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When Are Transistors Passive?*

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The paper presents results on the stability and dynamic behavior under large signal conditions of networks consisting of transistors and sources connected to a linear, passive, memoryless subnetwork. The transistors' model incorporates various nonlinearities. A characteristic common to the main results of the paper is that they relate to properties of the transistors alone and, hence, are independent of the passive part of the network.

Sufficient conditions are obtained for asymptotic and bounded input-bounded output stability. The conditions impose restrictions on some of the physical constants of the transistors' model. These conditions have an interesting physical interpretation in terms of temperature differentials in the transistor junctions. In particular, any transistor with the exponential type of static diode characteristic is passive only if the ratio of the junction temperatures lies inside an interval determined by the α's.

In the state space of the network there exists a well-defined region R specified by the transistors' model with the property that constant terminal states in R are independent of initial conditions. The region R is in a certain sense maximal.

I. INTRODUCTION AND DERIVATION OF THE DIFFERENTIAL EQUATION

The network considered is shown in Fig. 1 and it consists of a number of transistors connected to a subnetwork composed of voltage sources, current sources, and linear, passive, memoryless devices such as resistors and transformers. Sandberg[1] has analyzed the dynamic behavior of such networks and has defined a class which exhibit various features of stability. The results of this paper are essentially different in that they relate to properties of the transistors alone and, hence, are independent of the passive part of the network.

The analysis is concerned with certain network-theoretic properties of

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† The sequence of names was chosen by flipping a coin.

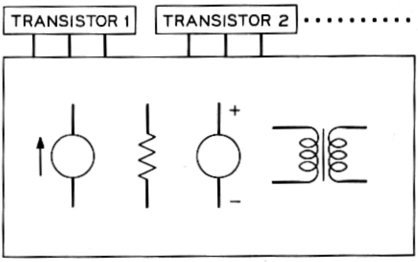

Fig. 1—General network containing transistors, sources, resistors, and transformers.

transistor models that are assumed in extensively used Network-Analysis Programs, such as NET 1, CIRPAK, CIRCUS, etc., (see Ref. 2). The intent of the paper is to study the large signal behavior of transistor networks using this model and certain generalizations of it. Each transistor is represented by a model of the type shown in Fig. 2 which takes into account the nonlinear dc properties as well as the presence of nonlinear junction capacitances. Six parameters are associated with the model: α_1, α_2, τ_1, τ_2, c_1, and c_2, all of which are positive; also $\alpha_1, \alpha_2 < 1$. The two nonlinear static diode functions are denoted by $f_1(\cdot)$ and $f_2(\cdot)$. Initially it is assumed that $f_1(\cdot)$ and $f_2(\cdot)$ are monotone, strictly increasing mappings of the interval $(-\infty, \infty)$ into itself; $f_1(0) = f_2(0) = 0$, and $f_1(\cdot)$ and $f_2(\cdot)$ are continuously differentiable on $(-\infty, \infty)$. Further assumptions are made about $f_1(\cdot)$ and $f_2(\cdot)$ in the course of the paper.

Suppose the network has n transistors and with the polarity indicated in Fig. 2 let v_{2i-1} and v_{2i} respectively denote the emitter-to-base voltage and collector-to-base voltage of the ith transistor. Similarly, let i_{2i-1}

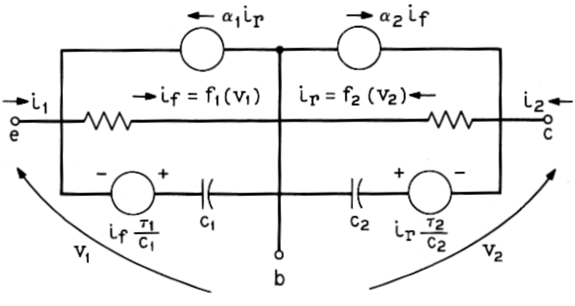

Fig. 2—Transistor model.

