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A Combinatorial Analysis of the Main Distributing Frame: Spare Requirements for Conversion to Preferential Assignment from Random Assignment

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This paper presents a combinatorial analysis of a mathematical model of the Main Distributing Frame (MDF). Results are found concerning the amount of additional spare needed to convert a randomly connected MDF to a preferentially assigned mode of operation. The analysis is first performed for an MDF having two sections and serving one class of service. The results are subsequently extended to an MDF with multiple sections and several classes of service.

I. INTRODUCTION

The Main Distributing Frame (MDF) is that equipment in a central office building whose main function is to permit the flexible interconnection of cable from outside the building to central office equipment inside the building. This primary function accounts for most of the terminal capacity of a typical MDF. The remaining capacity serves a wide variety of other cross-connecting functions ranging from tying together two outside plant cable pairs to cross-connecting two or more pieces of central office equipment.

The conventional MDF is a double-sided steel structure with protectors or terminal strips mounted on one side, and terminal strips on the other. These are referred to as the "verticals" and "horizontals" due to their mounting orientation. The vertical side is the part of the MDF where the outside cable is terminated. The horizontal side is the part of the MDF where cables, which connect to the equipment of a central office (mainly line and trunk equipment), are usually terminated. To provide the maximum capacity of interconnections as well as complete flexibility in the connection of any outside plant equipment to any central office equipment, administration of the MDF must result in small numbers of long cross-connections. This can only be accomplished by having the outside plant terminals opposite or nearly opposite the central office equipment terminals. The MDF at present is handled on a completely manual basis by framemen and framewomen who connect and disconnect the large number of wires which terminate on the MDF.

Cables terminated on the vertical side of the MDF are connected to the horizontal side by wires referred to as "cross-connections" or "jumpers." Where the outside plant terminals are not directly opposite the central office equipment terminals, these wires are run along horizontal shelves in order to connect these vertical and horizontal terminals together. Presently, assignment of outside plant terminals to central office equipment terminals is on a random basis, i.e., no attempt is made to make assignments so that jumper lengths are kept to a practical minimum. Therefore, when cross-connections become too numerous or too long they accumulate excessively, become unmanageable, and exceed the capacity (jumper volume) of the horizontal shelves.

One proposed method of keeping cross-connections short is the use of preferential assignment. In this mode of operation the MDF is divided into several zones, and records are kept which show in which zones cables and central office equipment are terminated. When connections are to be made, for example, between a cable pair and line circuit, a cable pair is first selected based on outside plant economics and availability. Then, line equipment is selected which is located in the same zone as the cable pair or in the closest zone to it. This method reduces jumper pileup and increases the MDF capacity. There arises the question of how many additional spare terminals and, therefore, how much central office equipment is required to convert from a randomly connected MDF to a preferentially assigned mode of operation. Although some people may feel that a prohibitive amount of spare would be needed, the following sections show that under the specified conditions a relatively small amount of spare is required by this conversion.

The approach used in this analysis is the following. Initially, it is assumed that there is a single class of service on the MDF. Given the initial number of wires which cross from one zone to another, and a model for the introduction of preferential assignment, the probability of exhausting spare in at least one zone can be found exactly for a 2-zone MDF and bounded for the general case. Since running a few jumpers between zones is not considered a problem, this probability is an upper bound on the probability of the MDF encountering trouble while converting to preferential assignment. For a given probability of MDF trouble, we can, therefore, find an upper bound on the amount of spare equipment required by conversion. If the number of zone crossings is not known for a randomly connected MDF, the probability of exhausting spare can be averaged over the joint probability of the number of crossovers of each kind.

When dealing with several classes of service,* each service can be treated as a separate MDF with its own zone structure. To do this we make a simplifying assumption about the manner in which changes in service occur.

It should be noted that the analysis performed here finds the additional spare required initially for conversion to preferential assignment. However, once the conversion is completed, all that is needed is some small amount of spare to effectively administer service changes.

II. DISTRIBUTION OF CROSSOVERS IN A 2-ZONE MDF

In this section and in the next section, a model is developed for a 2-zone MDF. This model can be used to determine how much spare is required for conversion to preferential assignment. In later sections the more general and more useful case of multizone MDF is considered.

As illustrated in Fig. 1, zone 1 and zone 2 of the MDF terminate N_1 and N_2 outside plant cable pairs (side O) respectively. On the inside plant side (side I) there are $N_1 + S_1$ and $N_2 + S_2$ terminals

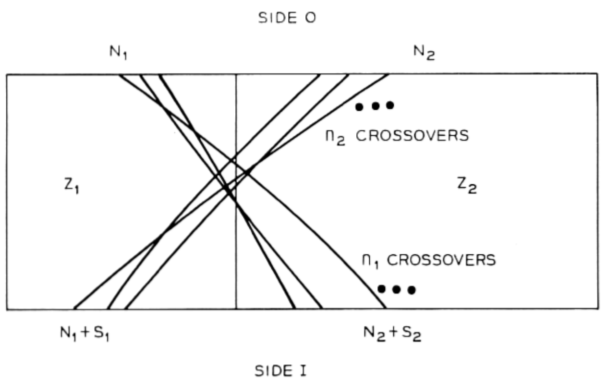

Fig. 1—Model of a 2-zone MDF.

* This term usually refers to the class of subscriber's line. Here this term will be used in a more general sense and will refer to various classes of all equipment or cable pairs terminated on the MDF.

respectively. S_1 and S_2 are the nominal number of spare equipment terminals available for each zone of the MDF.

First, assume that the MDF is used to capacity (i.e., all $N_1 + N_2$ outside plant terminals are connected to equipment or inside plant terminals). Further, assume that the terminals are randomly connected. We wish to find the probability $p(n_1, n_2)$ of having exactly n_1 jumpers crossing from side O of zone 1 to side I of zone 2, and exactly n_2 jumpers crossing from side O of zone 2 to side I of zone 1. This situation is illustrated in Fig. 1. After finding a suitable model to account for preferential assignment, the probability of exhausting spare at any time during the implementation of preferential assignment will be found using $p(n_1, n_2)$. This will be done in Section III.

