comparison of measured and computed distribution tails of within unit (50 pair) far-end crosstalk. The system is 2702 feet long, it has 4 splices, and is composed of 22-gauge pulp cable. The measured tails are based on measurements of in-place cable in Scranton, Pennsylvania. The computed tails are based on a random selection of 1-kHz factory capacitance unbalance data. (On normal probability paper.)

higher frequencies. In conclusion it seems that voice-frequency capacitance data can be used as a basis for simulating the crosstalk performance of a system up to at least the high kHz frequency range.

We are now in a position to give an abstract definition of the outside plant. Let L index over $[0, \infty)$. Let $(\Omega, \mathcal{A}, p)$ be a probability space. By an outside plant subsystem we mean any $2n \times 2n$ complex matrix stochastic process $X_L(\omega)$ of the form

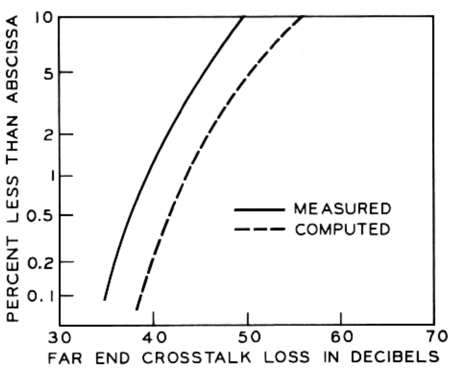

Fig. 7—Comparison of measured and computed distribution tails of 3.15-MHz within unit (50 pair) far-end crosstalk on 1000 feet of 22-gauge pulp cable. The computed distribution is based on 1-kHz capacitance unbalance data. (On normal probability paper.)

$$X_L(\omega) = \begin{bmatrix} \mathfrak{a}(L)I & \mathfrak{b}(L)I \\ \mathfrak{c}(L)I & \mathfrak{a}(L)I \end{bmatrix} (I + \epsilon Y(L, \omega)),$$

where $\begin{pmatrix} \mathfrak{a}(L) & \mathfrak{b}(L) \\ \mathfrak{c}(L) & \mathfrak{a}(L) \end{pmatrix}$ is the chain matrix characterization of a physically realizable, linear, passive, bilateral, symmetric two port and $Y(L, \omega)$ has a multivariate Gaussian distribution with mean 0. Let χ denote an independent set of all such $X_L(\omega)$. Let X^α and X^β belong to χ. By $X^\alpha \cdot X^\beta$ we mean the ordinary matrix product with the ϵ^2 term neglected. Since the product of 2×2 symmetric matrices with determinant one is again a symmetric matrix with determinant one and since the sum of independent Gaussian matrices with mean zero is again a Gaussian matrix with mean zero it follows that χ is closed under this multiplication. We note that χ clearly contains the identity matrix and that the multiplication we have defined is associative. A collection of elements closed under an associative multiplication rule and containing the multiplicative identity element is called a semigroup. Our definition of χ is a bit too general so we present the characters of interest: the cables, the bridged taps, and the load coils.

Cable Representation

Equation (5) defines the cable. The integrals involving capacitance and inductance unbalance are to be considered as stochastic integrals with respect to the Brownian motion process. The statistics of the capacitance portion of the cable process are known if the statistics of the matrix $(\int_0^L c_{ij}(x, \omega)\, dx)$ are known. Recall the way the stochastic integral was defined. The cable matrix has mean

$$\begin{bmatrix} \cosh \Gamma L I & -z_0 \sinh \Gamma L I \\ -z_0^{-1} \sinh \Gamma L I & \cosh \Gamma L I \end{bmatrix}.$$

The dispersion matrix involves terms of the form

$$\int_0^L \frac{(\phi(x)\bar{\psi}(x) + \bar{\phi}(x)\psi(x))}{2}\, dx k_{lm,rs}\,,$$

where $\phi(x)$ and $\psi(x)$ are deterministic hyperbolic functions of the primary constants and

$$k_{lm,rs} = E\left\{\left[\int_0^L dC_{lm}(x, \omega) \int_0^L dC_{rs}(x, \omega)\right]\right\}$$

which can be determined from capacitance unbalance data.

