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Performance of a System of Mutually Synchronized Clocks

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This paper is concerned with mutual sychronization—a scheme for synchronizing a nationwide network of clocks for an integrated digital transmission and switching communications system. Described is an approach to the problem of determining values for the design parameters of a one-sided, linear phase averaging scheme with no filters. Two different sets of performance objectives are considered. The primary results concern the bounds which the effects of delay change force on the parameters which describe the inherent clock stability. Specifically, if a performance objective is no slips, and a limit is imposed on the amount of buffer storage, then an upper bound is forced on the allowable random drift in the free-running frequencies of the clocks. Alternatively, if an objective is that the slip rate not exceed some specified rate, again with a limited buffer size, then an upper bound is forced on the rate of random drift. Both bounds depend on the network configuration with the so called "dumbbell" configuration representing the worst case. A numerical example is included.

I. INTRODUCTION

This paper is concerned with mutual synchronization—a scheme for synchronizing a nationwide network of clocks for an integrated digital transmission and switching communication system. The clock at a switching center in such a system determines the rate of flow of data bits through the switch and to the output links. Buffer stores can make allowances for small, temporary differences in the received rate (determined by a distant clock and delay change) and the local switch clock rate. However, if a frequency difference persists then a buffer will occasionally overflow (or underflow) causing deletion (or repetition) of data bits from the output stream. Either such error is referred to as a slip. In this paper a synchronous network is one in which there are either no slips or, alternatively, in which the slip rate does not exceed some given rate.

Mutual synchronization was conceived about 10 years ago[1] as an alternative to a master-slave synchronous timekeeping plan for large telephone networks. This new plan aimed at eliminating the possible need to reorganize a master-slave network in the event of failure of a timing link or the master clock. Previous analyses have been largely concerned with questions of stability,[2] equilibrium frequency,[3] dynamic response,[4,5] and control strategies.[6]

Questions relating to the system's ability to meet specific performance objectives, e.g., no slips, have been largely ignored. It has been noted that since changes in phase differences between clocks remain finite, sufficiently large buffers could be placed in each communications link to avoid such slips. However, the relationships between required buffer size and clock stability, delay changes, network configuration, and control parameters have not been established. It is to this question that this paper is addressed. A simple control strategy is considered, viz., one-sided, linear phase averaging with no filters. Two different sets of performance objectives are considered and the results are expressed as bounds which are forced on parameters which describe the inherent clock stability. A statement of these performance objectives and a description of the network model follow.

II. ALTERNATIVE PERFORMANCE OBJECTIVES

2.1 *Performance Objective #1*

Let a performance objective be no slips. Assume, also, a given amount of buffer storage (of sufficient size to account at least for changes in link delay). As an additional objective, the variation of system frequency (i.e., that common frequency at which all clocks operate in equilibrium) must not exceed the bandwidth limitations of transmission and switching equipment (which might typically be of the order of one part in 10^5). To meet the first objective, changes in phase difference between any pair of clocks must not exceed a given value. Due to the effect of delay change on clock phase and system frequency, this requirement will be seen to force an upper bound on the allowable random drift in a clock's free-running frequency (i.e., that frequency at which it would operate in the absence of a control input). The second objective forces an upper bound on the allowable total drift of a clock's free-running frequency.

2.2 *Performance Objective #2*

As an alternative, suppose slips are permitted to occur but at no more than a given rate. In addition, assume the same objective, as in

Section 2.1, on variations of system frequency. Again, a given amount of buffer storage is assumed. In this case frequency differences between pairs of clocks (that persist long enough for several slips to actually occur) must not exceed some preset value. This value will depend on the slip rate requirement and the amount of buffer storage and might typically be of the order of one part in 10^9. Due to the effects of delay change, this requirement will be seen to force an upper bound on the allowable rate of random drift of any clock's free-running frequency.

Changes in equilibrium phase differences occur as a result of changes in the free-running frequencies of the clocks or the transmission delays between clocks. Frequency differences between pairs of clocks are nonequilibrium effects which occur while the free running frequencies or the transmission delays are changing. The magnitude of both these effects depends critically upon the network configuration. The "dumbbell" configuration, consisting of two equal size groups of mutually synchronized clocks with full interconnection within a group but only a single intergroup link (Fig. 1), appears to represent the worst case.

