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# Analysis of Dependence Effects in Telephone Trunking Networks

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*Since theoretical and computational difficulties often preclude exact solution of telephone trunking network problems, approximate methods are naturally used. A typical approach is to determine link* blocking probabilities and from them calculate point-to-point blocking probabilities by invoking independence assumptions. Although the link blocking probabilities may be quite accurate, the point-to-point blocking probability calculations will, in some cases, suffer from the independence assumptions. This paper presents a method of taking dependence into account for certain networks by approximating conditional probabilities which reflect the dependence. The approximations avoid the problems of dealing with the large sums associated with this problem.*

## I. INTRODUCTION

The exact analysis of telephone trunking networks often leads to severe computational problems due, e.g., to the large number of possible states. Approximate methods are thus naturally used. There has been much success in approximately calculating link blocking probabilities but less in determining point-to-point blocking probabilities. Errors in the point-to-point blocking probabilities can be caused by independence assumptions.

The purpose of this paper is to take advantage of existing techniques for approximating link blocking probabilities (which are quite accurate) and develop an approach for taking link dependences into account. In particular, we present a method for approximating the appropriate conditional probabilities for cases where the traffics are Poisson (or close to Poisson). The extension of the approach to the case of distinctly non-Poisson processes, such as arise in overflows, will be reported in Ref. 1. To obtain point-point blocking probabilities for non-Poisson processes,

* We use link to denote trunk group.

one must also solve for the blocking seen by individual traffics when more than one traffic is offered to a link (the equivalent random method gives the blocking seen by the combined traffic). This is also treated in Ref. 1.

To clarify the role of independence assumptions, suppose there is common traffic on links 1 and 2. Let $A_1$ and $A_2$ be the events that the common traffic is blocked on links 1 and 2, respectively. Then the probability of being blocked on either link 1 or on link 2, $P\{A_1 \cup A_2\} = P\{A_1\} + P\{A_2\} - P\{A_1 \cap A_2\}$, is often approximated by $P\{A_1 \cup A_2\} \cong P\{A_1\} + P\{A_2\} - P\{A_1\}P\{A_2\}$ or even by $P\{A_1 \cup A_2\} \cong P\{A_1\} + P\{A_2\}$. The last approximation is clearly accurate if $P\{A_1\} + P\{A_2\} \gg P\{A_1 \cap A_2\}$ and the first is accurate if $P\{A_1 \mid A_2\} \cong P\{A_1\}$ or if both $P\{A_1 \cap A_2\}$ and $P\{A_1\}P\{A_2\}$ are relatively small. On the other hand, it is easy to give examples where the neglect of dependence leads to non-negligible errors.

The errors due to neglecting dependence naturally depend strongly on mutual traffic. To see this in a transparent case, let us reconsider the two-link network mentioned above with each link having the same number of trunks. $P\{A_1 \mid A_2\}$ could vary from $P\{A_1\}$ (when the mutual traffic is zero) to unity (when there is no traffic on link 2 not shared with link 1). In the latter extreme case, assuming independence would give $P\{A_1\} + P\{A_2\} (1 - P\{A_1\})$ for $P\{A_1 \cup A_2\}$ compared to the correct answer $P\{A_1\}$. Thus, assuming independence could, in this case, conceivably overestimate the point-to-point blocking probabilities by something approaching 100 percent. In fact, it is possible to overestimate a point-to-point blocking probability on $m$ tandem links by almost as much as $m$ times by assuming independence.

## II. BASIC APPROACH

For simplicity of explanation, first consider the situation shown in Fig. 1. $\lambda_1$, $\lambda_2$, and $\lambda_{12}$ are the parameters of mutually independent Poisson processes. Holding times are mutually independent exponential random variables with unity mean (or the mean is the time unit) here and throughout this paper. Also, throughout the paper we shall assume that lost calls are cleared and that the system is in equilibrium. Links 1 and 2 have $N_1$ and $N_2$ trunks, respectively. Assume $N_1 \geqq N_2$.

Let $A_1$ and $A_2$ denote the events of link 1 and link 2 being blocked, respectively (i.e., there are $N_1$ calls up on link 1 and $N_2$ calls up on link 2).

All of the equilibrium state probabilities may actually be expressed in closed form using, e.g., the form of solution given in Ref. 2, Section 7. For example,

$$P\{A_1 \cup A_2\} = \frac{\displaystyle\sum_{i_{12}=0}^{N_2} \frac{\lambda_{12}^{i_{12}}}{i_{12}!} \sum_{i_1=0}^{N_1-i_{12}} \frac{\lambda_1^{i_1}}{i_1!} \sum_{i_2=0}^{N_2-i_{12}} \frac{\lambda_2^{i_2}}{i_2!}}{\displaystyle\sum_{\substack{i_1+i_{12}\leqq N_1\\ i_2+i_{12}\leqq N_2}} \frac{\lambda_1^{i_1}}{i_1!} \frac{\lambda_2^{i_2}}{i_2!} \frac{\lambda_{12}^{i_{12}}}{i_{12}!}}, \tag{1}$$

where the indices are nonnegative integers (throughout the paper). However, the use of such exact results very quickly becomes impractical for all but the smallest problems even for large computers because of the number of computations required.