and i_{2i} respectively denote the emitter current and the collector current of the ith transistor. An identical scheme of subscripting is used to define $f_i(\cdot)$, c_i, and τ_i for $i = 1, 2, \cdots, 2n$. v, i, and $F(\cdot)$ are $2n$-column vectors formed by arranging $\{v_i\}$, $\{i_i\}$, and $\{f_i(\cdot)\}$ respectively. In applying the current law to the transistors' model it follows that

$$i = \frac{d}{dt} C(v) + TF(v) \tag{1}$$

where

$$[C(v)]_j = c_j v_j + \tau_j f_j(v_j)$$

and $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ the direct sum of n 2 by 2 matrices T_j in which

$$T_j = \begin{bmatrix} 1 & -\alpha_1^j \\ -\alpha_2^j & 1 \end{bmatrix}$$

for $j = 1, \cdots, n$.

The subnetwork to which the transistors are connected is assumed to impose a constraint of the form

$$i = -Gv + b \tag{2}$$

in which G is the conductance matrix and hence $G \geqq 0$,* and b is an element of all real bounded continuous $2n$-vector-valued functions of t on $[0, \infty)$.

From (1) and (2),

$$\frac{d}{dt} C(v) + TF(v) + Gv = b. \tag{3}$$

The above equation is also derived in Ref. 1. Since all of the c_i and τ_i are positive and each of the $f_i(\cdot)$ is continuous and monotone increasing, $C_i^{-1}(\cdot)$ exists; obviously $C^{-1}(\cdot)$ is also strictly monotone increasing and $C^{-1}(0) = 0$.

II. ASYMPTOTIC STABILITY FOR THE UNFORCED SYSTEM

In this section we show how to prove an intuitively reasonable result concerning the asymptotic stability of the system. A Lyapunov function with a simple energy interpretation is introduced and used to prove

* $G \geqq 0$ indicates that G is a positive semidefinite matrix. Unless otherwise stated, G will only be assumed to be positive semidefinite.

stability. Let

$$L(v) \triangleq \sum_{i=1}^{2n} \int_0^{C_i(v_i)} C_i^{-1}(\sigma)\, d\sigma. \tag{4}$$

Since $C_i(\cdot)$ defines the charge-voltage relationship in the nonlinear junction capacitances, $L(v)$ is the total stored electrical energy in the network. It is easily verified that (*i*) $L(v) \geqq 0$, equality holding only when $v = 0$, and (*ii*) $L(v) \to \infty$ as $||v|| \to \infty$. Thus $L(v)$ is a Lyapunov function if $(d/dt)L(v) < 0$. For the unforced network

$$\frac{d}{dt} L(v) = -v^t TF(v) - v^t Gv. \tag{5}$$

Clearly if $v^t TF(v) \geqq 0$ with equality holding only if $v = 0$ then $L(v)$ is a Lyapunov function since G has been assumed to be positive semidefinite. It then follows from a well-known result in stability theory[3] that $v(t) \to 0$ as $t \to \infty$.

Note that for constant v the term $v^t TF(v)$ expresses the net power flow into the transistors and if all the transistors are passive then certainly $v^t TF(v) > 0$. However it is shown later that in certain abnormal conditions the transistors are not all passive and the judicious choice of the passive network allows one to extract energy from the transistors. To show this we first prove the following:

Lemma 1: *Given two real* $2n$*-vectors* x *and* y *such that* $x^t y > 0$, *there exists a* $G > 0$ *for which the following holds*

$$y = Gx. \tag{6}$$

Proof. Define a $2n$ by $2n - 1$ matrix Z such that the columns of Z span the $(2n - 1)$-dimensional subspace orthogonal to x. Then $Z^t x = 0$. Since by assumption y is not orthogonal to x, y is not an element of the range space of Z and, hence, the columns of Z together with y span E^{2n}. For these reasons the following construction of G,

$$G = \frac{1}{x^t y} yy^t + ZZ^t, \tag{7}$$

suffices to prove the lemma.

When for some v_1, $v_1^t TF(v_1) < 0$ then by Lemma 1 there exists a $G > 0$ such that

$$TF(v_1) + Gv_1 = 0. \tag{8}$$

Hence with initial condition $v(0) = v_1$ the solution of the network equation (3) with $b \equiv 0$ is,

$$v(t) \equiv v_1 \tag{9}$$

and the system is not globally asymptotically stable.

