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Bending Losses of the Asymmetric Slab Waveguide

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The bending losses of the asymmetric slab waveguide are computed. The computation is based on the knowledge of the exact form of the solution of Maxwell's equations of the bent structure and the additional assumption that the field near the bent waveguide can be approximated by the field of the straight waveguide. The result of this theory is in good agreement with an existing theory. It appears that the bending loss formula can be used to estimate the bending losses of the round optical fiber if the mode parameters entering the formula are replaced by the corresponding mode parameters of the round fiber. We present curves that allow the numerical evaluation of the bending loss of the lowest order even TE mode of the symmetric slab waveguide.

I. INTRODUCTION

E. A. J. Marcatili has shown that a bent slab waveguide loses power by radiation.[1] His analysis is based on a solution of the eigenvalue equation of the bent waveguide. It is possible to derive the expression for the bending losses from an approximate theory that is much simpler than the solution of the eigenvalue equation. We use this method to derive the formula for the bending losses of an asymmetric slab waveguide. The symmetric slab waveguide is, of course, included in this treatment as a limiting case. The result of this approximate theory is in very good agreement with the theory of Marcatili. Furthermore, if the parameters of the HE_{11} mode of the round optical fiber are used in the slab waveguide formula, loss values are obtained that agree well with experimental loss values for this mode.[2]

The bending loss theory presented in this paper is based on the following idea. A bent slab waveguide can conveniently be described in a cylindrical coordinate system whose axis coincides with the center of curvature of the waveguide (Fig. 1). The solution of Maxwell's equations

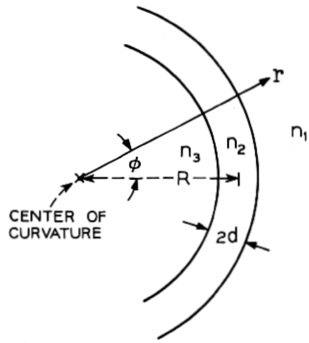

Fig. 1—Bent slab waveguide with a cylindrical coordinate system centered at the center of curvature.

in the cylindrical coordinate system is known so that the shape of the field distribution of the curved waveguide is known except for an undetermined amplitude factor and for the value of the order number of the cylinder function of the solution. Both of these unknown parameters can be obtained if we assume that the field in the vicinity of the waveguide must be similar to the field of the straight guide as long as the radius of curvature is large. The approximate solution is used to calculate the power that is radiated from the waveguide so that the power loss per unit length caused by the waveguide curvature can be determined. This procedure leads to a simple equation for the curvature loss. The theory breaks down when the curvature is so severe that the field near the waveguide can no longer be approximated by the field of the straight guide. The limits of applicability of the curvature loss theory can be expressed by inequalities for the waveguide parameters.

II. THE FIELD OF THE STRAIGHT ASYMMETRIC SLAB WAVEGUIDE

The field of the straight asymmetric slab waveguide is obtained as the solution of a straightforward boundary value problem. The geometry of the structure is shown in Fig. 2. We assume that there is no field variation in the y direction so that the waves of the structure are simple TE and TM modes. We limit our discussion to TE modes. The field is then given by the following equations:[3]

$$E_y = Ae^{-\gamma(x-d)} \qquad d \leqq x < \infty, \qquad (1a)$$

$$E_y = A\cos\kappa(x-d) - \frac{\gamma}{\kappa}\sin\kappa(x-d) \qquad -d \leqq x \leqq d, \qquad (1b)$$

$$E_y = A\left(\cos 2\kappa d + \frac{\gamma}{\kappa}\sin 2\kappa d\right)e^{\theta(x+d)} \qquad -\infty \leqq x \leqq -d. \tag{1c}$$

The amplitude of the field can be expressed by the power P carried by the field:

$$A = 2\kappa\left\{\frac{\omega\mu_o P}{\beta\left(2d + \frac{1}{\gamma} + \frac{1}{\theta}\right)(\kappa^2 + \gamma^2)}\right\}^{\frac{1}{2}}. \tag{2}$$