The link blocking probabilities can usually be quite well approximated. For example, the following reduced load equations,

$$P_1 = B(N_1, \lambda_1 + \lambda_{12}(1 - P_2)), \tag{2}$$

$$P_2 = B(N_2, \lambda_2 + \lambda_{12}(1 - P_1)), \tag{3}$$

$$P\{A_1\} \cong P_1, \tag{4}$$

$$P\{A_2\} \cong P_2, \tag{5}$$

($B(N, \lambda)$ is the Erlang $B$ formula) are often sufficiently accurate as they stand.* See Ref. 3 for much more discussion of link blocking probabilities.

Once the link blocking probabilities are determined (however they are determined) all that is left to calculate is $P\{A_1 \mid A_2\}$ in order to determine $P\{A_1 \cup A_2\}$. We first write down this probability exactly and then show a simple practical approximation. (Approximating the sums becomes even more important when considering more complicated dependences as in Section III.) To this end, note that

$$P\{A_1 \mid A_2\} = \frac{\displaystyle\sum_{i_{12}=0}^{N_2} \frac{\lambda_{12}^{i_{12}}}{i_{12}!} \frac{\lambda_2^{N_2-i_{12}}}{(N_2 - i_{12})!} \frac{\lambda_1^{N_1-i_{12}}}{(N_1 - i_{12})!}}{\displaystyle\sum_{i_{12}=0}^{N_2} \frac{\lambda_{12}^{i_{12}}}{i_{12}!} \frac{\lambda_2^{N_2-i_{12}}}{(N_2 - i_{12})!} \sum_{i_1=0}^{N_1-i_{12}} \frac{\lambda_1^{i_1}}{i_1!}}$$

$$= \frac{\displaystyle\sum_{i=0}^{N_2} \binom{N_2}{i} p^i q^{N_2-i} \frac{\lambda_1^{N_1-i}}{(N_1 - i)!}}{\displaystyle\sum_{i=0}^{N_2} \binom{N_2}{i} p^i q^{N_2-i} \sum_{i_1=0}^{N_1-i} \frac{\lambda_1^{i_1}}{i_1!}} \tag{6}$$

* It may easily be shown that these $P_1$ and $P_2$ agree exactly with $P\{A_1\}$ and $P\{A_2\}$ to first-order terms in $\lambda_{12}$. Although the $P_i$ usually overestimate the $P\{A_i\}$, they can underestimate (seriously, in extreme cases) when $\lambda_{12}$ is very large.

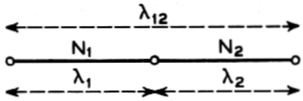


Fig. 1—Two-link network.

where

$$p = \frac{\lambda_{12}}{\lambda_{12} + \lambda_2}, \tag{7}$$

$$q = 1 - p. \tag{8}$$

Observe that (6) is the ratio of two sums each of which is essentially a Bernstein polynomial (see Ref. 4). However, a Bernstein polynomial is of the form $\sum_{i=0}^{n} \binom{n}{i} p^i q^{n-1} f(i/n)$ but our $f$ is actually a function of $i$ or of $n(i/n)$. Each sum is also the expectation of a function of a binomially distributed random variable. Using the following interpolation formula (see Ref. 5, p. 178),

$$f(i) \cong f(\bar{\imath}) + (i - \bar{\imath})\tfrac{1}{2}[f(\bar{\imath} + 1) - f(\bar{\imath} - 1)] + (i - \bar{\imath})^2\tfrac{1}{2}[f(\bar{\imath} + 1) - 2f(\bar{\imath}) + f(\bar{\imath} - 1)], \tag{9}$$

and taking expectations yields

$$\sum_{i=1}^{n} \binom{n}{i} p^i q^{n-1} f(i) \cong f(np) + \frac{npq}{2}[f(np + 1) - 2f(np) + f(np - 1)]. \tag{10}$$

(10) is the usual expression for the mean and variance approximation to an expectation of a function of a random variable but with the second derivative replaced by a central difference. Using (10) on the numerator and denominator of (6) yields

$$P\{A_1 \mid A_2\} \cong P_a\{A_1 \mid A_2\} = \frac{1 + \dfrac{N_2pq}{2}\left[\dfrac{\lambda_1}{N_1 - N_2p + 1} - 2 + \dfrac{N_1 - N_2p}{\lambda_1}\right]}{\dfrac{1}{B(N_1 - N_2p, \lambda_1)} + \dfrac{N_2pq}{2}\left[\dfrac{\lambda_1}{N_1 - N_2p + 1} - 1\right]}. \tag{11}$$

Interpolation, such as given on page 571 of Ref. 3, may be used to evaluate the Erlang $B$ formula for a nonintegral number of trunks or an approximation for Erlang $B$ in integral form can be used.

Note that we use the same type of approximation on the numerator and denominator, which are closely related, so that the ratio can be more accurate than either taken separately.

(11) shows quantitatively what we expect qualitatively, namely, that the dependence effect gets small as the common traffic gets small. In particular, when $\lambda_{12} = 0$, $P\{A_1 \mid A_2\} = B(N_1, \lambda_1) = P\{A_1\}$ and when $\lambda_2 = 0$, $P\{A_1 \mid A_2\} = B(N_1 - N_2, \lambda_1)$ (recall that we assumed $N_1 \geqq N_2$).