The above results may be summarized in the following:

Theorem 1: If $v^tTF(v) > 0 \ \forall v \neq 0$ then the network is globally asymptotically stable. Furthermore, if there exists a v_1 such that $v_1^tTF(v_1) < 0$ then there exists a $G > 0$ for which the system is not globally asymptotically stable.

2.1 *Sufficient Conditions for Asymptotic Stability*

Consideration is given to the positivity of $v^tTF(v)$. Due to the quasi-diagonal structure of T we need only consider in detail the behavior of

$$\varphi(v_{2i-1}, v_{2i}) \triangleq (v_{2i-1}, v_{2i}) \begin{bmatrix} 1 & -\alpha_1^i \\ -\alpha_2^i & 1 \end{bmatrix} \begin{bmatrix} f_{2i-1}(v_{2i-1}) \\ f_{2i}(v_{2i}) \end{bmatrix}. \tag{10}$$

For notational convenience α_1^i, α_2^i, v_{2i-1}, v_{2i}, $f_{2i-1}(\cdot)$, and $f_{2i}(\cdot)$ are respectively denoted by α_1, α_2, v_1, v_2, $f_1(\cdot)$, and $f_2(\cdot)$.

Note that when $v_1v_2 \leqq 0$, $\varphi(v) \geqq 0$ and $\varphi(v) = 0$ only if $v_1 = v_2 = 0$. Let $v_1 \neq 0$, $v_2 = \rho v_1$, and $\rho \geqq 0$. Then

$$\varphi(v) = v_1f_1(v_1) - \alpha_1v_1f_2(v_2) + v_2f_2(v_2) - \alpha_2v_2f_1(v_1) \tag{11}$$

and

$$g(\rho) \triangleq \frac{\varphi(v)}{v_1f_1(v_1)} = (1 - \rho\alpha_2) + \frac{f_2(\rho v_1)}{f_1(v_1)}(\rho - \alpha_1). \tag{12}$$

If $\rho = \alpha_1$ then $g(\rho) > 0$ since $1 - \alpha_1\alpha_2 > 0$. If $\rho \neq \alpha_1$, $g(\rho) = 0$ if and only if

$$\frac{1 - \rho\alpha_2}{\alpha_1 - \rho} = \frac{f_2(\rho v_1)}{f_1(v_1)}. \tag{13}$$

$(1 - \rho\alpha_2)/(\alpha_1 - \rho)$ is plotted as a function of ρ in Fig. 3. The salient features of the function for $\rho \geqq 0$ are:

$$\left.\begin{aligned} \frac{1 - \rho\alpha_2}{\alpha_1 - \rho} &\geqq \frac{1}{\alpha_1} \quad \text{for} \quad 0 \leqq \rho < \alpha_1 \\ &\leqq 0 \quad \text{for} \quad \alpha_1 < \rho \leqq \frac{1}{\alpha_2} \\ &< \alpha_2 \quad \text{for} \quad \frac{1}{\alpha_2} < \rho < \infty \end{aligned}\right\}. \tag{14}$$

Now $[f_2(\rho v_1)/f_1(v_1)]$ is a monotone, strictly increasing function of ρ and

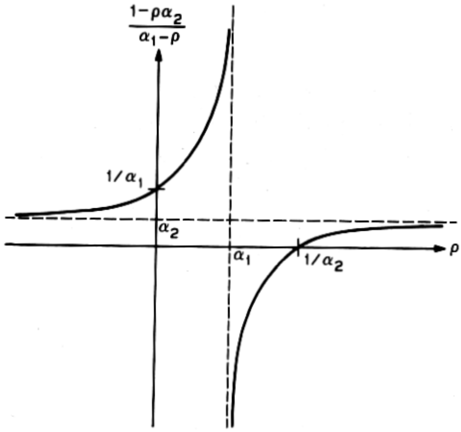

Fig. 3—Plot of $(1 - \rho\alpha_2)/(\alpha_1 - \rho)$.