Given Γ, Z_0, f, and capacitance unbalance data the chain matrix can be sampled. For example, to get a sample of

$$-\epsilon Z_0 \int_0^L \sinh 2\Gamma x \; \eta_{ij} (x, \omega) \, dx$$

merely multiply the sample of $\int_0^L c_{ij}(x, \omega) \, dx$ by

$$\Theta = \sqrt{\frac{\epsilon^2 \mid 2\pi f Z_0 \mid^2}{L} \int_0^L \sinh 2\Gamma x \, \overline{\sinh 2\Gamma x} \, dx}.$$

This follows from the fact that if $X(\omega)$ is normally distributed with mean zero and variance V and Θ is a constant then $\Theta X(\omega)$ is normally distributed with mean zero and variance $\mid \Theta \mid^2 V$ and from the definition of stochastic integral.

The Representation of a Cable in the Open Bridged Tap Arrangement

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a cable matrix. By the representation of an open-circuited bridged tap we mean the first two terms in the power series in ϵ for $\begin{pmatrix} I & 0 \\ A^{-1}C & I \end{pmatrix}$.

The Representation of a Load Coil

By the representation of a load coil we mean the random matrix

$$\begin{bmatrix} [1+(R+i2\pi fL)(G+i2\pi fC)]I & -(R+i2\pi fL)I \\ -(G+i2\pi fC)[R-i2\pi fL)(G+i2\pi fC)+2]I & [1+(R+i2\pi fL)(G+i2\pi fC)]I \end{bmatrix}$$

$$-\sqrt{-1}\,\epsilon 2\pi f \begin{bmatrix} \bigcirc & \begin{matrix} & 0 & \\ & & 0 & M_{ij}(\omega) \\ & & \cdot^{\cdot^{\cdot}} & \\ & M_{ij}(\omega) & & 0 \end{matrix} \\ \begin{matrix} 0 & & \\ & 0 & M_{ij}(\omega) \\ & \cdot^{\cdot^{\cdot}} & \\ M_{ij}(\omega) & & 0 \end{matrix} & \bigcirc \end{bmatrix},$$

where M_{ij} is Gaussian with mean zero.

Let $X(\omega) = \begin{pmatrix} a & b \\ c & a \end{pmatrix} (I + \epsilon Y(\omega))$ be a system. Let $\mathbf{V}_S$ and $\mathbf{V}_R$ denote voltage stimuli and let $\mathbf{v}_S$ and $\mathbf{v}_R$ denote voltage responses. Let $(y_{S1}, y_{S2}, \cdots, y_{Sn})^t = \mathbf{y}_S$ and $(y_{R1}, y_{R2}, \cdots, y_{Rn})^t = \mathbf{y}_R$ denote the vector terminal admittances. The basic relation is

$$\begin{bmatrix} \mathbf{v}_R \\ (\mathbf{v}_R - \mathbf{V}_R)^t, \mathbf{y}_R) \end{bmatrix} = X(\omega) \begin{bmatrix} \mathbf{v}_S \\ (\mathbf{V}_S - \mathbf{v}_S)^t, \mathbf{y}_S) \end{bmatrix}.$$

For ϵ sufficiently small we can solve for the random variables $\mathbf{v}_R$ and $\mathbf{v}_S$. It is easy to show that these random variables are asymptotic to a multivariate normal distribution as ϵ tends to zero.

Consider the problem of characterizing the near-end crosstalk performance of a certain loop. Here $\mathbf{V}_R = \mathbf{0}$, $\mathbf{V}_S = (0, 0, \cdots 0, 2, 0, \cdots 0)^t$ where the 2 is in the ith position. $v_{R1}, v_{R2}, \cdots, v_{R(i-1)}, v_{R(i+1)}, \cdots, v_{Rn}$ are the far-end crosstalks and $v_{S1}, v_{S2}, \cdots, v_{S(i-1)}, v_{S(i+1)}, \cdots, v_{Sn}$ are the near-end crosstalks. There is a set of these random variables for each i. For each $\omega \epsilon \Omega$ there corresponds a distribution of near-end and far-end crosstalk. The set of all these distributions characterize the crosstalk behavior of the system. We illustrate the stochastic nature of systems. To accomplish this, capacitance unbalance data on nine 22-gauge PIC cables was selected at random from Western Electric's cable manufacturing plant at Hawthorne, Illinois. In an unbiased manner, this data was used to simulate three 9-kft systems having two splices each. The three far-end crosstalk distributions are shown in Fig. 8. There were only twenty-five pairs in this system. Apparently, as the number of splices and/or the number of pairs is increased the distributions of different realizations of the same system tend to look more and more alike. Further study of the implication of the stochastic nature of outside plant systems is needed.