To examine the approximation, we find it convenient to first consider $P_a\{A_1^c \mid A_2\}$ where superscript $c$ denotes complement. Actually, since $P\{A_1 \cup A_2\} = P\{A_1\} + P\{A_1^c \mid A_2\}P\{A_2\}$, $P\{A_1^c \mid A_2\}$ is really the crucial quantity. With $N_1 \geqq N_2$, we have

$$P_a\{A_1^c \mid A_2\} = 1 - P_a\{A_1 \mid A_2\}$$
$$= \frac{\dfrac{1}{B(N_1 - N_2p, \lambda_1)} - 1 + \dfrac{N_2pq}{2}\left[1 - \dfrac{2(N_1 - N_2p)}{\lambda_1}\right]}{\dfrac{1}{B(N_1 - N_2p, \lambda_1)} + \dfrac{N_2pq}{2}\left[\dfrac{\lambda_1}{N_1 - N_2p + 1} - 2\right]}. \tag{12}$$

If we let $N_1 = N_2 = N$, then

$$P_a\{A_1^c \mid A_2\} = \frac{\dfrac{1}{B(Nq, \lambda_1)} - 1 + \dfrac{N_2pq}{2}\left[1 - \dfrac{2Nq}{\lambda_1}\right]}{\dfrac{1}{B(Nq, \lambda_1)} + \dfrac{Npq}{2}\left[\dfrac{\lambda_1}{Nq + 1} - 2\right]}. \tag{13}$$

(12) could have been derived directly using a mean and second central difference approximation just as (6) was approximated by (11). The terms multiplied by $N_2pq/2$ in both the numerator and denominator are then the second central difference terms. If we hold $\lambda_1$ and $p$ fixed and let $N$ get large, it can be shown that the second central difference terms get small compared to the mean terms suggesting that the approximation is accurate for large $N$. However, since $P_a\{A_1 \mid A_2\} \to 0$ as $N \to \infty$ (with $\lambda_1$ and $p$ fixed), investigation of this type of convergence is of limited practical value.

It is probably of more interest to examine the approximation for fixed $p$ and to let $\lambda_1$ get large with $N$. To this end, let $\lambda_1 = kN$ with $k > q$. For sufficiently large $n$ and $a > n$, $B(n, a) \cong 1 - n/a$ so that

$$B(Nq, kN) \cong 1 - \frac{q}{k}. \tag{14}$$

Hence, we obtain for large $N$,

$$P_a\{A_1 \mid A_2\} = \frac{1 + \dfrac{Npq}{2}\left[\dfrac{kN}{Nq+1} - 2 + \dfrac{Nq}{kN}\right]}{\dfrac{1}{1 - \dfrac{q}{k}} + \dfrac{Npq}{2}\left[\dfrac{kN}{Nq+1} - 1\right]}$$

$$\cong B(Nq, kN) \tag{15}$$

which has the interesting interpretation that with link 2 full, the number of trunks on link 1 to handle the $\lambda_1$ traffic is reduced by the average number of calls on link 2 which are common to link 1 (we shall elaborate on this below). Although (15) is intuitively appealing and obviously correct for $p = 0$ or 1, its accuracy should be examined for $p \in (0, 1)$. Observe that for large $N$, the variance terms dominate.

By elaborating on the interpretation alluded to above, we can see why $B(Nq, \lambda_1)$ should tend to overestimate $P\{A_1 \mid A_2\}$. With $D_i$ the event that $i$ calls from the $\lambda_{12}$ traffic are up,

$$D_i = \{i\ \lambda_{12} \text{ calls up}\}, \tag{16}$$

observe that

$$P\{A_1 \mid A_2\} = \sum_{i=0}^{N_2} P\{A_1 \mid A_2 \cap D_i\}P\{D_i \mid A_2\} \tag{17}$$

$$= \sum_{i=0}^{N_2} B(N_1 - i, \lambda_1)P\{D_i \mid A_2\}. \tag{18}$$

Now, *if* the $\lambda_{12}$ calls were never blocked on link 1, then $P\{D_i \mid A_2\} = \binom{n}{i}p^i q^{n-i}$. With $N_1 = N_2 = N$, $\lambda_1 = kN$, it may be shown that $\sum_{i=0}^{N_2} B(N_1 - i, \lambda_1)\binom{n}{i}p^i q^{n-i} \to B(N_1 - N_2 p, \lambda_1) = B(Nq, \lambda)$ as $N \to \infty$. $B(N_1 - N_2 p, \lambda_1)$ is the blocking probability on link 1 when the number of trunks is reduced by the conditional mean of trunks occupied by $\lambda_{12}$ calls. Hence, under these conditions, $B(Nq, \lambda_1)$ approximates $P\{A_1 \mid A_2\}$ by ignoring the blocking of $\lambda_{12}$ calls on link 1. If we take this blocking into account, we would expect that for large $N$, $P\{A_1 \mid A_2\} < B(Nq, \lambda_1)$. This will be seen to be the case below.

Another approximation for $P\{A_1 \mid A_2\}$ is

$$P_b\{A_1 \mid A_2\} = \frac{r_1^{-N_1}(q + pr_1)^{N_2}e^{\lambda_1 r_1}/\sqrt{2\pi\beta_1}}{r_2^{-N_1}(q + pr_2)^{N_2}e^{\lambda_1 r_2}/(1 - r_2)\sqrt{2\pi\beta_2}} \tag{19}$$

where

$$r_1 = \frac{\sqrt{(\lambda_1 q - N_1 p + N_2 p)^2 + 4\lambda_1 N_1 pq} - \lambda_1 q + N_1 p - N_2 p}{2\lambda_1 p}, \tag{20}$$

$$\beta_1 = N_1 - N_2\left(\frac{pr_1}{q + pr_1}\right)^2, \tag{21}$$

$$\beta_2 = N_1 - N_2\left(\frac{pr_2}{q + pr_2}\right)^2 + \left(\frac{r_2}{1 - r_2}\right)^2, \tag{22}$$

and $r_2$ is the solution between 0 and 1 of the cubic

$$\lambda_1 pr_2^3 + (N_2 p - p - N_1 p + \lambda_1 q - \lambda_1 p)r_2^2 + (-\lambda_1 q + N_1 p - N_1 q - N_2 p - q)r_2 + N_1 q = 0. \tag{23}$$

This approximation is due to D. L. Jagerman (see the derivation in the Appendix).