$[f_2(\rho v_1)/f_1(v_1)]|_{\rho=0} = 0$. Clearly if

$$\left.\begin{aligned} &\frac{f_2(\alpha_1 v_1)}{f_1(v_1)} \leqq \frac{1}{\alpha_1} \\ \text{and} \quad &\frac{f_2\left(\dfrac{1}{\alpha_2} v_1\right)}{f_1(v_1)} \geqq \alpha_2 \end{aligned}\right\} \forall\, v_1 \neq 0 \tag{15}$$

then no solution of (13) exists. Since $g(\rho)$ is continuous in ρ, $g(0) > 0$, and $g(\rho) \neq 0\ \forall\ \rho > 0$, it follows that $g(\rho) > 0\ \forall\ \rho \geqq 0$. Hence sufficient conditions for the positivity of $v^t TF(v)$ are:

$$\left.\begin{aligned} &\frac{f_{2i}(\alpha_1^i v)}{f_{2i-1}(v)} \leqq \frac{1}{\alpha_1^i} \\ \text{and} \quad &\frac{f_{2i}(v)}{f_{2i-1}(\alpha_2^i v)} \geqq \alpha_2^i \end{aligned}\right\} \forall\, v \neq 0 \quad \text{and} \quad i = 1, 2, \cdots, n. \tag{16}$$

Remark:

$$\left.\begin{aligned} \frac{f_{2i}(\rho v)}{f_{2i-1}(v)} &< \frac{1 - \rho\alpha_2}{\alpha_1 - \rho} \quad \text{for} \quad 0 < \rho \leqq \alpha_1 \\ &> \frac{1 - \rho\alpha_2}{\alpha_1 - \rho} \quad \text{for} \quad \frac{1}{\alpha_2} \leqq \rho < \infty \end{aligned}\right\} \forall\, v \neq 0 \tag{17}$$

is both necessary and sufficient for $v^t TF(v) > 0$. However, the practical value of such a condition is limited.

2.2 *Necessary and Sufficient Conditions for the Exponential Type of Nonlinearities*

The functions $f_i(\cdot)$ in the conventional Ebers-Moll model and in the charge-control model[4] are of the following form:

$$f_i(v) = a_i\,(\exp \lambda_i v - 1). \tag{18}$$

Such exponential nonlinearities are subsumed in a class of nonlinearities with the following properties:

(*i*) $f_i(v_i) = a_i f(\lambda_i v_i)$, $i = 1, 2, \cdots, 2n$, where a_i and λ_i are positive constants.
(*ii*) $f(\cdot)$ is a monotone, strictly increasing function.
(*iii*) $f(0) = 0$.
(*iv*) $\lim_{v \to -\infty} f(v) = -1$.
(*v*) $\lim_{v \to \infty} [f(\rho v)/f(v)] = 0$ for all ρ where $0 < \rho < 1$.

When the transistors' static diode characteristics fall under this class then the following is true:

necessary and sufficient conditions for $v^t TF(v) \geqq 0$ ($v^t TF(v) = 0$ only when $v = 0$) are

$$\alpha_1^i \leqq \frac{a_{2i-1}}{a_{2i}}, \frac{\lambda_{2i-1}}{\lambda_{2i}} \leqq \frac{1}{\alpha_2^i} \tag{19}$$

for $i = 1, 2, \cdots, n$.

Sufficiency follows from the definitions and (16). For the necessity part consider the case when $v_{2i-1} \to -\infty$ with the remaining $2n - 1$ variables fixed. It is necessary that the coefficient of v_{2i-1} in the expansion of $v^t TF(v)$ approach a nonpositive value, i.e.,

$$\lim_{v \to -\infty} \{f_{2i-1}(v) - \alpha_1^i f_{2i}(v)\} \leqq 0 \tag{20}$$

for $i = 1, 2, \cdots, n$.
Similarly,

$$\lim_{v \to -\infty} \{f_{2i}(v) - \alpha_2^i f_{2i-1}(v)\} \leqq 0 \tag{21}$$

for $i = 1, 2, \cdots, n$. Hence

$$-a_{2i-1} + \alpha_1^i a_{2i} \leqq 0$$

and,

$$-a_{2i} + \alpha_2^i a_{2i-1} \leqq 0$$

or

$$\alpha_1^i \leqq \frac{a_{2i-1}}{a_{2i}} \leqq \frac{1}{\alpha_2^i} \tag{22}$$

for $i = 1, 2, \cdots, n$. To obtain the remaining conditions consider equation (12) in the notation of this section. By property (v),

$$\lim_{v_{2i-1}\to\infty} \frac{f_{2i}(\rho v_{2i-1})}{f_{2i-1}(v_{2i-1})} = 0$$

for all ρ such that $\rho < \dfrac{\lambda_{2i-1}}{\lambda_{2i}}$ and $i = 1, 2, \cdots, n$.