A factor $\exp[i(\omega t - \beta z)]$ has been suppressed. The constants and parameters appearing in equations (1) and (2) are defined as follows:

ω = $2\pi f$, radian frequency,
P = power carried by the mode,
μ_o = magnetic permeability of free space,
ϵ_o = electric permittivity of free space,
β = propagation constant,
$2d$ = slab thickness,

$$\kappa = (n_2^2k^2 - \beta^2)^{\frac{1}{2}}, \tag{3}$$

$$\gamma = (\beta^2 - n_1^2k^2)^{\frac{1}{2}}, \tag{4}$$

$$\theta = (\beta^2 - n_3^2k^2)^{\frac{1}{2}}, \tag{5}$$

$$k = \omega(\epsilon_o\mu_o)^{\frac{1}{2}}, \tag{6}$$

n_1 = refractive index in the region $d < x$,
n_2 = refractive index in the region $-d < x < d$,
n_3 = refractive index in the region $-\infty < x < -d$.

The magnetic field components are obtained from the equations

$$H_x = -\frac{i}{\omega\mu_o}\frac{\partial E_y}{\partial z} \tag{7}$$

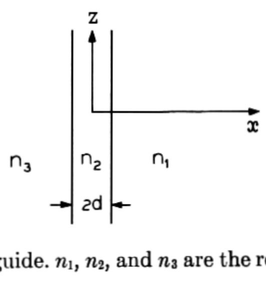

Fig. 2—Straight slab waveguide. n_1, n_2, and n_3 are the refractive indices of the three media.

and

$$H_z = \frac{i}{\omega\mu_o}\frac{\partial E_y}{\partial x}. \tag{8}$$

The boundary conditions (requirement of continuity of the transverse electric and magnetic field components at the two interfaces) lead to the eigenvalue equation for the propagation constant β,

$$\tan 2\kappa d = \frac{\gamma + \theta}{\kappa\left(1 - \frac{\gamma\theta}{\kappa^2}\right)}. \tag{9}$$

III. THE FIELD OF THE CURVED STRUCTURE

The solution of Maxwell's equations in the region $(R + d) < r < \infty$ can be expressed as follows:

$$E_y = BH_\nu^{(2)}(n_1 kr)e^{-i\nu\phi}. \tag{10}$$

The Hankel function of the second kind and of order ν represents an exact solution of Maxwell's equations in the coordinate system shown in Fig. 1. There is no field variation in the direction of the axis of the cylindrical polar coordinate system of Fig. 1. (In the coordinate system used in Fig. 2 the direction of the polar coordinate axis would be y.) The radial distance r is measured from the center of curvature of the bent waveguide. The Hankel function of the second kind is required since at infinite distance, $r \to \infty$, an outward traveling wave must result. With our time dependence, exp $(i\omega t)$, the field of equation (10) satisfies this requirement.

The order number ν need not be an integer in this case since we need not require periodicity of the field as a function of the polar angle ϕ. If we were interested in an exact solution of the problem of the mode traveling along a curved waveguide we would obtain the value of ν as the solution of an eigenvalue equation. In our approximate treatment we assume that the field near the waveguide can still be approximately described by the field of the straight structure. We can use the coordinate system of the straight guide (Fig. 2) to describe the curved guide. The z axis of the straight coordinate system becomes bent and we have the relation

$$z = R\phi. \tag{11}$$

The function exp $(-i\nu\phi)$ is equivalent to the propagation factor exp $(-i\beta z)$ of the straight waveguide so that we have the approximate relation

$$\beta = \frac{\nu}{R}. \tag{12}$$

Since R is much larger than the wavelength of the field, ν is a very large number. Using ν as defined by (12) in (10) constitutes the first approximation. It remains to obtain a relation between the amplitude factor B and the power that is carried by the guided mode. To achieve this we use an approximation for the Hankel function that is valid for very large order number ν in the region

$$\nu > n_1 kr. \tag{13}$$

The desired approximation can be found in Ref. 4.