For $N_1 = N_2 = N$, $\lambda_1 = kN$, and $N$ large,

$$P_b\{A_1 \mid A_2\} \cong 1 - \frac{\sqrt{k^2q^2 + 4kpq} - kq}{2kp}. \tag{24}$$

Table I shows some numerical values of the exact and approximate conditional probabilities along with the values of the approximations for large $N$. It is seen that, although $P_b\{A_1 \mid A_2\}$ does not always surpass the simpler $P_a\{A_1 \mid A_2\}$, it is, on the whole, superior and its behavior with increasing $N$ is clearly better. We dwelled on $P_a\{A_1 \mid A_2\}$ because it is useful in many cases and the mean and variance approach is easily

TABLE I—NUMERICAL VALUES

| $p$ | $\lambda_1 = N$ | $P\{A_1 \mid A_2\}$ | $P_a\{A_1 \mid A_2\}$ | $P_b\{A_1 \mid A_2\}$ | % error in $P_a\{A_1 \mid A_2\}$ | % error in $P_b\{A_1 \mid A_2\}$ |
|---|---|---|---|---|---|---|
| 0.1 | 5 | 0.3256 | 0.3246 | 0.3404 | −0.3 | 4.5 |
| 0.1 | 10 | 0.26087 | 0.26091 | 0.2732 | 0.015 | 4.7 |
| 0.1 | 15 | 0.2296 | 0.2287 | 0.2402 | −0.4 | 4.6 |
| 0.1 | 20 | 0.2101 | 0.2096 | 0.2197 | −0.2 | 4.5 |
| 0.1 | 25 | 0.1966 | 0.1957 | 0.2052 | −0.4 | 4.4 |
| 0.1 | ∞ | | 0.1 | 0.092 | | |
| 0.5 | 5 | 0.5017 | 0.4926 | 0.5125 | −1.8 | 2.15 |
| 0.5 | 10 | 0.4579 | 0.4635 | 0.4645 | 1.2 | 1.4 |
| 0.5 | 15 | 0.4386 | 0.4587 | 0.4431 | 4.6 | 1.0 |
| 0.5 | 20 | 0.4274 | 0.4599 | 0.4307 | 7.6 | 0.8 |
| 0.5 | 25 | 0.4200 | 0.4621 | 0.4226 | 10.0 | 0.6 |
| 0.5 | ∞ | | 0.5 | 0.382 | | |
| 0.9 | 5 | 0.7806 | 0.8161 | 0.7827 | 4.5 | 0.25 |
| 0.9 | 10 | 0.7521 | 0.8258 | 0.7535 | 9.8 | 0.19 |
| 0.9 | 15 | 0.7416 | 0.8412 | 0.7421 | 13.4 | 0.11 |
| 0.9 | 20 | 0.7360 | 0.8525 | 0.7366 | 15.8 | 0.07 |
| 0.9 | 25 | 0.7326 | 0.8604 | 0.7330 | 17.4 | 0.05 |
| 0.9 | ∞ | | 0.9 | 0.718 | | |

extendable to cases where extreme accuracy is not required (as in Section III). It can be easily seen that a percentage error in the conditional probability typically causes a smaller percent error in the point-to-point blocking probability.

## III. MORE COMPLICATED DEPENDENCES

The extension of the basic idea of Section II is often rather straightforward. For example, consider the network in Fig. 2 which is complicated enough to illustrate the ingredients of a general approach.

Assume (for the sake of being concrete) that $N_1 \geqq N_3 \geqq N_2$. All of the traffics are mutually independent Poisson. Let $A_1$, $A_2$, $A_3$ represent the events of links, 1, 2, 3 being blocked, respectively. Then the following are approximations to conditional probabilities:

$$P\{A_1 \mid A_2\} \cong \frac{\sum_{i=0}^{N_2} \binom{N_2}{i} p_{12}^i q_{12}^{N_2-i} \frac{\lambda_1^{N_1-i}}{(N_1-i)!}}{\sum_{i=0}^{N_2} \binom{N_2}{i} p_{12}^i q_{12}^{N_2-i} \sum_{j=0}^{N_1-i} \frac{\lambda_1^j}{j!}}, \tag{25}$$

$$P\{A_3 \mid A_2\} \cong \frac{\sum_{i=0}^{N_2} \binom{N_2}{i} p_{32}^i q_{32}^{N_2-i} \frac{\lambda_3^{N_3-i}}{(N_3-i)!}}{\sum_{i=0}^{N_2} \binom{N_2}{i} p_{32}^i q_{32}^{N_2-i} \sum_{j=0}^{N_3-i} \frac{\lambda_3^j}{j!}}, \tag{26}$$

$$P\{A_1 \mid A_3\} \cong \frac{\sum_{i=0}^{N_3} \binom{N_3}{i} p_{13}^i q_{13}^{N_3-i} \frac{(\bar{\lambda}_1)^{N_1-i}}{(N_1-i)!}}{\sum_{i=0}^{N_3} \binom{N_3}{i} p_{13}^i q_{13}^{N_3-i} \sum_{j=0}^{N_1-i} \frac{(\bar{\lambda}_1)^j}{j!}}, \tag{27}$$

where

$$p_{12} = \frac{\lambda_{12} + \lambda_{13}(1 - P_3)}{\lambda_{12} + \lambda_{13}(1 - P_3) + \lambda_2 + \lambda_{23}(1 - P_3)}, \qquad q_{12} = 1 - p_{12}, \tag{28}$$