It is assumed that performance objectives must be met without knowledge of network configuration; specifically, all clocks and timing links are to be considered "equal," regardless of their location in the network. Design parameters are determined, then, so that objectives are met in the dumbbell configuration.

III. CONTROL MODEL

In one-sided phase averaging the frequency of each office clock is controlled by an average of the observed phases of signals received from distant clocks, as measured with reference to the phase of the local clock. The frequency of each clock responds then to both a change

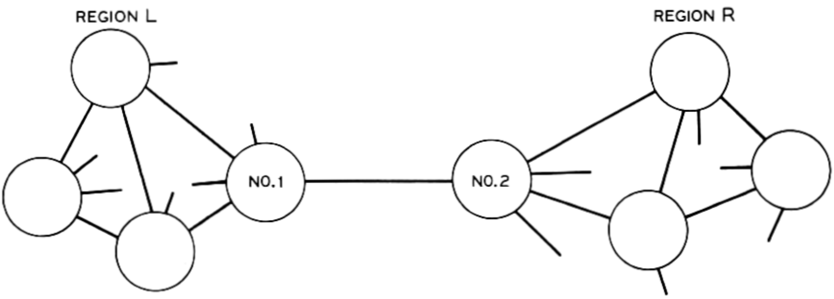

Fig. 1—Dumbbell configuration.

in the phase of a distant clock and to a change in the transmission delay from a distant clock. (Two-sided controls, in which the results of phase comparisons are transmitted back, on a special data link, to control the distant clocks, can eliminate the effect of delay on frequency.[5,6]) The general control equation is

$$f_i(t) = F_i(t) + K_i \sum_{j=1}^{N} a_{ij}\{p_j[t - \tau_{ij}(t)] - p_i(t) + \phi_{ij}\}. \tag{1}$$

In this equation: N is the number of clocks; f_i is the frequency of the ith clock; F_i is the free-running frequency of the same clock (frequency in the absence of a control input); K_i is a control gain with the dimensions of reciprocal time; a_{ij} are averaging coefficients such that

$$a_{ij} \geqq 0, \qquad \sum_{j=1}^{N} a_{ij} = 1 \qquad i = 1, 2, \cdots, N;$$

p_i is the phase of the ith clock, related to frequency by

$$\frac{dp_i}{dt} = f_i \ ;$$

τ_{ij} is the transmission delay encountered by a pulse arriving at the ith office, from the jth, at time t; and ϕ_{ij} is a reference phase whose value depends upon initial conditions.

IV. SYSTEM FREQUENCY

Previous studies[3] have shown that if the system parameters remain constant and there exists at least one clock which distributes timing control to every other clock (either directly or indirectly via some intermediate clocks) then the system will asymptotically approach an equilibrium state in which all clocks run at a common frequency—called the system frequency. Under these conditions the system equations can be solved algebraically for the system frequency; the result is[3]

$$f = \frac{\sum_{i=1}^{N} b_i\left(F_i + K_i \sum_{j=1}^{N} a_{ij}\phi_{ij}\right)}{\sum_{i=1}^{N} b_i\left(1 + K_i \sum_{j=1}^{N} a_{ij}\tau_{ij}\right)}, \tag{2}$$

where the weighting coefficients b_i depend upon the gains and the averaging coefficients. In the general case b_i is the cofactor of any element in the ith row of the matrix $K(I\text{-}A)$, where K is the diagonal matrix with diagonal elements K_i, I is the identity matrix, and A is the matrix of the averaging coefficients a_{ij}. The proof of these results is found in Ref. 3.

The effects of delay change and free-running frequency drift on the system frequency will be considered separately, allotting to each one-half the maximum-allowable variation.