In order to calculate $p(n_1, n_2)$ we assume all possible ways, W_0, to wire the MDF are equally likely (random connections). $p(n_1, n_2)$ is, then, the ratio of $W(n_1, n_2)$, the total number of ways to connect the MDF with n_1 and n_2 crossovers of the respective types, to W_0.

First, let us find W_0. As a first step, we choose at random $N_1 + N_2$ terminals of side I to be used. We can do this in $C(T, N)$* ways where $T = N_1 + S_1 + N_2 + S_2$ and $N = N_1 + N_2$. In addition, we will use the following notation: $S = S_1 + S_2$, $T_1 = S_1 + N_1$, and $T_2 = S_2 + N_2$. Now let us permute the N terminals we are using; there are $N!$ ways. Since the two operations of choosing and permuting are independent, the total number of ways to connect the MDF is the product of the separate enumerations, or

$$W_0 = N!\,C(T, N) = T!/S!. \tag{1}$$

Next, $W(n_1, n_2)$ is found as follows. First, let us choose n_2 terminals from side I of zone 1, which will accommodate jumpers from side O of zone 2. There are $C(T_1, n_2)$ distinct ways to make this choice. Likewise, there are $C(T_2, n_1)$ distinct ways of making the analogous choice for zone 2. Now zone 1 has $N_1 - n_1$ unused terminals on side O which will require direct connection to the remaining terminals of side I of zone 1. There are $T_1 - n_2$ unused terminals on side I of zone 1 (n_2 have been used to connect to side O of zone 2). Therefore, there are $C(T_1 - n_2, N_1 - n_1)$ possible ways to make direct connection in zone 1. Similarly, the direct connection in zone 2 can be made in any of $C(T_2 - n_1, N_2 - n_2)$ ways. Thus far, N_1 terminals from side I ($N_1 - n_2$ from zone 1 and n_2 from zone 2) have been chosen for con-

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* $C(n, k) = \binom{n}{k} = n!/k!(n-k)!$

nection to the N_1 terminals on side O of zone 1. Upon choosing the N_1 terminals, each permutation of the chosen terminals corresponds to a unique configuration. There are $N_1\,!$ such permutations. Similarly, there are $N_2\,!$ permutations for the N_2 terminals of side O of zone 2. Since each of the enumerations listed above is independent of the others, the total number of configurations, $W(n_1, n_2)$, is the product of each of the individual enumerations or

$$W(n_1, n_2) = C(T_1, n_2)C(T_2, n_1)C(T_1 - n_2, N_1 - n_1) \cdot C(T_2 - n_1, N_2 - n_2)N_1!\,N_2!.$$

Upon expanding into factorials, simplifying, and regrouping, we have

$$W(n_1, n_2) = C(N_1, n_1)C(N_2, n_2)C(S, S_1 + n_1 - n_2)\frac{T_1!\,T_2!}{S!}. \qquad (2)$$

$p(n_1, n_2)$ is the ratio $W(n_1, n_2)/W_0$ or

$$p(n_1, n_2) = C(N_1, n_1)C(N_2, n_2)C(S, S_1 + n_1 - n_2)/C(T, T_1). \qquad (3)$$

III. PROBABILITY OF EXHAUSTING SPARE IN A 2-ZONE MDF

Now suppose that a 2-section MDF has n_1 and n_2 crossovers of the two types discussed above; and that it is decided to begin using preferential assignment. Note that simplifying assumptions are made that the MDF is filled to capacity (i.e., all $N_1 + N_2$ outside plant terminals are connected to equipment or inside plant terminals) and that all lines have the same class of service.*

With preferential assignment, the desire is to keep all jumpers within the zone, i.e., not to have crossovers from one zone to another. Suppose a change occurs for a jumper which does not cross the boundary. If no spare is available in that zone, the change cannot occur (unless the boundary is crossed, which we will assume is not allowed). However, if a spare is available, there will be just as much spare available after the change. Clearly then, a change for such a line does not affect the total amount of available spare. Therefore, since our goal is to find the probability of running out of spare while implementing preferential assignment, we need only consider the $n_1 + n_2$ lines that cross the boundary between sections.

Since we start with n_1 crossovers from side O of zone 1 to side I of

* It is shown in a later section that under certain restricting conditions, the results for a single class of service can be extended to the case of many classes of service.

zone 2 and with n_2 crossovers from side O of zone 2 to side I of zone 1, the number of remaining spare terminals (on side I) is $S_1 + n_1 - n_2$ for zone 1 and $S_2 + n_2 - n_1$ for zone 2. The process of changing service for the $n_1 + n_2$ lines can be described as a random walk starting at (0, 0) with $n_1 + n_2$ epochs. At each epoch (corresponding to a change for one of the $n_1 + n_2$ lines) we move to the right and one step up if one of the n_1 lines has a change, and a step down if one of the n_2 lines has a change. We denote the location of a point on the path by (h_r, r) where h_r is the height at epoch r. This situation is described in Fig. 2. The final point of the random walk will, in all cases, be $(n_1 - n_2, n_1 + n_2)$ as illustrated in Fig. 2. Each step up in the random walk brings zone 1 one step closer to running out of spare and zone 2 one step further from running out of spare, and conversely for steps down. The probability of exhausting spare, $\Pr\{\text{exhaust}/n_1, n_2\}$, is then the probability that a randomly chosen path from (0, 0) to $(n_1 - n_2, n_1 + n_2)$ (there are $C(n_1 + n_2, n_1)$ such paths) touches or crosses one or both of the boundaries at height $S_1 + n_1 - n_2$ and height $-S_2 + n_1 - n_2$.