$$H_\nu^{(2)}(n_1 kr) = -i \frac{e^{\nu(\alpha - \tanh \alpha)}}{\sqrt{\frac{\pi}{2} \nu \tanh \alpha}} \tag{14}$$

with

$$\cosh \alpha = \frac{\nu}{n_1 kr}. \tag{15}$$

We obtain the hyperbolic tangent of α by the relation

$$u = \tanh \alpha = \frac{[\cosh^2 \alpha - 1]^{\frac{1}{2}}}{\cosh \alpha} = \frac{\left[\beta^2 - \left(n_1 k \frac{r}{R}\right)^2\right]^{\frac{1}{2}}}{\beta}. \tag{16}$$

Equation (12) was used to replace ν by β. Near the axis of the waveguide we can use the approximation $r/R \approx 1$ so that we obtain from (4), (12), and (16)

$$\nu \tanh \alpha = \gamma R. \tag{17}$$

This approximation is adequate for the denominator of (14). The expression in the exponent must be approximated more accurately. We use the x coordinate to describe the radial distance from the center of the waveguide core and write

$$\frac{r}{R} = 1 + \frac{x}{R}. \tag{18}$$

This x coordinate corresponds directly to the x axis of the straight guide as shown in Fig. 2. Using (4) and (16) we obtain

$$u \approx \frac{\left(\gamma^2 - 2(n_1k)^2 \dfrac{x}{R}\right)^{\frac{1}{2}}}{\beta} \approx \frac{\gamma}{\beta}\left[1 - \left(\frac{n_1k}{\gamma}\right)^2 \frac{x}{R}\right]. \tag{19}$$

We can express α by a well-known relation between the inverse hyperbolic tangent function and the natural logarithm[4]

$$\alpha = \tanh^{-1} u = \tfrac{1}{2} \ln \left(\frac{1+u}{1-u}\right). \tag{20}$$

The logarithm can be expanded in an infinite series

$$\alpha = u + \tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + \ u^7 + \cdots. \tag{21}$$

From (19) we obtain approximately

$$u^p = \left(\frac{\gamma}{\beta}\right)^p - p\left(\frac{n_1k}{\gamma}\right)^2 \frac{x}{R}\left(\frac{\gamma}{\beta}\right)^p. \tag{22}$$

Substitution of (22) into (21) yields

$$\alpha = \frac{\gamma}{\beta} + \frac{1}{3}\left(\frac{\gamma}{\beta}\right)^3 + \frac{1}{5}\left(\frac{\gamma}{\beta}\right)^5 + \cdots - \left(\frac{n_1k}{\gamma}\right)^2 \frac{x}{R}\frac{\gamma}{\beta}\left[1 + \frac{\gamma^2}{\beta^2} + \left(\frac{\gamma^2}{\beta^2}\right)^2 + \left(\frac{\gamma^2}{\beta^2}\right)^3 + \cdots\right]. \tag{23}$$

The first series can again be expressed by the logarithmic function. The second part of (23) contains a simple geometric series. We thus obtain the approximation

$$\alpha = \tfrac{1}{2} \ln \left[\frac{1 + \dfrac{\gamma}{\beta}}{1 - \dfrac{\gamma}{\beta}}\right] - \frac{(n_1k)^2}{\gamma\beta} \frac{1}{1 - \dfrac{\gamma^2}{\beta^2}} \frac{x}{R}. \tag{24}$$

With the help of (4), (19), and (24) we can form the expression

$$\alpha - \tanh \alpha = \alpha - u = \tfrac{1}{2} \ln \left[\frac{1 + \dfrac{\gamma}{\beta}}{1 - \dfrac{\gamma}{\beta}}\right] - \frac{\gamma}{\beta} - \frac{\gamma}{\beta}\frac{x}{R}. \tag{25}$$