V. EFFECTS OF DELAY CHANGE ON SYSTEM FREQUENCY (BOUND ON GAIN)

To obtain some feel for the effects of delay change on the system frequency consider first a network of two clocks with equal control gains and suppose that the transmission delay between them increases by an amount $\Delta\tau$. By symmetry each clock is subjected to the same influence and, hence, signals will arrive $\Delta\tau$ seconds later at *both* clocks. The corresponding change in both observed phases is, then, $f\tau - (f + \Delta f)(\tau + \Delta\tau) \approx -(f\Delta\tau + \tau\Delta f)$, where Δf is the (as yet undetermined) change in the system frequency. But this change is equal to the gain K times change in observed phase; hence,

$$\Delta f \approx \frac{-Kf\Delta\tau}{1 + K\tau}.$$

To estimate the effects of delay changes in a more general network suppose that all gains K_i are equal and that identical delay changes $\Delta\tau$ occur on all links at the same time. The above reasoning, with the identical conclusion, applies also to this case as may be verified from equation (2). That is, with $K_i = K$ and $\tau_{ij} = \tau$,

$$f + \Delta f = \frac{1}{1 + K(\tau + \Delta\tau)} \frac{\sum_{i=1}^{N} b_i(F_i + K\phi_i)}{\sum_{i=1}^{N} b_i},$$

from which

$$\Delta f \approx \frac{-Kf\Delta\tau}{1 + K\tau} = -\frac{K\tau}{1 + K\tau} f \frac{\Delta\tau}{\tau}.$$

Since the fractional change in delay, $\Delta\tau/\tau$, is typically several orders of magnitude greater than the allowable fractional change in f, it follows that K must be chosen so that $K\tau$ is much less than unity; hence, neglecting the sign of the change,

$$\Delta f \approx Kf\Delta\tau. \tag{3}$$

If the maximum allowable variation of system frequency (due to delay change) is $\frac{1}{2}\Delta f_s$, then K must satisfy the upper bound

$$K \leqq K_s = \frac{1}{2\Delta\tau_s} \frac{\Delta f_s}{f_o}, \tag{4}$$

where $\Delta\tau_s$ is the maximum delay change (assumed to occur at the same time on all links) and f_o is the nominal system frequency. This is one of two upper bounds on the gain which will lead to an upper bound on either the total random drift or rate of random drift in a clock's free-running frequency.

VI. EFFECT OF CLOCK DRIFT ON SYSTEM FREQUENCY

With $K\tau \ll 1$, the change in the system frequency is, approximately,

$$\Delta f \approx \frac{\Sigma b_i \Delta F_i}{\Sigma b_i}.$$

Thus, if the drifts are systematic, i.e., all in the same direction, then (assuming a simple average over N clocks) each clock must have the property that its total free-running frequency drift does not exceed the allotted variation in the system frequency (i.e., $\frac{1}{2}\Delta f_s$). If, however, the drifts are not correlated then the standard deviation of the system frequency is (again assuming a simple average over N clocks) inversely proportional to the square root of the number of clocks. In this case the allowed drift can, for large networks, be somewhat greater than $\frac{1}{2}\Delta f_s$.

VII. DUMBBELL CONFIGURATION

The magnitude of the effects of delay and clock drift on the phase and frequency differences between pairs of clocks, unlike their effects on system frequency, depends critically upon the network configuration. The "dumbbell" configuration, illustrated in Fig. 1, appears to represent the worst case. In this configuration the N offices are divided into two equal groups, with a direct timing link from every clock to every other clock in the same group, but only one timing link (the bar of the dumbbell) connecting one group to the other. In this configuration changes in the free-running frequencies or the delays can force a relatively large change in the phase difference between clocks in opposite halves. An intuitive feeling for such an effect may be obtained by supposing that each bar clock has but one "vote" out of $N/2$ in determining the common frequency of the clocks in its half of the dumbbell, while signals on the bar link have but one of $N/2$ votes in determining the frequency of a bar clock. On this basis one might guess then that a given change $\Delta\phi$ in the phase difference between the bar clocks would tend to produce a change of magnitude $K/(N/2)^2 \; |\Delta\phi|$ in the common frequency of the clocks in each half (one half up, the other down). Assuming this to be true, suppose then that all the delays in one half should increase by an amount $\Delta\tau$, thus tending to decrease the frequency of the clocks in that half by $Kf\Delta\tau$ [equation (3)], while the opposite occurs in the other half. By symmetry the system frequency will not change. Hence, to compensate for the tendency of each half to change frequency by an amount $\pm Kf\Delta\tau$, the phase change $\Delta\phi$ must be such that

$$\frac{K}{(N/2)^2} \mid \Delta\phi \mid = Kf\Delta\tau, \quad \text{or}$$

$$\mid \Delta\phi \mid = \frac{N^2}{4} f \mid \Delta\tau \mid .$$

This result will be formally verified below. The dumbbell is not the only configuration which can give rise to the factor N^2. For example, the phase difference between the end clocks in a bilateral chain also grows, under similar conditions, as the square of the number of clocks. However, the dumbbell is worse in the sense that this dependence on network size holds for any two clocks in opposite halves.