More precisely stated, our problem is to determine the probability

$$\Pr\{\text{exhaust}/n_1, n_2\} = 1 - \Pr\{-S_2 + n_1 - n_2 < h_r < S_1 + n_1 - n_2 \quad \text{for} \quad r = 1, 2, \cdots, n_1 + n_2\}. \tag{4}$$

As discussed in Appendix A, this problem is equivalent to a form of the ballot problem[1] and the solution to our problem becomes

$$\Pr\{\text{exhaust}/n_1, n_2\} = 1 - \sum_k [C(n_1 + n_2, n_2 - kS) - C(n_1 + n_2, S_1 + n_1 + kS)]/C(n_1 + n_2, n_1) \tag{5}$$

where the summations are over all integers (positive, negative, and zero) for which the summands exist.

If we know n_1 and n_2 in advance this is the desired result. However, if we do not have this knowledge (e.g., if we do not wish to count all of the crossovers) this result should be weighted by the joint probability of n_1 and n_2. Now let us assume that we have no *a priori* information about n_1 and n_2. To find the probability of exhausting spare, equation (5) is weighted by $p(n_1, n_2)$, i.e.,

$$\Pr\{\text{exhaust}\} = 1 - \sum_{n_1} \sum_{n_2} \Pr\{\text{exhaust}/n_1, n_2\} p(n_1, n_2). \tag{6}$$

Substituting equations (3) and (4) into equation (6) and simplifying yields

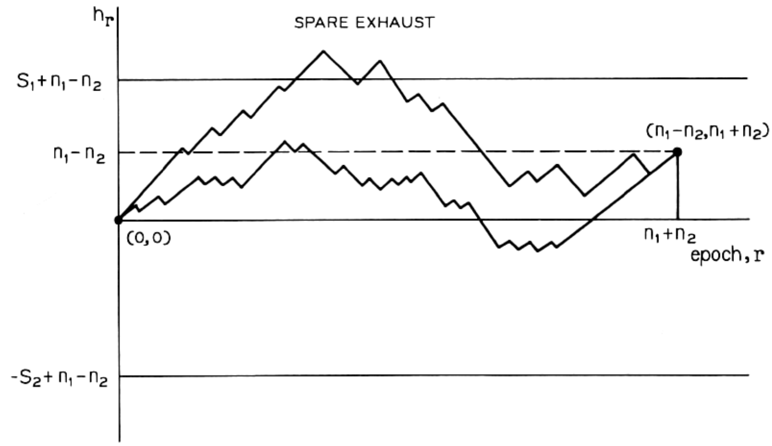

Fig. 2—Random walk model for a 2-zone MDF.

$$\Pr\{\text{exhaust}\} = 1 - \sum_{n_1}\sum_{n_2}\left\{\sum_k [C(n_1+n_2, n_2-kS) - C(n_1+n_2, S_1+n_1+kS)] \times \frac{C(N_1, n_1)C(N_2, n_2)C(S, S_1+n_1-n_2)}{C(n_1+n_2, n_1)C(T, T_1)}\right\}. \quad (7)$$

Expanding binomial coefficients in terms of factorials, simplifying, and regrouping yields

$$\Pr\{\text{exhaust}\} = 1 - \frac{1}{C(N_1+N_2, N_1)C(T, T_1)} \times \sum_{n_1}\sum_{n_2}\sum_k [C(N_1+N_2, N_1+kS)C(S, S_1+n_1-n_2) \times C(N_2-kS, n_2-kS)C(N_1+kS, N_1-n_1) - C(N_1+N_2, T_1+kS)C(S, S_1+n_1-n_2) \times C(T_1+kS, N_1-n_1)C(N_2-S_1-kS, n_2-S_1-kS)]. \quad (8)$$

At this point the order of summation is interchanged, summing first with respect to n_1 then with respect to n_2. In each case the basic combinatorial identity of Appendix B [equation (23)] is used with the final result:

$$\Pr\{\text{exhaust}\} = 1 - \frac{\sum_k [C(T, T_1-kS)C(N, N_2-kS) - C(T, N_1-kS)C(N, T_1+kS)]}{C(T, T_1)C(N, N_1)}. \tag{9}$$

As an example of the result of this equation, see Fig. 3. The curves are plots of Pr{exhaust} versus nominal percent spare, s ($s = S/N$), for 2-zone (equal size) MDFs with 10^4, 10^5, and 10^6 lines* and one class of service.

Equation (9) gives the probability of exhausting spare in one or both of the two zones. To find the probability of exhausting in a single zone, say zone 1, we wish to find the probability

$$\Pr\{Z_1 \text{ exhausts}\} = \sum_{n_1} \sum_{n_2} \Pr\{Z_1 \text{ exhausts}/n_1, n_2\} p(n_1, n_2).$$

The probability conditioned on n_1 and n_2 can be written

$$\Pr\{Z_1 \text{ exhausts}/n_1, n_2\} = 1 - \Pr\{h_r < S_1 + n_1 - n_2 \quad \text{for} \quad r = 1, 2, \cdots, n_1 + n_2\}. \tag{10}$$

As discussed in Appendix A this is a one-sided ballot problem[1] with solution

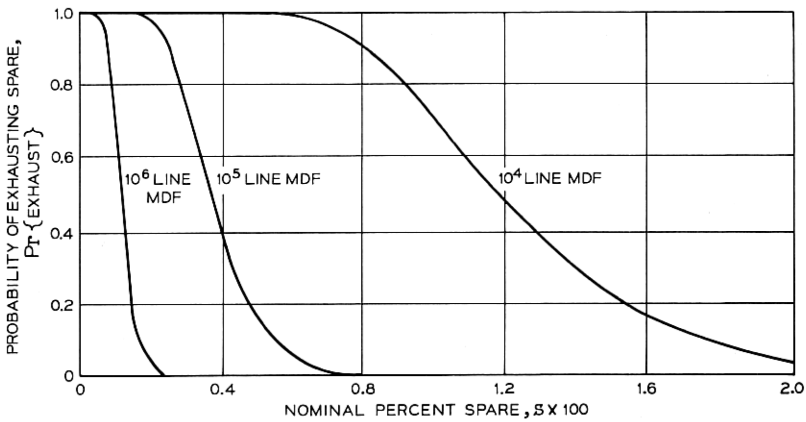

Fig. 3—Probability of exhausting spare vs nominal percent spare for a 2-zone (equal size) MDF.