Hence, a necessary condition for the positivity of $v^t TF(v)$ is: $1 - \rho\alpha_2^i > 0$ is implied by $\rho < \lambda_{2i-1}/\lambda_{2i}$; i.e.,

$$\frac{\lambda_{2i-1}}{\lambda_{2i}} \leqq \frac{1}{\alpha_2^i} \tag{23}$$

for $i = 1, 2, \cdots, n$. Repeating the argument with the roles of v_{2i-1} and v_{2i} interchanged yields

$$\alpha_1^i \leqq \frac{\lambda_{2i-1}}{\lambda_{2i}} \tag{24}$$

for $i = 1, 2, \cdots, n$. This concludes the proof.

Remark 1: It is clear from the derivation that the conditions stated in equation (19) with the ratios replaced by the appropriate limits are necessary for the positivity of $v^t TF(v)$ when $\{f_i(\cdot)\}$ are general monotone, strictly increasing functions.

Remark 2: If condition (19) is violated then there exists a v_1, $\| v_1 \| < \infty$, such that $v_1^t TF(v_1) < 0$. The proof is as indicated below.

Consider equation (12) and say $\lambda_1/\lambda_2 > 1/\alpha_2$; then for ρ such that $1/\alpha_2 < \rho < \lambda_1/\lambda_2$ it follows from property (v) that there exists a V such that for $v_1 > V$, $g(\rho) < 0$. Likewise, for $a_1/a_2 > 1/\alpha_2$ and $\rho(\rho > 0)$ such that $(1 - \alpha_1(a_2/a_1)) + \rho(a_2/a_1 - \alpha_2) < 0$ it follows from property (iv) that there exists a V such that for $v_1 < V$, $g(\rho) < 0$. For the remaining possibilities the proof follows by interchanging the roles of v_1 and v_2 in the definition of $g(\rho)$ and proceeding as above.

III. PASSIVITY

This section is devoted to a discussion on condition (19) which has been shown to be both necessary and sufficient for the passivity of transistors for which the static diode characteristics are of the form

$$f_i(v_i) = a_i(\exp \lambda_i v_i - 1) \tag{25}$$

in which a_i and λ_i are positive constants. When the temperature in the transistor is uniform, then from the well-known physical model,[5] $\lambda_i = (q/KT)$ where q, K, and T are respectively the electron charge, Boltzman's constant, and temperature. However, when a temperature differential exists in the transistor junctions, for instance in a p-n-p transistor the temperature in the neighborhoods of the p-n and n-p junctions are respectively T_1 and T_2, then the same physical model holds[6] and

$$\lambda_{2i-1} = \frac{q}{KT_1} \quad \text{and} \quad \lambda_{2i} = \frac{q}{KT_2}$$

so that

$$\frac{\lambda_{2i-1}}{\lambda_{2i}} = \frac{T_2}{T_1}. \tag{26}$$

Hence when the ratio of the temperatures at the base-collector and emitter-base junctions of at least one transistor, say the ith, lies outside the interval $[\alpha_1^i, 1/\alpha_2^i]$ then there exists a $2n$-vector v_1 such that $v_1^t TF(v_1) < 0$. Furthermore, by Lemma 1, there exists a positive definite matrix G such that the solution of the network equation (3) with $b \equiv 0$ and initial condition v_1 is $v(t) \equiv v_1$. Then,

$$\left.\begin{array}{c}\text{power delivered to the}\\ \text{passive subnetwork}\end{array}\right\} = v_1^t G v_1 = -v_1^t TF(v_1) > 0.$$

The temperature dependence of the a_i's may similarly be exploited to deliver power. The phenomenon of differential heating of transistors as described above is the basis of the thermocouple effect and thermoelectric generators.[5]

IV. BOUNDED INPUT—BOUNDED OUTPUT STABILITY

It is shown in this section that for a positive definite matrix G, bounded inputs, and passive transistors there exists a bounded neighborhood of the origin which, loosely speaking, is sure to contain the forced response of the network. The norm used is the Euclidean norm.