The conclusions pertaining to the dumbbell configuration are based on the following assumptions:

(*i*) The gain K_i at each clock has the same value K.

(*ii*) Each control link (shown in Fig. 1) is assigned the same weight; hence, for $N/2 \gg 1$, all (nonzero) a_{ij} are approximately equal to $2/N$.

(*iii*) The delays on all links within region R and within region L (see Fig. 1) are approximately equal, respectively, to τ_R and τ_L and the difference $\mid \tau_R - \tau_L \mid$ changes at a constant rate $\dot{\tau}_D$ for $t_{\dot{\tau}}$ seconds. The maximum change $\Delta\tau_D$ persists for $t_{\Delta\tau}$ seconds at which point the difference returns to its original value (Fig. 2). Such a situation might reflect an effect of a cold front moving successively through the two regions.

(*iv*) The free-running frequency of each clock drifts at a constant rate of $\pm\dot{F}$ for $t_{\dot{F}}$ seconds, with the maximum drift $\pm\Delta F$ per-

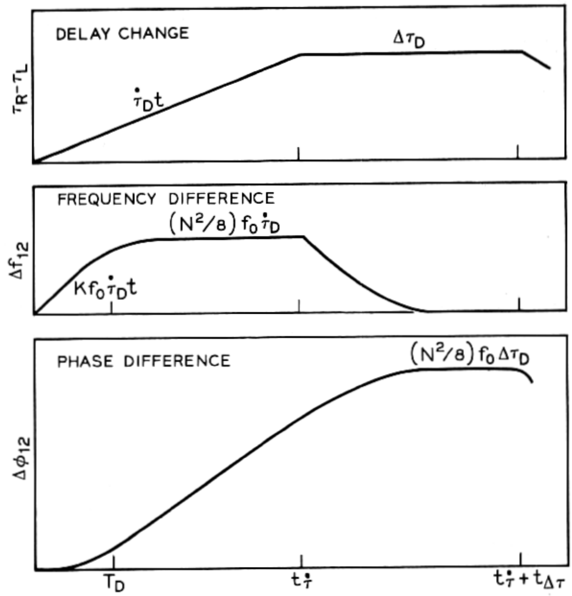

Fig. 2—Response of dumbell to delay change.

sisting for $t_{\Delta F}$ seconds. The drifts are uncorrelated and hence the standard deviation of the difference $| F_R - F_L |$ of the numerical averages of the free-running frequencies in each half grows at a rate $2\dot{F}/\sqrt{N}$ up to maximum change of $2\Delta F/\sqrt{N}$.

To gain some insight into the results to be presented, the effects of a step change in delay will be considered first. Thus, suppose that the delay on each link within region R suddenly increases by an amount $\Delta\tau$, while the opposite occurs on each link within region L—with no change in the bar-link delay. By symmetry there will be no change in the system frequency nor in the phase differences between any pair of clocks both within region R or both within region L (*Note:* R and L excludes the bar clock). Consider then, any one of the clocks within region R; the following statements apply:

(*i*) Signals from each of the other $(N/2 - 2)$ clocks within R arrive later by an amount $\Delta\tau$, thus tending to make this clock run slower.

(*ii*) To balance this tendency, signals from bar clock #2 must arrive earlier by an amount $(N/2 - 2)\Delta\tau$; i.e., the phase of bar clock

#2 must advance with respect to that of each clock within R by an amount $(N/2 - 1)f\Delta\tau$.

Next consider bar clock #2; the following statements apply:

(*i*) Signals from each of the $(N/2 - 1)$ clocks within R arrive later by an amount $(N/2)\Delta\tau$, thus tending to make this clock run slower.