* Note that "line" is being used here to mean cable termination of any kind on the MDF.

$$\Pr\{Z_1 \text{ exhausts}/n_1, n_2\} = \frac{C(n_1 + n_2, S_1 + n_1)}{C(n_1 + n_2, n_1)}.$$

Again, if we know n_1 and n_2 exactly, then this is the desired solution. However, we assume no such *a priori* knowledge, and upon multiplying by $p(n_1, n_2)$ and summing over n_1 and n_2 (as we did for Pr{exhaust}), the result is

$$\Pr\{Z_1 \text{ exhausts}\} = \frac{C(T_2 + N_2, N_2 - S_1)}{C(T_2 + N_2, N_2)}. \tag{11}$$

Figure 4 shows plots of equation (11) for zones which comprise one-half of an MDF with 10^4, 10^5, and 10^6 lines and one class of service.

In the case of a 2-zone MDF, an exact solution has been found for the probability of exhausting spare in the entire MDF and for the probability that a particular zone exhausts spare. As will be shown in the next two sections, the general solution for the probability for exhausting spare in an M-section MDF is difficult to find. Instead, the result [equation (11)] for a particular section can be used to find upper and lower bounds on the desired probability.

IV. DISTRIBUTION OF CROSSOVERS IN AN M-ZONE MDF

In this section the distribution of crossovers for an M-zone MDF with a single class of service is found in much the same way as it was found for a 2-zone MDF in Section II.

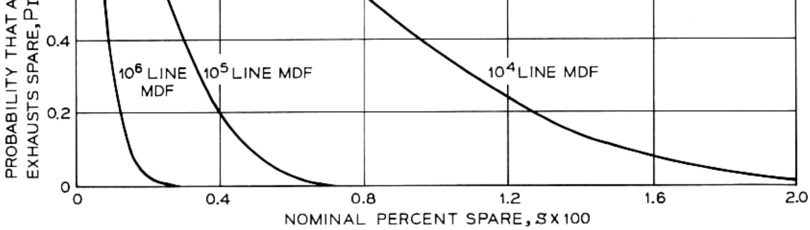

Fig. 4—Probability that a particular zone exhausts spare vs nominal percent spare for a 2-zone (equal size) MDF.

The configuration of the M-zone MDF is illustrated in Fig. 5. For simplicity, the following notation will be used:

$$T_i = N_i + S_i ,$$

$$S = \sum_{i=1}^{M} S_i ,$$

$$T = \sum_{i=1}^{M} T_i .$$

In addition, we define n_{ij} to be the number of crossovers from side O of zone i to side I of zone j with $n_{ii} = 0$ by definition. In this case the number of distinct ways to wire the MDF is again given by W_0 of equation (1), using the more general definitions of T and S above.

Now we must determine for a given matrix $\{n_{ij}\}$ of crossovers the number, $W(\{n_{ij}\})$, of distinct connections of the MDF that have $\{n_{ij}\}$ crossovers of each type. Let us concentrate on the ith zone. Side I of this zone must supply the other zones with $\Sigma_k\, n_{ki}$ terminals to connect to side O of those zones. The number of ways to do this is

$$(T_i \,; n_{1i} , n_{2i} , \cdots , n_{Mi} , T_i - \sum_k n_{ki})^*$$

where the summation is from $k = 1$ to $k = M$. As far as direct connections are concerned, $N_i - \Sigma_k\, n_{ik}$ terminals must be chosen from the $N_i + S_i - \Sigma_k\, n_{ki}$ terminals remaining on side I. This may be done in $C[T_i - \Sigma_k\, n_{ki} , N_i - \Sigma_k\, n_{ik}]$ ways. In addition, after choosing the particular terminals to be connected, the N_i terminals on side O of zone i can be permuted in $N_i!$ ways. Each of the choices mentioned above

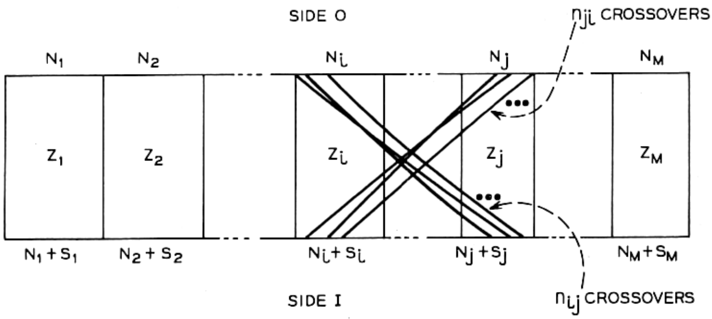

Fig. 5—Model of an M-zone MDF.