The approximations (12), (14), (17), and (25) allow us to express the E_y component (10) in the following approximate form:

$$E_y = -iB \frac{\exp\left\{\left[\dfrac{1}{2}\dfrac{\beta}{\gamma} \ln \left[\dfrac{1 + \dfrac{\gamma}{\beta}}{1 - \dfrac{\gamma}{\beta}}\right] - 1\right]\gamma R\right\}}{\sqrt{\dfrac{\pi}{2}\gamma R}} e^{-\gamma x} e^{-i\beta z}. \tag{26}$$

The functional dependence of (26) on the coordinate x coincides with (1a).

For large values of R the field of the curved structure near the waveguide is approximately equal to that of the straight guide. The factor $\exp(-i\beta z)$ appearing in (26) was omitted from (1a). Comparison of (1a) and (26) allows us, with the help of (2), to determine the amplitude coefficient of the field,

$$B = 2i\kappa e^{\gamma d}\left\{\frac{\frac{\pi}{2}\omega\mu_o\gamma RP}{\beta\left(2d + \frac{1}{\gamma} + \frac{1}{\theta}\right)(\kappa^2 + \gamma^2)}\right\}^{\frac{1}{2}} \cdot \exp\left\{-\left[\frac{1}{2}\frac{\beta}{\gamma}\ln\frac{1 + \frac{\gamma}{\beta}}{1 - \frac{\gamma}{\beta}} - 1\right]\gamma R\right\}. \tag{27}$$

Very far from the waveguide, $r \gg R$, the Hankel function can be expressed by its approximation for large argument[4] so that (10) assumes the form

$$E_y = B\sqrt{\frac{2}{\pi n_1 kr}}\, e^{-in_1kr}e^{i(2\nu+1)\pi/4}e^{-i\beta z}. \tag{28}$$

Equation (28) is very interesting. It shows that far from the waveguide the field is very different from the field of the straight structure. Whereas the field of the curved structure decays exponentially near the waveguide it assumes the form of a radiation field far from the guide. This behavior of the exact field solution (10) of the curved structure explains why curved dielectric waveguides lose power by radiation.

IV. THE BENDING LOSS FORMULA

Since we know the field far from the waveguide it is now easy to calculate the power loss caused by the fact that energy is radiated away from the waveguide. The amplitude of the radiation field is independent of the z coordinate. The z dependent factor in (28) determines only the phase of the field. The power loss suffered by the field at a given position z can thus be calculated by the radial power flow at the same position z, even though the contribution to this radiation may have come from a point z_1 with $z_1 \ll z$, because each length element of the guide contributes an equal amount of radiation. An element of unit length on the axis of the waveguide is projected on an element of arc length

$$L = \frac{r}{R} \tag{29}$$

at a distance $r - R$ from the waveguide. The power loss 2α per unit length of waveguide is thus

$$2\alpha = \frac{r}{R}\frac{S_r}{P}. \tag{30}$$

S_r is the r component of the Poynting vector,

$$S_r = -\tfrac{1}{2}E_y H_\phi^* = \tfrac{1}{2}n_1\sqrt{\frac{\epsilon_o}{\mu_o}}\,|E_y|^2, \tag{31}$$

and P is the power carried by the mode. Using (27), (28), and (31) we obtain from (30)

$$2\alpha = \frac{2\gamma\kappa^2 e^{2\gamma d} e^{-U}}{(n_2^2 - n_1^2)k^2\beta\left(2d + \dfrac{1}{\gamma} + \dfrac{1}{\theta}\right)} \tag{32}$$

with

$$U = \left\{\frac{\beta}{\gamma}\ln\left|\frac{1 + \dfrac{\gamma}{\beta}}{1 - \dfrac{\gamma}{\beta}}\right| - 2\right\}\gamma R \approx \frac{2}{3}\frac{\gamma^2}{\beta^2}\gamma R. \tag{33}$$