(*ii*) To balance this tendency, the signal from bar clock #1 must, then, arrive earlier by an amount $(N/2 - 1)(N/2)\Delta\tau$, i.e., the phase of bar clock #1 must advance with respect to that of bar clock #2 by approximately $(N^2/4)f\Delta\tau$ as suggested above.

An estimate of the time required to reach equilibrium may be obtained by noting that the initial effect of the step change is to decrease the frequency of bar clock #2 by $Kf_0\Delta\tau$ and to increase that of bar clock #1 by the same amount—resulting in an initial frequency difference of $2Kf_0\Delta\tau$. If this difference persisted, an interval of duration $N^2/8K$ would be required to reach equilibrium. The actual approach to equilibrium is an exponential with time constant $T_D = N^2/8K$.

The response to a ramp change in delay could, if the system equations were linear, be derived directly from the response to a step change. That is, a ramp change is the integral of a step change and hence, were the system linear, the response to a ramp change would be the integral of the response to a step change. Although the systems equations are not linear in the delays, it is assumed that changes are slow enough so that a linear approximation is justified. The frequency difference Δf_{12} between the bar clocks is then shown, in the Appendix, to be given by the solution of the following linear differential equation:

$$\ddot{\Delta f}_{12} + K\dot{\Delta f}_{12} + \frac{8K^2}{N^2}\Delta f_{12} = K(\dot{F}_R - \dot{F}_L) - Kf_o(\dot{\tau}_R - \dot{\tau}_L), \qquad (5)$$

where f_o is the nominal value of the system frequency.

In what follows, the effects of delay and free-running frequency changes will again be considered separately.

VIII. EFFECT OF DELAY CHANGE (ADDITIONAL BOUNDS ON GAIN)

The solution of equation (5), while the delay difference $\tau_R - \tau_L$ is changing at a constant rate $\dot{\tau}_D$, is shown in the Appendix to be

$$\Delta f_{12} \approx \frac{N^2}{8} f_o\dot{\tau}_D(1 - e^{-t/T_D}), \qquad (6)$$

where $T_D = N^2/8K$ is the previously suggested time constant. The complete solution for Δf_{12} and for the change $\Delta\phi_{12}$ in the phase difference between the bar clocks is sketched in Fig. 2 (for the case when both $t_{\dot{\tau}}$ and $t_{\Delta\tau}$ are large compared with T_D).

If the "equilibrium" frequency difference

$$\Delta f \mid \max = \frac{N^2}{8} f_o \dot{\tau}_D \tag{7}$$

does not exceed the maximum allowable difference $\frac{1}{2}\Delta f_D$, i.e., if

$$N \leqq N_f = \left(\frac{4\Delta f_D}{f_0 \dot{\tau}_D}\right)^{\frac{1}{2}} \tag{8}$$

then no restriction need be placed on the gain K. However, if $N > N_f$ then an upper bound, determined from equation (6) with $t = t_{\dot{\tau}}$, is forced on K. For $(N^2/8) f_o \dot{\tau}_D \gg \frac{1}{2}\Delta f_D$ the bound is

$$K \leqq K_f = \frac{\Delta f_D}{2 f_o \Delta \tau_D}; \quad \text{for} \quad N \gtrapprox 3N_f . \tag{9}$$

Similarly, if the "equilibrium" change in phase difference

$$\Delta\phi \mid \max = \frac{N^2}{8} f_o \Delta\tau_D \tag{10}$$

is less than the allowable change $\frac{1}{2}\Delta\phi_D$, i.e., if

$$N \leqq N_\phi = \left(\frac{4\Delta\phi_D}{f_o \tau_D}\right)^{\frac{1}{2}} \tag{11}$$

then K may be chosen arbitrarily. However, if N exceeds N_ϕ an upper bound must be placed on K. When the equilibrium value is much larger than the allowable change, the least upper bound is easily determined. For, then, the time constant $T_D = N^2/8K$ must be made long compared with the interval over which the delay disturbance persists. The phase difference grows then nearly as if the two halves were not connected and hence approaches the value $K f_o t_D \Delta\tau_D$, which will not exceed the maximum allowable value provided K meets the upper bound

$$K \leqq K_\phi = \frac{\Delta\phi_D}{2 f_o t_D \Delta\tau_D}; \quad \text{for} \quad N \gtrapprox 3N_\phi \tag{12}$$

where $t_D = t_{\dot{\tau}} + t_{\Delta\tau}$.