Now

$$v^t TF(v) > 0 \quad \forall v \neq 0 \tag{27}$$

and let

$$|| b(t) || \leqq \mathrm{K}. \tag{28}$$

$L(v)$ is as defined in equation (4). For the forced system, $v \neq 0$,

$$\begin{aligned} \frac{d}{dt} L(v) &= -v^t TF(v) - v^t Gv + v^t b \\ &< -v^t Gv + | v^t b | \\ &\leqq -v^t Gv + \mathrm{K} || v || \quad \text{by Schwarz's inequality} \\ &\leqq -\lambda_{\min} || v ||^2 + \mathrm{K} || v || \end{aligned} \tag{29}$$

where $\lambda_{\min}$ is the smallest eigenvalue of G and is positive by assumption. Hence when $||v|| \geqq \mathrm{K}/\lambda_{\min}$, $(d/dt)L(v) < 0$. Thus there exists a T such that for $t > T$,

$$|| v(t) || < \frac{\mathrm{K}}{\lambda_{\min}}.$$

V. A PROPERTY OF TRANSISTOR NETWORKS

Let $R = \{v' \mid (v' - v)^t T[F(v') - F(v)] > 0 \ \forall v \neq v'\}$. Then: (*i*) if G is positive definite, v' and b' are constant vectors such that $TF(v') + Gv' = b'$, and $v' \in R$, then for all inputs which approach b' as $t \to \infty$, the state vector approaches v' independent of initial conditions; (*ii*) there exists a passive subnetwork for which the overall network is such that if the terminal state corresponding to an input approaching a constant vector is not in $\bar{R}$, the closure of R, then there exists another input approaching the same constant vector for which the terminal state is different.

Remark: It has been pointed out by Sandberg[1] that the independence of the steady state from initial conditions is a basic property, and in Ref. 1 it has been proved that if (T, G) belongs to a certain class then the region R in statement *i* extends to the entire state space of the network. However, the region R defined here is independent of G.

In switching circuits with "memory" the dependence on initial conditions is a salient feature. Statement (*i*) observes that the design of such circuits be such that the steady states lie outside the region R, i.e., the steady state bias voltages violate the conditions which define the region R. Of course, the fact that the transistors are passive merely implies that R is non-empty, and their use in switching circuits with "memory" is not precluded. Statement (*ii*) states that in a certain sense the region R is maximal.

Proof: (*i*)

$$TF(v') + Gv' = b' \tag{30}$$

and

$$\begin{aligned}\frac{d}{dt}C(v) + TF(v) + Gv - b' &= \frac{d}{dt}\{C(v) - C(v')\} \\ &\quad + T\{F(v) - F(v')\} + G(v - v') \\ &= b(t) - b' \end{aligned} \tag{31}$$

from (3) and (30). Define

$$\bar{v} \triangleq v - v', \tag{32a}$$

$$\bar{C}(\bar{v}) \triangleq C(v) - C(v'), \tag{32b}$$

and

$$\bar{F}(\bar{v}) \triangleq F(v) - F(v'). \tag{33}$$

From (31),

$$\frac{d}{dt}\bar{C}(\bar{v}) + T\bar{F}(\bar{v}) + G\bar{v} = b(t) - b'. \tag{34}$$

Note that $\bar{C}_i(\cdot)$ for $i = 1, 2, \cdots, 2n$ are monotone, strictly increasing mappings of the real interval $(-\infty, \infty)$ onto itself and $\bar{C}_i(0) = 0$; also there exists $\bar{C}_i^{-1}(\cdot)$. Since $v' \in R$, $\bar{v}'T\bar{F}(\bar{v}) > 0$. Also by assumption for any $\epsilon > 0$ there exists T such that

$$\| b(t) - b' \| \leqq \lambda_{\min}\,\epsilon \quad \text{for} \quad t \geqq T \tag{35}$$

where $\lambda_{\min}$ is the smallest eigenvalue of G. On applying the results on bounded input-bounded output stability, as stated in Section IV, to the system described by equation (34) it follows that there exists a T' such that for $t > T'$,

$$\| \bar{v}(t) \| = \| v(t) - v' \| < \frac{\lambda_{\min}\epsilon}{\lambda_{\min}} = \epsilon \tag{36}$$

and the proof is complete.