The approximation on the right-hand side of (33) holds for $\gamma/\beta \ll 1$. The relation $\kappa^2 + \gamma^2 = (n_2^2 - n_1^2)k^2$ was used to simplify (32). The range of validity of the bending loss formula (32) cannot be given precisely. We have already encountered the inequality (13) that was necessary for the approximation (14) to hold. A similar inequality can be stated for the field expressed by Bessel and Neumann functions inside of the waveguide. In order to be able to express the field inside of the curved waveguide by approximate expressions that reduce to the sine and cosine functions appearing in (lb) in the limit of large radius of curvature, we must require

$$\nu < n_2 kr \tag{34}$$

everywhere inside of the waveguide. We can express these conditions in the form

$$\beta > n_1 k\left(1 + \frac{d}{R}\right) \tag{35}$$

and

$$\beta < n_1 k\left(1 - \frac{d}{R}\right). \tag{36}$$

The validity of our theory becomes doubtful if one or both of these inequalities are violated. However, a comparison with Marcatili's theory[1] shows that our approximation is still quite good even in regions where (36) no longer holds.

The simple expression (32) for the bending loss of an asymmetric slab waveguide can be used only if the values of κ, γ, θ, and β are known. It is, of course, only necessary to determine one of these parameters from the eigenvalue equation (9) since they are all interconnected by the equations (3) through (6).

It is useful to point out that the loss equation (32) seems to be applicable to other types of waveguide than the one for which it was derived. I have compared experimental values[2] of bending losses of a round optical fiber with the loss predicted by (32). For such a comparison it is necessary to use the parameters κ, γ, etc., that apply to the waveguide to which the formula is to be applied. In case of the round fiber the parameters of the HE_{11} mode were used to compute the bending loss from (32). The reason for this choice of parameters is the fact that the parameter γ determines the decay behavior of the field outside of the waveguide. It is very important that the proper field decay is used, so that it is more logical to use the γ value of the round fiber instead of the value computed from (9), if (32) is to be used to compute the bending loss of the round fiber.

It is a curious fact that the loss formula (32) can also be obtained without use of the Hankel function appearing in (10) if we use a field of the form

$$E_y = \frac{C}{\sqrt{\gamma(r)r}} \exp\left\{-i \int_R^r \gamma(r)\, dr\right\} \tag{37}$$

with

$$\gamma(r) = \left[\beta^2 \frac{R^2}{r^2} - n_1^2 k^2\right]^{\frac{1}{2}}. \tag{38}$$

The validity of this claim can easily be checked by performing the integration. The factor in front of the exponential function is somewhat arbitrary. However, the exponential function itself admits of a physical interpretation.

The straight waveguide has a field that, outside of its core, behaves according to $\exp(-\gamma x)$. In the curved system x is naturally replaced by r. If we consider that the process of bending the waveguide is likely to lead also to a distortion of the phase fronts we can try to describe the separation of consecutive wavefronts by an r dependent wavelength

(λ_g is the wavelength of the straight guide),

$$\lambda(r) = \lambda_g \frac{r}{R}. \tag{39}$$

The propagation constant is related to the wavelength by the relation

$$\beta(r) = \frac{2\pi}{\lambda(r)} = \beta_g \frac{R}{r}. \tag{40}$$

By replacing the propagation constant in (4) with (40) we obtain (38). Since γ is now no longer a constant it is natural to replace γr by $\int \gamma \, dr$ and thus arrive at the form of the exponential function appearing in (37). By using (37) instead of (10) and proceeding exactly as shown in this paper we obtain (32) once more. It is interesting that we have thus obtained an approximation for the Hankel function that holds for the region where the order number is very nearly equal to the argument as well as for the region where the argument is much larger than the order number. Equation (38) shows clearly that $\gamma(r)$ changes from real to imaginary values as r increases.†

One might hope that a similar procedure would allow us to obtain approximate expressions for the bending loss of the round optical fiber. However, such attempts lead to equations that are not in agreement with experiment. On the other hand, the loss formula (32) agrees well with experiment[2] if the parameters of the round fiber are used.