IX. BOUNDS ON CLOCK STABILITY

It is assumed that the stabilities of the clocks under consideration are such that performance objectives would not be met if the halves of the dumbbell were not joined by the bar link (for small numbers of clocks this is nearly equivalent to assuming that objectives would not be met with all clocks operating independently). Thus, the interval during which the free-running frequencies drift and that for which the maximum drift persists must both be long compared with the dumbbell time constant T_D. Hence, the frequency difference between the bar clocks approaches $(N^2/8K) \mid \dot{F}_R - \dot{F}_L \mid$, while the change in phase difference approaches $(N^2/8K) \mid \Delta F_R - \Delta F_L \mid$. For the assumed model these numbers are random variables; however, for purposes of further discussion they will be replaced by the values of their standard deviation, i.e.,

$$\Delta f \mid \max = \frac{N^{\frac{3}{2}}}{8K} \dot{F} \tag{13}$$

and

$$\Delta \phi \mid \max = \frac{N^{\frac{3}{2}}}{8K} \Delta F. \tag{14}$$

If the performance objective is no slips then $\Delta\phi$ must not exceed $\frac{1}{2}\Delta\phi_D$ and hence, from equation (14), the random drift in each clock's free-running frequency must satisfy the bound

$$\Delta F < \frac{4K}{N^{\frac{3}{2}}} \Delta \phi_D ,$$

where K must not exceed K_s, as given by equation (4), and further, if $N \gtrapprox 3N_\phi$, must not exceed K_ϕ as given by equation (12). Assuming $K_\phi < K_s$ (see Section X), the maximum value of $\Delta F/f_o$ is

$$\Delta F/f_o \mid \max = \begin{cases} \dfrac{2}{N^{\frac{3}{2}} \Delta \tau_s} \dfrac{\Delta \phi_D}{f_o} \dfrac{\Delta f_s}{f_o} & N < N_\phi \\[2ex] \dfrac{2}{N^{\frac{3}{2}} t_D \Delta \tau_D} \left(\dfrac{\Delta \phi_D}{f_o} \right)^2 & N \gtrapprox 3N_\phi , \end{cases} \tag{15}$$

where

$$N_\phi = \left(\frac{4 \Delta \phi_D}{f_o \Delta \tau_D} \right)^{\frac{1}{2}}.$$

Similarly, if the performance objective is expressed as a maximum slip rate then Δf must not exceed $\frac{1}{2}\Delta f_D$ and hence, from equation (13), the rate of random drift in the free-running frequency of each clock must satisfy the bound

$$\dot{F} < \frac{4K}{N^{\frac{3}{2}}} \Delta f_D ,$$

where K must not exceed K_s and further, if $N \gtrsim 3N_f$, must not exceed K_f as given by equation (9). Assuming $K_f < K_s$ (see Section X) the maximum value of $\dot{F}/f_0$ is

$$\dot{F}/f_o \mid \max = \begin{cases} \dfrac{2}{N^{\frac{3}{2}}\Delta\tau_s} \dfrac{\Delta f_D}{f_o} \dfrac{\Delta f_s}{f_o} & N < N_f \\ \dfrac{2}{N^{\frac{3}{2}}\Delta\tau_D} \left(\dfrac{\Delta f_D}{f_o}\right)^2 & N \gtrsim 3N_f , \end{cases} \tag{16}$$

where

$$N_f = \left(\frac{4\Delta f_D}{f_o \dot{\tau}_D}\right)^{\frac{1}{3}}.$$

X. NUMERICAL EXAMPLE

Assume that all links are approximately 300 miles in length with a nominal delay of 6.5 microseconds per mile and a delay variation of 0.035 percent per degree Fahrenheit (these values apply to pulp-insulated cable). In considering the effect of delay change on the system frequency assume a temperature variation of ±40°F, leading to $(\Delta\tau)_s \approx 30\ \mu s$. As the worst case condition in the dumbbell configuration, assume that the temperature difference between the two halves changes at a rate of about 2°F per day for about 1 week, leading to $\dot{\tau}_D \approx 2 \times 10^{-11}$ and $\Delta\tau_D \approx 10\ \mu s$; and assume that the maximum difference persists for about 1 week.