* $(a;\, a_1 , a_2 , \cdots , a_n) = a!(a_1 !\, a_2 ! \cdots a_n !)^{-1}$.

is independent for each zone. Therefore, the number of ways to connect the MDF with $\{n_{ij}\}$ crossovers is the product of the three numbers enumerated above for each zone, viz.,

$$W(\{n_{ij}\}) = \prod_{i=1}^{M} C(T_i - \sum_k n_{ki}, N_i - \sum_k n_{ik})N_i ! \cdot (T_i ; n_{1i}, n_{2i}, \cdots, n_{Mi}, T_i - \sum_i n_{ki}).$$

The probability, $p(\{n_{ij}\})$, of having $\{n_{ij}\}$ crossovers is equal to the ratio $W(\{n_{ij}\})/W_0$ or upon regrouping

$$p(\{n_{ij}\}) = \frac{(S; S_1 + \sum_k (n_{1k} - n_{k1}), S_2 + \sum_k (n_{2k} - n_{k2}), \cdots, S_M + \sum_k (n_{Mk} - n_{kM}))}{(T; T_1, T_2, \cdots, T_M)} \times \prod_{i=1}^{M} (N_i ; n_{i1}, n_{i2}, \cdots, n_{iM}, N_i - \sum_k n_{ik}). \tag{12}$$

Equation (12) is, then, the distribution of crossovers for an M-zone MDF. Equation (12) reduces to equation (3) for $M = 2$, in which case $n_{12} = n_1$ and $n_{21} = n_2$.

V. PROBABILITY OF EXHAUSTING SPARE IN AN M-ZONE MDF

The remarks about the random walk model of Section III can be generalized in the following way. Let there be a random walk for each zone starting at (0, 0). The number of epochs is equal to the total number of crossovers, viz., $\Sigma_i \Sigma_j n_{ij}$. For zone i, the height of the random walk at epoch r is h_{ir}. Each of the M random walks moves one step to the right at each epoch. In addition, each random walk moves up each time spare is reduced and down when spare is increased in its corresponding zone. Thus, at each epoch one walk moves up, one moves down, and the others exhibit no change in height. Each random walk has associated with it a boundary, $S_i + \Sigma_j n_{ij} - \Sigma_j n_{ji}$, and there is a further restriction that each walk reach the final point $[\Sigma_i \Sigma_j n_{ij}, \Sigma_j n_{ij} - \Sigma_j n_{ji}]$. The probability that the MDF exhausts is

$$\Pr\{\text{exhaust}\} = 1 - \Pr\{h_{ir} < S_i + \sum_j (n_{ij} - n_{ji}) \quad \text{for all } i = 1, 2, \cdots, M \quad \text{and for all } r = 1, 2, \cdots, \sum_i \sum_j n_{ij}\}. \tag{13}$$

This problem is equivalent to an M candidate ballot problem as described in Appendix C. This problem, unlike the simpler ballot problems of Appendix A, is as yet unsolved. However, upper and lower bounds for equation (13) can be found, as shown in the next section.

VI. BOUNDS ON THE PROBABILITY OF EXHAUSTING SPARE IN AN M-ZONE MDF

In this section, the result [equation (11)] for the probability of exhausting spare in a single zone is used to find upper and lower bounds on the probability of exhausting spare in an M-zone MDF.

If we denote the M zones by $Z_1, Z_2, \cdots, Z_M$, we have

$$\Pr\{\text{exhaust}\} = \Pr\{Z_1 \text{ or } Z_2 \text{ or } \cdots \text{ or } Z_M \text{ exhausts}\}. \tag{14}$$

For simplicity let us assume that zones are indistinguishable,* i.e., that $N_1 = N_2 = \cdots = N_M$ and that $S_1 = S_2 = \cdots = S_M$. Now equation (14) can be rewritten as[2]

$$\begin{aligned}\Pr\{\text{exhaust}\} &= M \Pr\{Z_1 \text{ exhausts}\} - C(M, 2) \Pr\{Z_1 \text{ and } Z_2 \text{ exhaust}\}\\ &\quad + C(M, 3) \Pr\{Z_1 \text{ and } Z_2 \text{ and } Z_3 \text{ exhaust}\} + \cdots\\ &\quad + (-1)^{M-1} \Pr\{Z_1 \text{ and } Z_2 \text{ and } \cdots \text{ and } Z_M \text{ exhaust}\}.\end{aligned} \tag{15}$$

Successive partial sums of equation (15) oscillate about Pr {exhaust} and in particular

$$\begin{aligned}M \Pr\{Z_1 \text{ exhausts}\} \geqq \Pr\{\text{exhaust}\} &\geqq M \Pr\{Z_1 \text{ exhausts}\}\\ &\quad - \frac{M(M-1)}{2} \Pr\{Z_1 \text{ and } Z_2 \text{ exhaust}\}.\end{aligned} \tag{16}$$

Given a randomly wired MDF, a zone is less likely to exhaust if it is known that another zone has exhausted, i.e.,

$$\Pr\{Z_1 \text{ exhausts}/Z_2 \text{ exhausts}\} \leqq \Pr\{Z_1 \text{ exhausts}\}$$

and so

$$\Pr\{Z_1 \text{ and } Z_2 \text{ exhaust}\} \leqq \Pr\{Z_1 \text{ exhausts}\} \Pr\{Z_2 \text{ exhausts}\}.$$

Substituting into equation (16) and recalling that $\Pr\{Z_1 \text{ exhausts}\} = \Pr\{Z_2 \text{ exhausts}\}$ (since the zones are of equal size) yields

$$M \Pr\{Z_1 \text{ exhausts}\} \geqq \Pr\{\text{exhaust}\} \geqq M \Pr\{Z_1 \text{ exhausts}\}$$

* This assumes an even spread of equipment, which is not necessarily true in practice but can be approximately achieved through retermination of equipment. Retermination to achieve spreading has been done to enhance the effectiveness of preferential assignment.

$$- \frac{M(M-1)}{2} (\Pr \{Z_1 \text{ exhausts}\})^2. \qquad (17)$$

Equation (11) gives $\Pr\{Z_1 \text{ exhausts}\}$ for a 2-zone MDF. One can consider the M-zone MDF as having two zones, one with N_1 terminals and S_1 spares and the second with $(M-1)N_1$ terminals and $(M-1)S_1$ spares. Thus, in equation (17),

$$\Pr \{Z_1 \text{ exhausts}\} = \frac{C((M-1)(T_1+N_1), (M-1)N_1 - S_1)}{C((M-1)(T_1+N_1), (M-1)N_1)} \qquad (18)$$

and with the substitution, equation (17) gives the resulting upper and lower bounds for the probability of exhausting spare in any section of a multizone MDF with a single class of service.