VI. CONCLUSIONS

A mathematical model of outside plant wire pair cable systems that includes crosstalk is given. The voice-frequency version of the model has been used to develop a computer simulation capability. It appears that this simulation capability can be extended to at least high kHz frequencies. Quality assurance data is the key to this simulation effort. This same data can be used to provide the basis for optimal twist length selection.

VII. ACKNOWLEDGMENTS

The work presented in this paper is an outgrowth of the many discussions the author had with the late F. L. Schwartz. Thanks are also due to S. F. Stumpf and Mrs. R. E. Parris who did the computer programming and to M. F. Veverka who did most of the experimental work.

S. O. Rice urged the author to investigate V. R. Saari's work[6] in the light of matrix techniques. The development of the analogs between coupled lines and nonuniform, uncoupled lines originated from such an investigation. The cooperation of W. N. Bell and A. K. Jain who

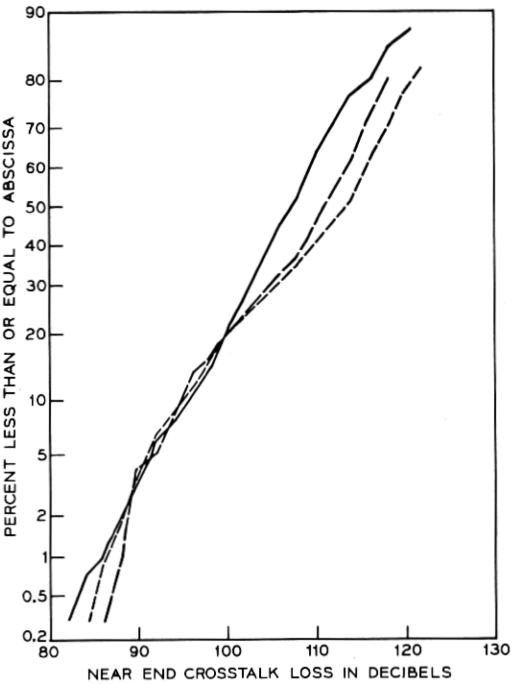

Fig. 8—Three separate realizations of the 1-kHz near-end crosstalk distribution of a nominally specified system. The nominal specification is a 25-pair, 22-gauge PIC system which is composed of three 3000-foot sections. The terminations are 600 Ω. (On normal probability paper.)

pursued the ideas concerning the statistical model and kindly permitted me to mention their unpublished results is greatly appreciated.

In closing, the author expresses his thanks to numerous individuals who planned, executed, or relayed the measurements used here.

APPENDIX

We shall give a detailed proof that $NEXT(i, j)$ and $N\hat{E}XT(i, j)$ are asymptotically equivalent. In what follows we shall lose no generality by assuming that $i = 1$ and $j = 2$. The solution to equation (2) is given by

$$T(x) = \exp\left[-\begin{pmatrix} 0 & zI \\ yI & 0 \end{pmatrix} x\right] \cdot$$

$$\Omega_0^x \left\{ \epsilon \exp\left[+\begin{pmatrix} 0 & zI \\ yI & 0 \end{pmatrix} \xi\right] \begin{pmatrix} 0 & \eta(\xi) \\ \zeta(\xi) & 0 \end{pmatrix} \exp\left[-\begin{pmatrix} 0 & zI \\ yI & 0 \end{pmatrix} \xi\right] \right\} . \tag{6}$$

(See Ref. 7 for the definition and basic properties of $\Omega_0^x(\cdot)$ and the proof that (6) is the solution. Equation (6) offers another method from which equations (4) and (5) can be obtained.) Because of the way ϵ appears in $\Omega_0^x(\cdot)$ and because $\Omega_0^x(\cdot)$ converges we can view $\Omega_0^x(\cdot)$ and hence $T(x)$ as a convergent power series in ϵ. From elementary considerations it can be shown that

$$\exp\left[-\begin{bmatrix} 0 & zI \\ yI & 0 \end{bmatrix} L\right] = \begin{bmatrix} \cosh \Gamma LI & -z_0 \sinh \Omega LI \\ -\dfrac{1}{z_0}\sinh \Gamma LI & \cosh \Gamma LI \end{bmatrix}.$$