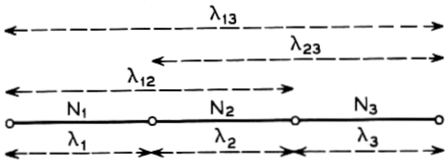


Fig. 2—Three-link network.

$$p_{32} = \frac{\lambda_{23} + \lambda_{13}(1 - P_1)}{\lambda_{23} + \lambda_{13}(1 - P_1) + \lambda_2 + \lambda_{12}(1 - P_1)}, \qquad q_{32} = 1 - p_{32}, \quad (29)$$

$$p_{13} = \frac{\lambda_{13}(1 - P_2)}{\lambda_{13}(1 - P_2) + \lambda_3 + \lambda_{23}(1 - P_2)}, \qquad q_{13} = 1 - p_{13}, \quad (30)$$

$$\overline{\lambda_1} = \lambda_1 + \lambda_{12}\left(1 - \frac{P\{A_3 \mid A_2\}P_2}{P_3}\right), \quad (31)$$

and the $P_i$ are approximations to the link blocking probabilities. In (31), the approximation to $P\{A_3 \mid A_2\}$ is used (and we are obviously assuming $P_3 > 0$).

The rationale behind (25)–(27) is based on (6) at least in the case where the $P_i$ are small so that the carried traffics are not too far from Poisson. We have already discussed approximating the sums of (25)–(27) in Section II.

But for the network on Fig. 2 we should also calculate $P\{A_1 \cup A_2 \cup A_3\}$, the blocking probability seen by the $\lambda_{13}$ traffic. Since

$$\begin{aligned} P\{A_1 \cup A_2 \cup A_3\} &= P\{A_1\} + P\{A_2\} + P\{A_3\} \\ &\quad - P\{A_1 \mid A_2\}P\{A_2\} - P\{A_1 \mid A_3\}P\{A_3\} \\ &\quad - P\{A_3 \mid A_2\}P\{A_2\} + P\{A_1 \cap A_3 \mid A_2\}P\{A_2\}, \end{aligned} \quad (32)$$

we see that we need only discuss the evaluation of $P\{A_1 \cap A_3 \mid A_2\}$, the other quantities already having been treated.

Just as we derived (6), we can show that (exactly)

$$P\{A_1 \cap A_3 \mid A_2\} = \frac{\displaystyle\sum_{i_{12}+i_{23}+i_{13}+i_2=N_2} \frac{N_2!\, p_{12}^{i_{12}} p_{23}^{i_{23}} p_{13}^{i_{13}} p_2^{i_2}}{i_{12}!\, i_{23}!\, i_{13}!\, i_2!} \frac{\lambda_3^{N_3-i_{23}-i_{13}}}{(N_3 - i_{23} - i_{13})!} \frac{\lambda_1^{N_1-i_{12}-i_{13}}}{(N_1 - i_{12} - i_{13})!}}{\displaystyle\sum_{i_{12}+i_{23}+i_{13}+i_2=N_2} \frac{N_2!\, p_{12}^{i_{12}} p_{23}^{i_{23}} p_{13}^{i_{13}} p_2^{i_2}}{i_{12}!\, i_{23}!\, i_{13}!\, i_2!} \sum_{i_3=0}^{N_3-i_{23}-i_{13}} \frac{\lambda_3^{i_3}}{i_3!} \sum_{i_1=0}^{N_1-i_{12}-i_{13}} \frac{\lambda_1^{i_1}}{i_1!}} \quad (33)$$

where now

$$p_{12} = \frac{\lambda_{12}}{\lambda_{12} + \lambda_{23} + \lambda_{13} + \lambda_2}, \quad (34)$$

$$p_{23} = \frac{\lambda_{23}}{\lambda_{12} + \lambda_{23} + \lambda_{13} + \lambda_2}, \quad (35)$$

$$p_{13} = \frac{\lambda_{13}}{\lambda_{12} + \lambda_{23} + \lambda_{13} + \lambda_2}, \tag{36}$$

$$p_2 = \frac{\lambda_2}{\lambda_{12} + \lambda_{23} + \lambda_{13} + \lambda_2}. \tag{37}$$