The loss formula (32) holds for all values of the refractive indices n_1, n_2, and n_3 for which mode guidance is possible. Small index differences are not required for (32) to be valid.

V. NUMERICAL EXAMPLES

Because of the large number of variables involved it is not possible to provide graphic displays for all possible applications. The loss formula (32) is sufficiently simple (except for the need of knowing the waveguide parameters κ, γ, etc.) so that loss values for cases of interest can easily be calculated. We provide curves that aid in computing the bending loss of the even TE mode of the symmetric slab waveguide.

Figure 3 is a comparison of our theory with the results of Ref. 1. The ordinate is the function $(2\Delta)^{\frac{1}{2}}\alpha R$ while $(8\Delta)^{\frac{1}{2}}2dn_2/\lambda$ is plotted on the abscissa. The parameter Δ is defined as $n_2 - n_1$ (we are using $n_1 = n_3$). The expression $36n_2R\Delta^{\frac{3}{2}}/\lambda$ assumes the constant value 60 for the curve of Fig. 3. The agreement with Marcatili's theory[1] is re-

† A discussion of bending losses based on a similar argument is presented in Ref. 5.

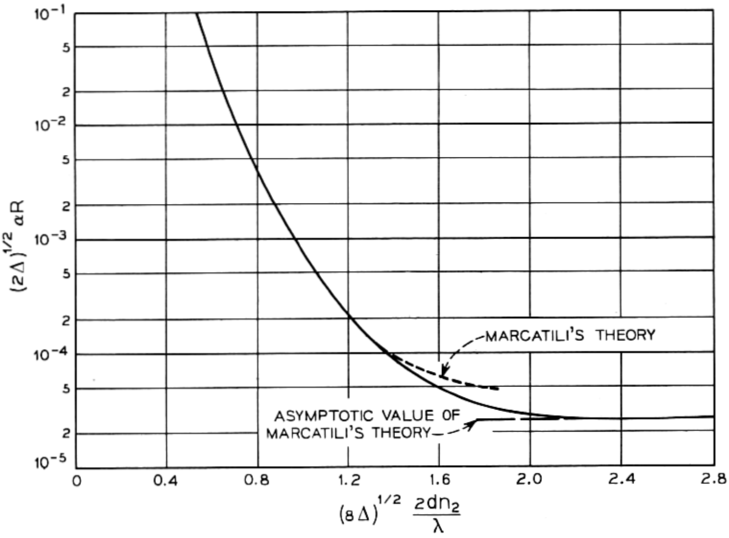

Fig. 3—Comparison between this theory and the theory of Ref. 1. $\Delta = n_2 - n_1$, $n_1 = n_3$, $36n_2R\Delta^{\frac{3}{2}}/\lambda = 60$.

markably good. The inequality (36) is violated for values of the abscissa that are larger than 1.7. This may explain the departure between the solid curve representing equation (32) and the dash-dotted curve representing Marcatili's theory. The fact that the solid curve actually touches the asymptotic value shown as a dotted line in Fig. 3 (this value is assumed for infinite values of the abscissa, its location as shown in the figure has no meaning) is probably an accident since the solid line increases again for larger values of the abscissa outside of the range shown in the figure.

We restrict ourselves to the case of the symmetric slab waveguide with $n_1 = n_3$. It is possible to express κ and γ (which equals θ in this special case) as functions of

$$V = (n_2^2 - n_3^2)k^{\frac{1}{2}}d. \tag{41}$$

The propagation constant β depends not only on V but also on the value of n_1k so that it is not possible to express the bending loss only in terms of V. However, in the special case that n_1 and n_2 are very nearly equal we have $\beta = n_1k$ so that we need not actually solve the eigenvalue equation to obtain the propagation constant. To aid in the evaluation of the bending loss formula we provide curves for $2\alpha\beta d^2e^U$ and for $d^3\beta^2U/R$ in

Fig. 4 as functions of V. The approximate expression given in (33) was used to express U. If β can be approximated by n_1k these two curves enable us to calculate the bending loss without any difficulty. For symmetric slab waveguides with a large value of $n_2 - n_1$ we can calculate β with the help of (4) from known values of γ and n_1k. A plot of γd as a function of V is provided in Fig. 5. The parameter γ is interesting in itself since it determines the exponential decay of the guided mode outside of the waveguide core.