(*ii*) It will be shown that if $v_\infty \notin \bar{R}$ then there exists a $G > 0$ such that

$$TF(v_\infty) + Gv_\infty = b_\infty, \tag{37a}$$

$$TF(v_1) + Gv_1 = b_\infty, \tag{37b}$$

and

$$v_\infty \neq v_1 . \tag{37c}$$

In (37a) v_∞ denotes the terminal vector corresponding to an input approaching the constant vector b_∞. The solution of the network equation (3) with $b \equiv b_\infty$ and initial condition v_1 is $v(t) \equiv v_1$.

The proof is by simply observing from the definition of R and Lemma 1 that there exist v_1 and $G > 0$ such that

$$T\{F(v_\infty) - F(v')\} + G(v_\infty - v') = 0. \tag{38}$$

At least when the static diode characteristics are of the conventional exponential type the region R is easily obtained as shown below:

$$(v' - v)^t T\{F(v') - F(v)\} = \bar{v}^t T D_{v'} F(\bar{v}) \tag{39}$$

where

$$\bar{v} = v' - v$$

and

$$D_{v'} = \text{diag}\,(\exp \lambda_1 v_1' , \exp \lambda_2 v_2' , \cdots , \exp \lambda_{2n} v_{2n}').$$

The necessary and sufficient conditions for the positivity of $\bar{v}^t T D_{v'} F(\bar{v})$

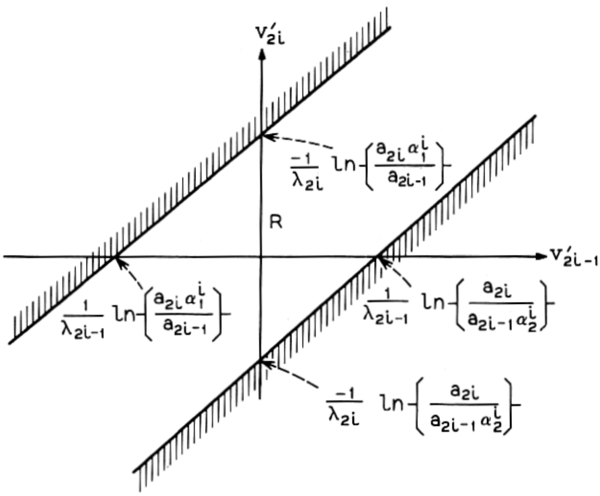

Fig. 4—Region R for the ith transistor (exponential diode characteristics).

are, from (19),

$$\alpha_1^i \leqq \frac{\lambda_{2i-1}}{\lambda_{2i}}, \qquad \frac{a_{2i-1} \exp(\lambda_{2i-1} v'_{2i-1})}{a_{2i} \exp(\lambda_{2i} v'_{2i})} \leqq \frac{1}{\alpha_2^i} \tag{40}$$

for $i = 1, 2, \cdots, n$. (40) is equivalent to

$$\left.\begin{aligned} \alpha_1^i \leqq \frac{\lambda_{2i-1}}{\lambda_{2i}} \leqq \frac{1}{\alpha_2^i} \\ \ln\left\{\frac{a_{2i}\alpha_1^i}{a_{2i-1}}\right\} \leqq (\lambda_{2i-1} v'_{2i-1} - \lambda_{2i} v'_{2i}) \leqq \ln\left\{\frac{a_{2i}}{a_{2i-1}\alpha_2^i}\right\} \end{aligned}\right\} \tag{41}$$

for $i = 1, 2, \cdots, n$. The region R is plotted in Fig. 4.

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