These sums may be approximated by recognizing that they are both expectations of functions of multinominal random variables. Thus,

$$\begin{aligned} E[f(i_{23}, i_{13}, i_{12})] &\cong f(\bar{\imath}_{23}, \bar{\imath}_{13}, \bar{\imath}_{12}) \\ &+ \tfrac{1}{2}\{[E(i_{23} - \bar{\imath}_{23})^2\, \delta_{23}^2 + E(i_{13} - \bar{\imath}_{13})^2\, \delta_{13}^2 + E(i_{12} - \bar{\imath}_{12})^2\, \delta_{12}^2] \\ &+ 2E[(i_{23} - \bar{\imath}_{23})(i_{13} - \bar{\imath}_{13})]\mu\delta_{23}\mu\delta_{13} \\ &+ 2E[(i_{23} - \bar{\imath}_{23})(i_{12} - \bar{\imath}_{12})]\mu\delta_{23}\mu\delta_{12} \\ &+ 2E[(i_{13} - \bar{\imath}_{13})(i_{12} - \bar{\imath}_{12})\mu\delta_{13}\mu\delta_{12}]\}f(\bar{\imath}_{23}, \bar{\imath}_{13}, \bar{\imath}_{12}) \end{aligned} \tag{38}$$

where $\bar{\imath}_{nm}$ here indicates expected value and where $\delta^2$ is a second central difference, e.g.,

$$\delta_{23}^2 f(\bar{\imath}_{23}, \bar{\imath}_{13}, \bar{\imath}_{12}) = f(\bar{\imath}_{23} + 1, \bar{\imath}_{13}, \bar{\imath}_{12}) - 2f(\bar{\imath}_{23}, \bar{\imath}_{13}, \bar{\imath}_{12}) + f(\bar{\imath}_{23} - 1, \bar{\imath}_{13}, \bar{\imath}_{12}) \tag{39}$$

and the operator $\mu\delta$ is given, e.g., by

$$\mu\delta_{13} f(\bar{\imath}_{23}, \bar{\imath}_{13}, \bar{\imath}_{12}) = \tfrac{1}{2}[f(\bar{\imath}_{23}, \bar{\imath}_{13} + 1, \bar{\imath}_{12}) - f(\bar{\imath}_{23}, \bar{\imath}_{13} - 1, \bar{\imath}_{12})] \tag{40}$$

so that, e.g.,

$$\begin{aligned} \mu\delta_{23}\mu\delta_{13} f(\bar{\imath}_{23}, \bar{\imath}_{13}, \bar{\imath}_{12}) &= \tfrac{1}{4}[f(\bar{\imath}_{23} + 1, \bar{\imath}_{13} + 1, \bar{\imath}_{12}) + f(\bar{\imath}_{23} - 1, \bar{\imath}_{13} - 1, \bar{\imath}_{12}) \\ &- f(\bar{\imath}_{23} - 1, \bar{\imath}_{13} + 1, \bar{\imath}_{12}) - f(\bar{\imath}_{23} + 1, \bar{\imath}_{13} - 1, \bar{\imath}_{12})]. \end{aligned} \tag{41}$$

The variances and covariances for the multinominal distribution are given on page 164 of Ref. 6. Thus,

$$\bar{\imath}_{jk} = N_2 p_{jk}, \qquad jk = 23, 13, 12, \tag{42}$$

$$\sigma_{jk}^2 \equiv E(i_{jk} - \bar{\imath}_{jk})^2 = N_2 p_{jk}(1 - p_{jk}), \qquad jk = 23, 13, 12, \tag{43}$$

$$\sigma_{jk,lm}^2 = E[(i_{jk} - \bar{\imath}_{jk})(i_{lm} - \bar{\imath}_{lm})] = -N_2 p_{jk} p_{lm}, \qquad jk, lm = 23, 13, 12 (jk \neq lm). \tag{44}$$

One obtains

$$P\{A_1 \cap A_3 \mid A_2\} \cong \Big\{1 + \frac{1}{2}\sigma_{23}^2\Big(\frac{\bar N_3}{\lambda_3} - 2 + \frac{\lambda_3}{\bar N_3 + 1}\Big)$$

$$+ \frac{1}{2}\sigma_{13}^2\Big(\frac{\bar N_3 \bar N_1}{\lambda_3\lambda_1} - 2 + \frac{\lambda_3\lambda_1}{(\bar N_3 + 1)(\bar N_1 + 1)}\Big) + \frac{1}{2}\sigma_{12}^2\Big(\frac{\bar N_1}{\lambda_1} - 2 + \frac{\lambda_1}{\bar N_1 + 1}\Big)$$

$$+ \frac{1}{4}\Big[\sigma_{23,13}^2\Big(\frac{\bar N_3(\bar N_3 - 1)}{\lambda_3^2}\frac{\bar N_1}{\lambda_1} - 2 + \frac{\lambda_3^2\lambda_1}{(\bar N_3 + 2)(\bar N_3 + 1)(\bar N_1 + 1)}\Big)$$

$$+ \sigma_{23,12}^2\Big(\frac{\bar N_3 \bar N_1}{\lambda_3\lambda_1} - 2 + \frac{\lambda_3\lambda_1}{(\bar N_3 + 1)(\bar N_1 + 1)}\Big)$$

$$+ \sigma_{13,12}^2\Big(\frac{\bar N_3 \bar N_1(\bar N_1 - 1)}{\lambda_3\lambda_1^2} - 2 + \frac{\lambda_3\lambda_1^2}{(\bar N_3 + 1)(\bar N_1 + 2)(\bar N_1 + 1)}\Big)\Big]\Big\}\Big/$$

$$\{B_3^{-1}B_1^{-1} + \tfrac{1}{2}\sigma_{23}^2(B_3^{-1}(-1)B_1^{-1} - 2B_3^{-1}B_1^{-1} + B_3^{-1}(1)B_1^{-1})$$

$$+ \tfrac{1}{2}\sigma_{13}^2(B_3^{-1}(-1)B_1^{-1}(-1) - 2B_3^{-1}B_1^{-1} + B_3^{-1}(1)B_1^{-1}(1))$$

$$+ \tfrac{1}{2}\sigma_{12}^2(B_3^{-1}B_1^{-1}(-1) - 2B_3^{-1}B_1^{-1} + B_3^{-1}B_1^{-1}(1))$$