Assume, also, that the system frequency corresponds to the digital sampling rate in D-type PCM channel banks, viz., 8×10^3 frames/second, and that it is to vary by no more than one part in 10^5 (which sets the total allowable free-running frequency drift). Furthermore, assume that $\Delta\phi_D = 1$ frame (which is ample allowance for the link delay change $f_o\Delta\tau_s \approx 1/4$ frame) and that Δf_D is not to exceed one part in 10^9.

Under these conditions, the various upper bounds on K are

$$K_s = 1/6\ s^{-1},$$

$$K_\phi = 5 \times 10^{-6}\, s^{-1},$$

$$K_f = 5 \times 10^{-5}\, s^{-1}.$$

If there are to be no slips (performance objective #1), then the maximum allowable random drift in any clock's free running is

$$\Delta F/f_o \mid \max = \begin{cases} \dfrac{4 \times 10^{-5}}{N^{\frac{3}{2}}} & N < 7 \\[2ex] \dfrac{2 \times 10^{-9}}{N^{\frac{3}{2}}} & N \gtrapprox 21. \end{cases}$$

The first bound corresponds roughly to the performance obtainable from relatively inexpensive crystal oscillators. However, the second bound, which must be met if the number of clocks exceeds about 20, implies atomic standards.

Alternatively, if it is the slip rate that is to be bounded (performance objective #2) then the rate of random drift in any clock's free running must not exceed

$$\dot{F}/f_o \mid \max = \begin{cases} \dfrac{6 \times 10^{-5}}{N^{\frac{3}{2}}} \Big/ \text{day} & N < 14 \\[2ex] \dfrac{2 \times 10^{-8}}{N^{\frac{3}{2}}} \Big/ \text{day} & N \gtrapprox 42. \end{cases}$$

Again, the first bound is met by relatively inexpensive crystal oscillators while the second, which applies if there are more than about 40 clocks, implies either the best present day crystals or atomic standards.

XI. CONCLUDING REMARKS

It should not be concluded that a mutually synchronized system must be designed as described above. For example, although the dumbbell configuration represents a worst case, it is perhaps too unlikely a case on which to base parameter specifications. Furthermore, the dependence of clock frequency on delay can be eliminated with two-sided controls[5] (which, however, requires a special data link). Alternatively, sufficient reliability without the problems associated with a large-dumbbell network might be obtainable from some type of hierarchical system—composed, for example, with clocks of varying degrees of stability[7] or of a small number of mutually synchronized clocks at a top level with lower levels consisting of a number of similar small groups each redundantly slaved to higher levels. Finally, it should be noted that the relatively large phase shift which may occur between the bar clocks in the dumbbell configuration could be reduced by increasing the "weight" given to the bar link (i.e., by increasing the averaging coefficients associated with this link). This, however, is not really in the original spirit of mutual synchronization, as it requires that the system configuration be known and used in determining the averaging coefficients. This introduces administrative complications comparable to those involved in reorganizing a master-slave network.

XII. ACKNOWLEDGMENTS

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APPENDIX

Phase and Frequency Difference in the Dumbbell

It is assumed that the solutions of equation (1) are continuous functions of the delays and that frequency changes are slow enough so that the approximations

$$p_i(t - \tau_i) \approx p_i(t) - f_i\tau_i$$

are justified. Furthermore, changes in $f_i\tau_i$ are assumed dominated by changes in delay (i.e., doppler shift) so that f_i, in this product, may be replaced by the nominal frequency f_o. Since the delay τ is small compared with the response time $1/K$, i.e., $K\tau \ll 1$, these approximations appear reasonable.

