THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 50, No. 8, October, 1971
Printed in U.S.A.

A Fast Amplitude Approximation For Quadrature Pairs

By G. H. ROBERTSON

(Manuscript received April 13, 1971)

This paper gives an algorithm from which an approximation to the amplitude of a quadrature pair can be obtained simply and without requiring more bits than are needed to represent the largest allowable amplitude, A.

If $A \cos \theta$ *and* $A \sin \theta$ *are a quadrature pair, the amplitude is*

$$A = (A^2 \cos^2 \theta + A^2 \sin^2 \theta)^{\frac{1}{2}}.$$

If AX *is the larger magnitude of the pair and* AY *the smaller, an approximation to* A *is*

$$A' = AX + \tfrac{1}{2}AY.$$

Computations using this algorithm with random values of θ *showed that when the true amplitude was* A*, the mean obtained was 1.087A, with standard deviation 0.031A.*

When A *consists of a signal appreciably contaminated with noise so that several estimates are required before the signal can be detected reliably, no significant degradation (*$\ll$*0.1 dB S/N) in detectability was found using the algorithm described above.*

I. INTRODUCTION

When a discrete Fourier transformation is applied to samples from a data segment the results are usually desired in the form of phase and amplitude estimates of the components into which the data is resolved. When FFT[1] algorithms are used to perform the transformation the information from which these estimates are obtained is usually produced in the form of quadrature pairs of numbers $A \cos \theta$, and $A \sin \theta$, where A is the amplitude and θ the phase of the estimate. Obviously

$$A = (A^2 \cos^2 \theta + A^2 \sin^2 \theta)^{\frac{1}{2}}. \tag{1}$$

The intermediate step in this computation of squaring the components of the pair requires in general twice as many bits as are required to represent A. Computing the squares requires appreciable time as does the final square root computation. The situation is aggravated by the fact that there are as many pair members as the original data samples.

This paper describes the use of an algorithm which gives an approximation that in many if not all cases is adequately accurate and greatly reduces the computation time as well as promoting efficiency in the use of hardware by not requiring more bits than are needed to represent the final estimates.

II. DISCUSSION

Let the larger of $\cos^2\theta$, $\sin^2\theta$, be represented by X^2 and the smaller by Y^2. Then

$$\begin{aligned} A &= AX\left(1+\frac{Y^2}{X^2}\right)^{\frac{1}{2}} \\ &= AX\left(1+\frac{1}{2}\frac{Y^2}{X^2}-\frac{1}{8}\frac{Y^4}{X^4}+\cdots\right), \end{aligned} \tag{2}$$

where $1 \geqq Y/X$.
An approximation to A is thus

$$A' = AX + \tfrac{1}{2}AY\left(\frac{Y}{X}\right), \tag{3}$$

which suggests the computationally more attractive form

$$A' = AX + \tfrac{1}{2}AY. \tag{4}$$

Note that AX and AY are magnitudes and (4) implies that the amplitude of a Fourier component is approximately the sum of the larger and half the smaller of its quadrature pair magnitudes.

III. EVALUATION

Curve (a) in Fig. 1 shows a plot of (4) as θ ranges from zero to $\pi/4$ where the curve is reflected to $\theta = \pi/2$. The pattern is then repeated for the remaining three quadrants. The corresponding plot for (3) is shown in curve (b).

From the samples displayed on this figure the mean and standard deviations were calculated and found to be:

For (3), mean = 1.01, standard deviation = 0.0157, and
for (4), mean = 1.087, standard deviation = 0.031.

The algorithm described by (3) thus gives more accurate estimates (as expected) than that described by (4) which, however, is easier to implement and requires less computation.

When the Fourier analysis is carried out in an attempt to detect signal components contaminated by noise it is important to determine how processing techniques affect the detectability of such signals. In order to check this a program was prepared to compute performance estimates when the algorithm based on (4) was used. These estimates were plotted on standard ROC curves[2] for a linear detector and no significant degradation in detectability was observed. Extensive computations using the proposed algorithm have produced spectrograms of noise-contaminated signals that are undistinguishable from those obtained conventionally.

Use of the algorithm described in equation (4) results in a phase-dependent variation of the detector output. However, when the phase is uniformly distributed in the range $(0, 2\pi)$ the amplitudes are quite densely clustered around the mean because of the shape of the modulation characteristic. Eighty-five percent of the amplitudes lie within a range of about 0.4 dB. The accuracy obtained using the fast amplitude algorithm based on (4) may thus be adequate even when the estimates are not corrupted by noise.

IV. CONCLUSION

An algorithm has been described for computing the approximate amplitudes of Fourier harmonics from their quadrature pairs. The

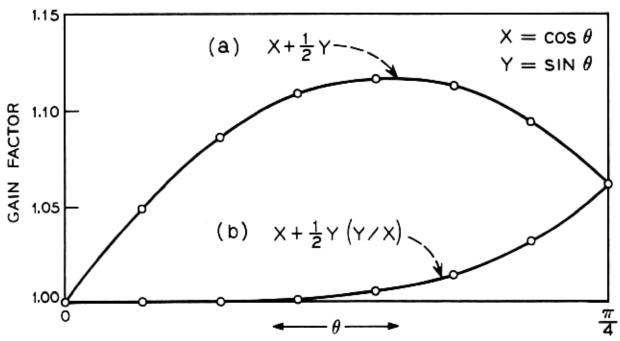

Fig. 1—Algorithm modulation characteristics.

algorithm is fast and does not require the use of more bits than are required to represent the largest permissible result. Use of the algorithm does not significantly degrade the detectability of signals contaminated by noise, compared with use of a perfect linear detector. Estimates obtained using the algorithm are amplitude modulated depending on their phase but the total modulation range is less than 1 dB. Representing the magnitude of the larger component of a quadrature pair by AX, and the smaller by AY, the algorithm is

$$A' = AX + \tfrac{1}{2}AY.$$

REFERENCES

1. Cooley, J. W., and Tukey, J. W., "An Algorithm for the Machine Calculation of Complex Fourier Series," Math. of Computation, *19,* No. 90 (April 1965), pp. 297–301.
2. Robertson, G. H., "Operating Charactertistics for a Linear Detector of CW Signals in Narrowband Gaussian Noise," B.S.T.J., *46,* No. 4 (April 1967), pp. 755–774.