For the manipulations we shall do here it will be convenient to partition $T(\epsilon)$ into square matrices, namely,

$$T(\epsilon) = \begin{bmatrix} A' & z_0 B' \\ \dfrac{C'}{z_0} & D' \end{bmatrix} = \sum_{i=1}^{\infty} \epsilon^i \begin{bmatrix} A_i' & z_0 B_i' \\ \dfrac{C_i'}{z_0} & D_i' \end{bmatrix}.$$

We are now prepared to determine the relation between $N\hat{E}XT(1, 2)$ and $NEXT(1, 2)$. Note we have

$$\begin{bmatrix} \mathbf{v}(L) \\ \dfrac{\mathbf{v}(L)}{z_0} \end{bmatrix} = T(\epsilon)\left[\begin{bmatrix} \mathbf{v}(0) \\ -\dfrac{\mathbf{v}(0)}{z_0} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{V}{z_0} \\ \vdots \\ 0 \end{bmatrix}\right]. \tag{7}$$

It will suffice to deal with (7) which we can rewrite as

$$(A' + D' - (B' + C'))\mathbf{v}(0) = (D' - B')\begin{bmatrix} V \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since $\lim_{\epsilon \to 0} (D' - B') = e^{\Gamma L} I$ is nonsingular, we conclude that for ϵ sufficiently small, say $\epsilon < \epsilon_1$, the matrix $(D' - B')$ is nonsingular. Thus for $\epsilon < \epsilon_1$, we can write

$$(I + (D' - B')^{-1}(A' - C'))v(0) = \begin{bmatrix} V \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now $\lim_{\epsilon \to 0} (I + (D' - B')^{-1}(A' - C')) = 2I$ thus for ϵ sufficiently small, say $\epsilon < \min(\epsilon_1, \epsilon_2)$ we can write

$$v(0) = (I + (D' - B')^{-1}(A' - C'))^{-1} \begin{bmatrix} V \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since D' and B' are convergent power series, we can write

$$D' - B' = e^{\Gamma L}[I + \epsilon(e^{-\Gamma L}[(D_1' - B_1') + \epsilon(D_2' - B_2') + \cdots])].$$

Thus for ϵ sufficiently small, say $\epsilon < \min(\epsilon_1, \epsilon_2, \epsilon_3)$ the inversion of $D' - B'$ is essentially the inversion of the identity plus an operator small enough to permit the Neumann inversion, see Ref. 8. Hence

$$\begin{aligned} (I + (D' - B')^{-1}(A' - C')) &= I + e^{-\Gamma L}(I - \epsilon e^{-\Gamma L}((D_1' - B_1') + \cdots) \\ &\quad + \epsilon^2 e^{-\Gamma L}((D_1' - B_1') + \cdots)^2 + \cdots) \\ &\quad \cdot e^{\Gamma L}(I + \epsilon e^{-\Gamma L}(A_1' - C_1') + \epsilon^2 e^{-\Gamma L}(A_2' - C_2') + \cdots) \\ &= 2I - \epsilon e^{-\Gamma L}(D_1' - B_1' + A_1' - C_1') + 0(\epsilon^2). \end{aligned}$$

Once again if we assume that ϵ is sufficiently small, say $\epsilon < \min(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ we can use the Neumann inversion to obtain

$$v(0) = \tfrac{1}{2}(I + 2\epsilon e^{\Gamma L}(A_1' + D_1' - B_1' - C_1') + 0(\epsilon^2)) \begin{bmatrix} V \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad (8)$$

Now we note that (8) can be viewed as a $2n \times 2n$ system resulting from (7) or a 2×2 system corresponding to (6), in the latter case, all quantities needed to be hatted. Evidently $\lim_{\epsilon \to 0} V(\epsilon) = 2$. Since $\hat{A}_1'$, $\hat{D}_1'$, $\hat{B}_1'$, and $\hat{C}_1'$ are merely the 2×2 upper right-hand corner submatrices in A_1', D_1', B_1', and C_1' it follows from (8) that the power series expressions for $N\hat{E}XT(i,\ j)$ and $NEXT(i,\ j)$ have identical coefficients of ϵ, (but not of ϵ^n, $n \geqq 2$). Thus $N\hat{E}XT(i, j)$ and $NEXT(i, j)$ are asymptotically equivalent.

An analogous proof goes through for $FEXT(i,\ j)$ and $F\hat{E}XT(i,\ j)$.

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