VI. CONCLUSIONS

The bending loss of an asymmetric slab waveguide has been calculated using an approximation that is based on the assumption that the field near the bent guide is still almost identical to the field of the straight guide. The results of this approximate theory are in good agreement with the bending loss theory of Marcatili[1] in the range of applicability of our theory. It is hard to apply a similar analysis to the bent round fiber because the exact form of the solutions of Maxwell's equations for the curved structure is not known. However, the bending loss formula obtained for the slab waveguide model yields good agreement with

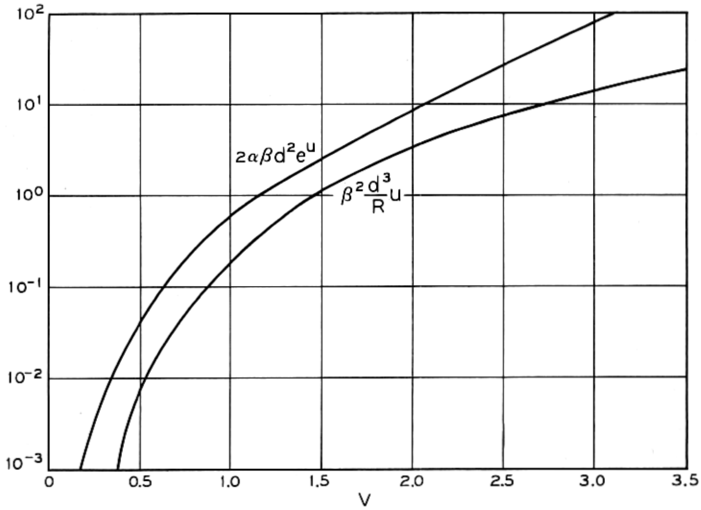

Fig. 4—The functions $2\alpha\beta d^2e^U$ and $d^3\beta^2U/R$ are plotted versus $V = (n_2 - n_1)^{\frac{1}{2}}kd$. $(n_1 = n_3.)$

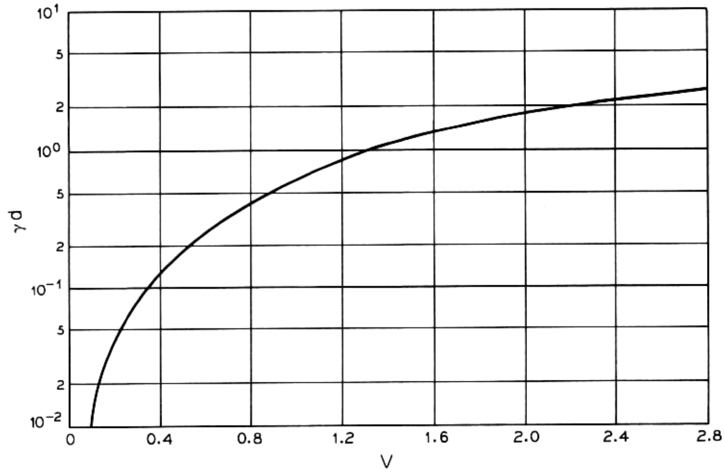

Fig. 5—Plot of γd as a function of V. ($n_1 = n_3$.)

experiment[2] if the mode parameters of the fiber mode are used in the loss formula instead of the mode parameters of the slab waveguide.

For the case of small index differences, curves that allow the determination of the bending loss of the lowest order symmetric TE mode of the symmetric slab waveguide are provided.

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