As an example of the results that can be obtained from equation (17), Fig. 6 shows plots of the upper and lower bounds on Pr{exhaust} for 10-section MDFs having 10^4, 10^5, and 10^6 lines. The upper bound is shown as a solid line and the lower bound is shown as a dashed line and is incompletely drawn. Figure 7 shows plots of bounds on Pr{exhaust} for 10^4-, 10^5-, and 10^6-line MDFs with 10^3-line zones. Pr{exhaust} for a 10^5-line MDF with 5, 10, and 20 zones is plotted in Fig. 8. The meaning of these results will be discussed in greater detail in Section VIII.

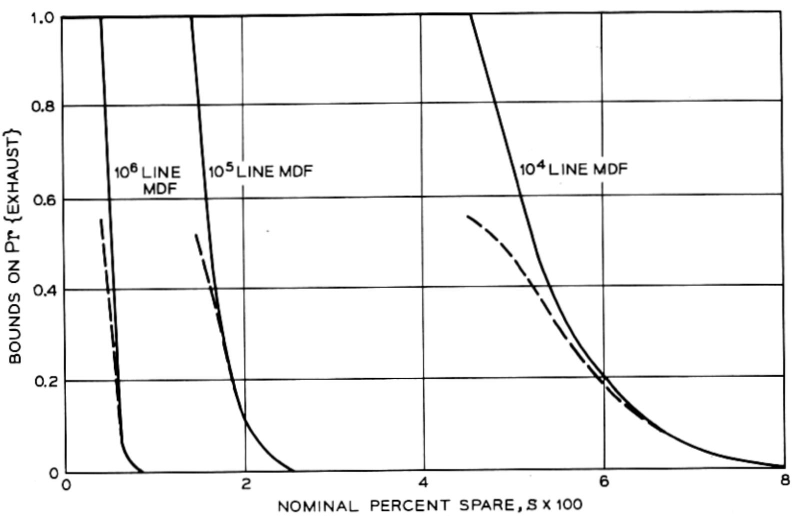

Fig. 6—Bounds on probability of exhausting spare for a 10-zone MDF, for 10^4, 10^5, and 10^6 lines (solid line—upper bound, broken line—lower bound).

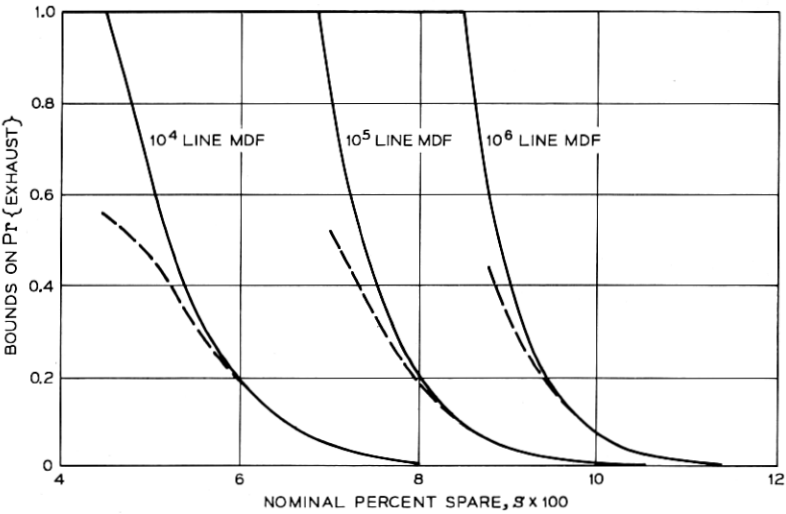

Fig. 7—Bounds on probability of exhausting spare for 10^4-, 10^5-, and 10^6-line MDFs with 10^3-line zones (solid line—upper bound, broken line—lower bound).

VII. SEVERAL CLASSES OF SERVICE

The analysis up to this point has considered a single class of service. The results for one class of service can be extended to the case of several classes of service, $CS_1, CS_2, \cdots, CS_P$, if we assume that a cable pair is always used for the same class of service. Then each class of service can be considered to comprise its own sub-MDF. This is also true for the less restrictive assumptions which follow.

Assume first that the working and spare equipment for each service is evenly distributed throughout the M zones of the MDF. In addition, assume that whenever a customer in zone Z_i requests a change in service from CS_j *to* CS_k, another customer in Z_i makes the symmetric request for change from CS_k to CS_j.*

With these assumptions, we can consider the single M-zone MDF with P classes of service to consist of P sub-MDFs each with a single class of service and M zones. Side O of each of the P sub-MDFs is comprised of the terminals equipped for that particular class of service. At any time, side I of each of the P sub-MDFs consists of all side I terminals of the total MDF which are connected to equipment of the particular class of service for that sub-MDF.

* This is unlikely to always be true in an instantaneous sense. However, it is reasonable in an average sense if the make-up of services on the total MDF is stationary.

As an example of the use of this analysis, consider the following case. A 100,000-line MDF has three classes of service. Class A serves 70,000 lines, class B serves 25,000 lines, and class C serves 5,000 lines. As described above, each class of service can be assumed, for the purpose of analysis, to comprise its own sub-MDF. Suppose a 10-zone preferential assignment procedure is introduced. After considering each class of service as having its own sub-MDF and applying the analysis of Section VI, we find the results illustrated in Fig. 9.

To assure negligible Pr{exhaust} during conversion to preferential assignment class A needs about 3 percent spare (2100 equipped spare terminals), class B requires about 5 percent spare (1250 equipped spare terminals), and class C requires about 12 percent spare (600 equipped spare terminals). Further, if there were only one class of service, Fig. 6 shows that about 2.5 percent spare would be required or 2500 equipped spare terminals as compared to 3950 for our example with three classes of service.