The control equation [equation (1)] for the left-hand bar clock may, then, be written in the form

$$\dot{f}_1 = \dot{F}_1 - Kf_o\dot{\tau}_L + K\{f_L' - f_1\} + \frac{K}{N/2}\{f_2 - f_1\}, \tag{17}$$

where f_L' is the arithmetic average of clock frequencies in region L (which excludes the bar clock) and, from equation (1), satisfies

$$\dot{f}_L' = \dot{F}_L' - Kf_o\dot{\tau}_L + \frac{K}{N/2}\{f_1 - f_L'\}. \tag{18}$$

Similarly, for the right-hand bar clock

$$\dot{f}_2 = \dot{F}_2 - Kf_o\dot{\tau}_R + K\{f_R' - f_2\} + \frac{K}{N/2}\{f_1 - f_2\}, \tag{19}$$

where

$$\dot{f}_R' = \dot{F}_R' - Kf_o\dot{\tau}_R + \frac{K}{N/2}\{f_2 - f_R'\}. \tag{20}$$

The network shown in Fig. 3 is equivalent in that it is described by the above set of equations.

With x and y denoting, respectively, the frequency differences $f_1 - f_2$ and $f_L' - f_R'$, it follows from equations (17) through (20) that

$$\dot{x} + K\left(1 + \frac{4}{N}\right)x = Ky + \dot{F}_1 - \dot{F}_2 - Kf_0(\dot{\tau}_L - \dot{\tau}_R) \tag{21}$$

and

$$\dot{y} + \frac{2K}{N}y = \frac{2K}{N}x + \dot{F}_L' - \dot{F}_R' - Kf_o(\dot{\tau}_L - \dot{\tau}_R). \tag{22}$$

Eliminating y from equation (21) it follows that x satisfies (for $N \gg 1$)

$$\ddot{x} + K\dot{x} + \frac{8K^2}{N^2}x = K(\dot{F}_L - \dot{F}_R) - K^2f_o(\dot{\tau}_L - \dot{\tau}_R) \tag{23}$$

and its solution, subject to the initial conditions $x(t = 0) = 0$ and $y(t = 0) = 0$ which implies $\dot{x}(t = 0) = (\dot{F}_1 - \dot{F}_2) - Kf_o(\dot{\tau}_L - \dot{\tau}_R)$, is

$$x(t) \approx \frac{N^2}{8}f_o(\dot{\tau}_L - \dot{\tau}_R)(1 - e^{-t/T_D}) + \frac{N^2}{8K}(\dot{F}_L - \dot{F}_R)$$
$$+ \frac{(\dot{F}_L - \dot{F}_1) - (\dot{F}_R - \dot{F}_2)}{K(1 - 8/N^2)}e^{-Kt}$$
$$+ \frac{(\dot{F}_1 - \dot{F}_2) - \dfrac{N^2}{8}(\dot{F}_L - \dot{F}_R)}{K(1 - 8/N^2)}e^{-t/T_D},$$

where $T_D = N^2/8K$.

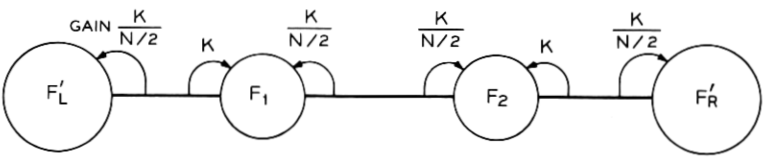

Fig. 3—Equivalent dumbbell configuration.

REFERENCES

1. Runyon, J. P., "Reciprocal Timing of Time Division Switching Centers," U. S. Patent No. 3,050,586, August 21, 1962.
2. Karnaugh, M., "A Model for the Organic Synchronization of Communications Systems," B.S.T.J., *45,* No. 10 (December 1966), pp. 1705–1736.
3. Brilliant, M. B., 'The Determination of Frequency in Systems of Mutually Synchronized Oscillators," B.S.T.J., *45,* No. 10 (December 1966), pp. 1737–1748.
4. Brilliant, M. B., "Dynamic Response of Systems of Mutually Synchronized Oscillators," B.S.T.J., *46,* No. 2 (February 1967), pp. 319–356.
5. Sandberg, I. W., "Some Properties of a Nonlinear Model of a System for Synchronizing Digital Transmission Networks," B.S.T.J., *48,* No. 9 (November 1969), pp. 2975–2997.
6. Candy, J. C., and Karnaugh, M., "Organic Synchronization: Design of the Controls and Some Simulation Results," B.S.T.J., *47,* No. 2 (February 1968), pp. 227–259.
7. Pierce, J. R., "Synchronizing Digital Networks," B.S.T.J., *48,* No. 3 (March 1969), pp. 615–636.