$$+ \tfrac{1}{4}[\sigma_{23,13}^2(B_3^{-1}(-2)B_1^{-1}(-1) - 2B_3^{-1}B_1^{-1} + B_3^{-2}(2)B_1^{-1}(1))$$

$$+ \sigma_{23,12}^2(B_3^{-1}(-1)B_1^{-1}(-1) - 2B_3^{-1}B_1^{-1} + B_3^{-1}(1)B_1^{-1}(1))$$

$$+ \sigma_{13,12}^2(B_3^{-1}(-1)B_1^{-1}(-2) - 2B_3^{-1}B_1^{-1} + B_3^{-1}(1)B_1^{-1}(2))]\} \tag{45}$$

where

$$\bar N_3 = N_3 - \bar\imath_{23} - \bar\imath_{13}\,, \tag{46}$$

$$\bar N_1 = N_1 - \bar\imath_{12} - \bar\imath_{13}\,, \tag{47}$$

and for $j = 3$ or 1,

$$B_j^{-1} = 1/B(\bar N_j\,,\,\lambda_j), \tag{48}$$

$$B_j^{-1}(-1) = B_j^{-1} - 1, \tag{49}$$

$$B_j^{-1}(-2) = B_j^{-1} - 1 - \frac{\bar N_j}{\lambda_j}\,, \tag{50}$$

$$B_j^{-1}(1) = B_j^{-1} + \frac{\lambda_j}{\bar N_j + 1}\,, \tag{51}$$

$$B_j^{-1}(2) = B_j^{-1} + \frac{\lambda_j}{\bar N_j + 1} + \frac{\lambda_j^2}{(\bar N_j + 2)(\bar N_j + 1)}. \tag{52}$$

In many cases, $P\{A_1 \cup A_2 \cup A_3\}$ may be sufficiently well approximated by using $B(\bar N_3\,,\,\lambda_3)B(\bar N_1\,,\,\lambda_1)$ for $P\{A_1 \cap A_3 \mid A_2\}$ since the last term in (32) is typically smaller than the preceding three which are, in turn, typically smaller than the first three.

## IV. DISCUSSION

We tried the method out in a number of examples and found the dependence analyses to improve accuracy considerably when dependence is indeed important and to be useful in indicating when the dependence effect can be safely ignored. This investigation was actually motivated by a study of the use of a satellite to carry telephone traffic. (This study will be described more fully elsewhere.) One mode of such use, called "variable destination," allocates up-channels (each channel consists of a number of trunks) from ground stations to the satellite while all ground stations can receive all that is transmitted downward from the satellite. An equivalent network for five ground stations (the outer nodes) is shown in Fig. 3. Between each pair of ground stations there is an offered traffic; thus, there are 10 (2-way) traffics. For various choices of the traffics and allocations, the dependence effect becomes significant and it was important to know when it needed to be taken into account and when it could safely be ignored (which leads to simplification in the study, particularly the optimization algorithms—an aspect of the study was to optimize the channel allocations to minimize the traffic blocked). The method of this paper was found quite useful in this regard. It should be noted that, although the $N_i$ are very large and results of the form of (1) lead to huge sums, the method was computationally simple.

The approach to dependence given here is one of approximations motivated by exact results in simple contexts. The results obtained thus far are promising and encourage further work in investigating the approximations and in extending the approach. Further experience is needed to determine the best way to apply the results to large net-

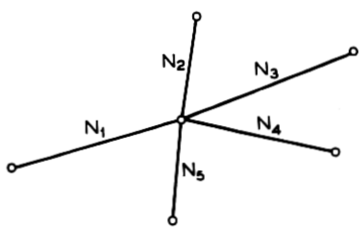


Fig. 3—Equivalent network for satellite variable destination mode.

works (see the last comment in Section III). Observe that improved accuracy (and, possibly, satisfactory accuracy from a practical viewpoint) in large networks can be achieved by considering only two-link or only two- and three-link dependences (already discussed in this paper). An inherent assumption that has been used is that the traffics are Poisson (or close to Poisson). As mentioned in Section I, the analysis of point-point blocking probabilities in conjunction with the use of the equivalent random method will be reported in Ref. 1.

## V. ACKNOWLEDGMENT

I am grateful to D. L. Jagerman for stimulating discussions about approximations.

## APPENDIX

In Section II, two approximations for $P\{A_1 \mid A_2\}$ were given. This Appendix derives the sum approximations for $P_b\{A_1 \mid A_2\}$. In particular, approximations for the following sums will be obtained:*

$$J_1 = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \frac{\lambda^{N-i}}{(N-i)!} = E\left[\frac{\lambda^{N-\xi}}{(N-\xi)!}\right], \tag{53}$$

$$J_2 = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \sum_{j=0}^{N-i} \frac{\lambda^j}{j!} = E\left[\sum_{j=0}^{N-\xi} \frac{\lambda^j}{j!}\right]. \tag{54}$$

The random variable $\xi$ is distributed according to the binomial law. One has

$$\frac{\lambda^{N-j}}{(N-j)!} = \frac{1}{2\pi i} \oint z^{-N+j-1} e^{\lambda z}\, dz, \tag{55}$$

$$\sum_{l=0}^{N-j} \frac{\lambda^l}{l!} = \frac{1}{2\pi i} \oint z^{-N+j-1} \frac{e^{\lambda z}}{1-z}\, dz \tag{56}$$

in which the integration is on a circle around the origin as center.