Thus, we can conclude that for a given MDF size, multiple classes of service require more spare equipment to convert to preferential assignment. Furthermore, just as a small MDF requires more percent spare than a larger MDF, a class of service provided to a small number of lines in an MDF requires more percent spare than a class serving

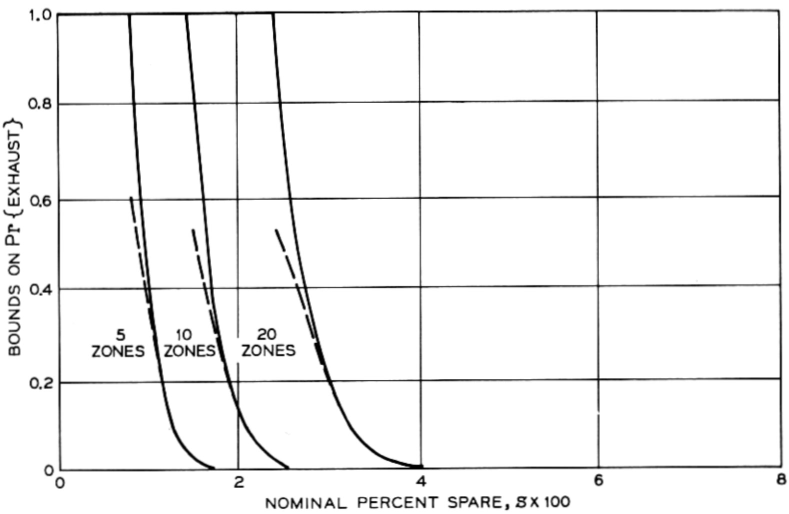

Fig. 8—Bounds on probability of exhausting spare for a 10^5-line MDF with 5, 10, and 20 zones (solid line—upper bound, broken line—lower bound).

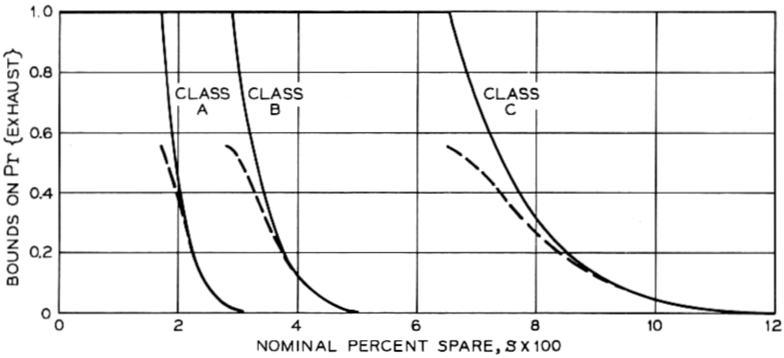

Fig. 9—Bounds on probability of exhausting spare in an MDF with three classes of service (solid line—upper bound, broken line—lower bound).

more lines. Thus, there is an economy of scale for spare equipment during conversion to preferential assignment.

VIII. DISCUSSION OF RESULTS

From Fig. 6 we see that for an MDF serving lines with the same class of service, Pr{exhaust} is negligible in a 10-zone, 10^6-line MDF if we have about 1 percent spare. For a 10-zone, 10^5-line MDF about 2.5 percent spare is sufficient and for a 10-zone, 10^4-line MDF about 8 percent spare is required. Thus, if we fix the number of zones, the required percent spare decreases as the size of the MDF increases. On the other hand, if, as in Fig. 7, zone size is fixed at say 10^3 lines, larger MDFs require more percent spare to attain the same value of Pr{exhaust}.

Figure 8 illustrates the effect of zone size on an MDF of a given size. For a 10^5-line MDF, 5 zones require about 1.75 percent spare for negligible Pr{exhaust}. If we have 10 zones, about 2.5 percent spare is required, while about 4 percent spare is required for 20 zones.

It should be remembered that an MDF will not be in trouble simply because spare has exhausted. Running a "few" jumpers across the boundary between zones is not considered a problem. Trouble occurs when this happens too often. Thus, the spare requirements stated above, based on Pr{exhaust}, are really upper bounds on the spare required for conversion to preferential assignment. We can heuristically say that the tightness of Pr{exhaust} as an upper bound on the probability of trouble increases with the size of the MDF. This is due to the fact that running jumpers between zones is more detrimental to larger MDFs.

In addition to analyzing a single class of service, a method for analyzing the case of multiple classes of service has been discussed. The results indicate that multiple classes increase the need for increased spare, since smaller classes require more percent spare than larger classes. It should be noted that the remark that Pr{exhaust} is a better upper bound for larger MDFs carries over to multiple classes of service. Specifically, Pr{exhaust} is a tighter upper bound for the probability of getting into trouble for a class of service which serves many lines rather than a few lines.

The example for the case of multiple classes of service treated three relatively small classes of service. In practice central offices employ thirty or more subscriber classes of service, and several hundred other classes of cross-connections. If all classes of service were of equal size, then spare requirements could become quite large. Usually, however, a few classes of service account for most of the terminations on the MDF, while the many remaining classes of service have a relatively small number of terminations. These less widely used classes would require a disproportionate amount of spare equipment. However, if less spare were provided for these classes, the resulting number of long cross-connections would be small if such classes constitute a small percentage of MDF terminations.

IX. CONCLUSIONS

Preferential assignment is a method to reduce the volume of wire in the MDF and thus increase the effective MDF capacity. The analysis presented here investigates the spare requirements for conversion of an MDF to preferential assignment from random assignment. Starting with a fully loaded, randomly connected MDF, conversion to preferential assignment first requires an even distribution of circuit types over the MDF. When, as discussed above, a few classes of service account for most of the terminations on the MDF, the overall percent spare required for conversion to preferential assignment is in the order of a few percent. If this assumption is not valid, then spare requirements may become relatively large. In either case the analysis presented here provides a useful method for estimating spare requirements.

X. ACKNOWLEDGMENTS

The author wishes to express his appreciation to P. J. Burke for his suggestions and, in particular, for pointing out Takács' solution to the 2-candidate ballot problem. The author is also grateful to C. W. Zebe for numerous discussions and suggestions. Thanks are also due to the reviewers for their constructive criticism of the original manuscript.

APPENDIX A

Ballot Problems and the Solution for a 2-Zone MDF

As stated in Section III, the problem of finding $\Pr\{\text{exhaust}/n_1, n_2\}$ is equivalent to the ballot problem (Ref. 1, problem 4) which states: In a ballot, candidate A scores a votes and candidate B scores b votes, and all the possible voting records are equally probable. Let $c - d < b - a < c$ where $0 < c < d$ are integers. Denote by α_r and β_r the number of votes registered for A and B, respectively, among the first r votes recorded. Find the probability

$$P = \Pr\{c - d < \beta_r - \alpha_r < c \quad \text{for} \quad r = 1, 2, \cdots, a + d\}. \tag{19}$$

Takács[1] shows the solution to be

$$P = \sum_k [C(a + b, a - kd) - C(a + b, a + c + kd)]/C(a + b, a). \tag{20}$$

To apply this solution to the problem at hand, let $b = n_1$, $a = n_2$, $c = S_1 + n_1 - n_2$, $d = S_1 + S_2$. Also, let β_r be the number of upward steps and α_r be the number of downward steps after r epochs in the random walk model of Fig. 2. Therefore, in the notation of Section III,

$$h_r = \beta_r - \alpha_r .$$

Making these substitutions, we have

$$\Pr\{-S_2 + n_1 - n_2 < h_r < S_1 + n_1 - n_2 \ \text{for} \ r = 1, 2, \cdots, n_1 + n_2\}$$
$$= \sum_k [C(n_1+n_2, n_2-kS) - C(n_1+n_2, S_1+n_1+kS)]/C(n_1+n_2, n_1).$$

This is the result reported in equations (4) and (5).

The problem of finding $\Pr\{\text{zone 1 exhausts}/n_1, n_2\}$ is equivalent to the one-sided ballot problem (Ref. 1, problem 3) which states: In a ballot, candidate A scores a votes and candidate B scores b votes, and all the possible voting records are equally likely. Let $b < a + c$ where c is a positive integer. Let α_r and β_r be the number of votes registered for A and B, respectively, among the first r votes recorded. Find the probability

$$Q_c(a, b) = \Pr \{\beta_r < \alpha_r + c \quad \text{for} \quad r = 1, 2, \cdots, a + b\}.$$

The solution is shown to be

$$Q_c(a, b) = 1 - \frac{C(b, c)}{C(a + c, c)}.$$

Upon making the substitutions that were made for the two-sided ballot problem, we have

$$\Pr \{h_r < S_1 + n_1 - n_2\} = 1 - \frac{C(n_1, S_1 + n_1 - n_2)}{C(S_1 + n_1, S_1 + n_1 - n_2)}.$$

After expanding binomial coefficients and regrouping, the result is

$$P\{h_r < S_1 + n_1 - n_2\} = 1 - \frac{C(n_1 + n_2, S_1 + n_1)}{C(n_1 + n_2, n_1)}$$

as reported in equations (10) and (11).

APPENDIX B

A Basic Combinatorial Identity

The basic combinatorial identity

$$\sum_i \binom{r}{i} \binom{s}{m - i} = \binom{r + s}{m} \tag{21}$$

(where r, s, m, and i are integers and the sum is over all integers for which the summand exists) is easily proven[3] by considering the equation

$$(1 + x)^r (1 + x)^s = (1 + x)^{r+s}.$$

Expansion of each term in a binomial series yields

$$\sum_i \binom{r}{i} x^i \sum_j \binom{s}{j} x^j = \sum_m \binom{r + s}{m} x^m. \tag{22}$$

Equating terms on each side of equation (22) with exponent m results in equation (21).

Equation (22) can be expressed in another, more general form. If we let $i = k + j$ with k and i integers, equation (22) becomes

$$\sum_i \binom{r}{k+j}\binom{s}{m-k-j} = \binom{r+s}{m},$$

where the sum is again for all integers for which the summand exists. If we further let $n = m - k$ the result is

$$\sum_i \binom{r}{k+j}\binom{s}{n-j} = \binom{r+s}{k+n}. \tag{23}$$

APPENDIX C

An M-Candidate Ballot Problem

In a ballot there are M candidates, viz., $C_1, C_2, \cdots, C_M$. Each of N voters casts a ballot which contains a vote in favor of one candidate and a vote against another candidate. Let b_i and a_i be the total pro and con votes for C_i. Let k_{ij} be the total number of ballots which have a vote for C_i and a vote against C_j. Thus $b_i = \Sigma_j k_{ij}$ and $a_i = \Sigma_j k_{ji}$. It is assumed that all possible voting records are equally probable. Let β_{ir} and α_{ir} be the number of pro and con votes, respectively, registered for C_i among the first r ballots recorded. Find the probability

$$P = \Pr\{\beta_{ir} - \alpha_{ir} < d_i \text{ for each } i = 1, 2, \cdots, M \text{ and for each } r = 1, 2, \cdots, \sum_i a_i\}. \tag{24}$$

Note that $\Sigma_i a_i = \Sigma_i b_i$.

This problem is equivalent to the random walk model of Section V if we let $h_{ir} = \beta_{ir} - \alpha_{ir}$, $d_i = S_i + \Sigma_j (n_{ij} - n_{ji})$, and $\{k_{ij}\} = \{n_{ij}\}$.

This problem is much more complex than the ballot problems of Appendix A, and has not, as far as the author knows, been solved.

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