Let $z = re^{i\theta}$, then

$$\frac{\lambda^{N-j}}{(N-j)!} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda r e^{i\theta} - (N-j) i\theta} r^{-(N-j)}\, d\theta, \tag{57}$$

$$\sum_{l=0}^{N-j} \frac{\lambda^l}{l!} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda r e^{i\theta} - (N-j) i\theta} \frac{r^{-(N-j)}}{1 - re^{i\theta}}\, d\theta. \tag{58}$$

* The derivations to follow are due to D. L. Jagerman who prepared this Appendix.

Computing the required expectations from (57) and (58), one obtains

$$J_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda r e^{i\theta} - Ni\theta - N \ln r} \phi(\theta - i \ln r) \, d\theta, \tag{59}$$

$$J_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\lambda r e^{i\theta} - Ni\theta - N \ln r} \left( \frac{\phi(\theta - i \ln r)}{1 - re^{i\theta}} \right) d\theta \tag{60}$$

in which the characteristic function $\phi(\tau)$ is given by

$$\phi(\tau) = E[e^{i\tau\xi}] = (q + pe^{i\tau})^n. \tag{61}$$

Thus

$$J_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{h_1(\theta)} \, d\theta, \tag{62}$$

$$J_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{h_2(\theta)} \, d\theta, \tag{63}$$

in which

$$h_1(\theta) = \lambda re^{i\theta} - Ni\theta - N \ln r + n \ln (q + pre^{i\theta}), \tag{64}$$

$$h_2(\theta) = \lambda re^{i\theta} - Ni\theta - N \ln r + n \ln (q + pre^{i\theta}) - \ln (1 - re^{i\theta}). \tag{65}$$

The expansions of $h_1(\theta)$ and $h_2(\theta)$ in powers of $\theta$ are

$$\begin{aligned} h_1(\theta) &= \lambda r - N \ln r + n \ln (q + pr) \\ &\quad + \left( \lambda r - N + \frac{npr}{q + pr} \right) i\theta \\ &\quad - \frac{1}{2} \left\{ \lambda r + n \frac{pr}{q + pr} - n \left( \frac{pr}{q + pr} \right)^2 \right\} \theta^2 + \cdots , \end{aligned} \tag{66}$$

$$\begin{aligned} h_2(\theta) &= \lambda r - N \ln r + n \ln (q + pr) - \ln (1 - r) \\ &\quad + \left( \lambda r - N + \frac{npr}{q + pr} + \frac{r}{1 - r} \right) i\theta \\ &\quad - \frac{1}{2} \left\{ \lambda r + n \frac{pr}{q + pr} - n \left( \frac{pr}{q + pr} \right)^2 + \frac{r}{1 - r} + \left( \frac{r}{1 - r} \right)^2 \right\} \theta^2 + \cdots . \end{aligned} \tag{67}$$

Using the method of Hayman (see Ref. 7) the functions $h_1(\theta)$, $h_2(\theta)$ will be made stationary at $\theta = 0$ by defining $r_1$, $r_2$ as follows:

$$\lambda r_1 - N + \frac{npr_1}{q + pr_1} = 0, \tag{68}$$

$$\lambda r_2 - N + \frac{npr_2}{q + pr_2} + \frac{r_2}{1 - r_2} = 0. \tag{69}$$

Thus the values of $r$ satisfy

$$\lambda pr_1^2 + (\lambda q - Np + np)r_1 - Nq = 0, \tag{70}$$

$$\lambda pr_2^3 + (np - p - Np + \lambda q - \lambda p)r_2^2 + (-\lambda q + Np - Nq - np - q)r_2 + Nq = 0. \tag{71}$$

For $r_1$, the following explicit formula is obtained:

$$r_1 = \frac{\sqrt{(\lambda q - Np + np)^2 + 4Npq} - \lambda q + Np - np}{2\lambda p}. \tag{72}$$

For $r_2$, the root between zero and one must be used. Let

$$\beta_1 = \lambda r_1 + \frac{npr_1}{q + pr_1} - n\left(\frac{pr_1}{q + pr_1}\right)^2, \tag{73}$$

$$\beta_2 = \lambda r_2 + n\,\frac{pr_2}{q + pr_2} - n\left(\frac{pr_2}{q + pr_2}\right)^2 + \frac{r_2}{1 - r_2} + \left(\frac{r_2}{1 - r_2}\right)^2, \tag{74}$$

then

$$h_1(\theta) = \lambda r_1 - N \ln r_1 + n \ln (q + pr_1) - \tfrac{1}{2}\beta_1\theta^2 + \cdots, \tag{75}$$

$$h_2(\theta) = \lambda r_2 - N \ln r_2 + n \ln (q + pr_2) - \ln (1 - r_2) - \tfrac{1}{2}\beta_2\theta^2 + \cdots. \tag{76}$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{1}{2}\beta\theta^2}\, d\theta \cong \frac{1}{\sqrt{2\pi\beta}} \tag{77}$$

in which the approximation becomes the more accurate the larger $\beta$ is, one now obtains

$$J_1 \cong r_1^{-N}(q + pr_1)^n \,\frac{e^{\lambda r_1}}{\sqrt{2\pi\beta_1}}, \tag{78}$$

$$J_2 \cong r_2^{-N}(q + pr_2)^n \,\frac{e^{\lambda r_2}}{(1 - r_2)\sqrt{2\pi\beta_2}}. \tag{79}$$

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