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Power Spectra of Multilevel Digital Phase-Modulated Signals

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A formula is derived for computing the power spectrum of multilevel digital phase-modulated signals. The results apply to arbitrary pulse shapes and probability distributions providing that the pulses do not overlap and are independent. This formula can be applied easily to compute power spectra of signals phase-modulated by various pulse shapes. Several examples are given for rectangular and raised-cosine pulses.

I. INTRODUCTION

Power spectra of digital angle-modulated signals can be calculated in many ways. The direct way of defining the power spectrum is to find the Fourier transform of a sample of the signal on a finite time interval T_o. The magnitude square of this Fourier transform is then divided by T_o and averaged over all possible values of the signal. The power spectrum is finally obtained by taking the limit of the previous result as T_o tends to infinity. Power spectra of binary frequency shift-keyed signals have been calculated by this method by W. R. Bennett and S. O. Rice.[1] R. R. Anderson and J. Saltz[2] have extended the analysis to multilevel digital frequency-modulated signals by using the same technique.

Power spectra of digital phase-modulated signals can also be obtained from the Fourier transform of the signal autocorrelation function. This second method has been used by L. Lundquist[3] to calculate the power spectra of signals phase-modulated by pulse stream $\sum_k a_k g(t - kT)$. The results obtained in his analysis apply to the case of overlapping pulses providing that the random discrete variables a_k are independent and have identical probability distributions.

In this paper, a general expression is derived for the power spectrum of multilevel digital phase-modulated signals by using the Fourier transform technique. The only restriction in these calculations is that the signal is modulated by independent non-overlapping pulses. Otherwise each level can be characterized by a different arbitrary pulse shape and have a different probability distribution. In order to simplify the spectral analysis, we show that a multilevel digital phase-modulated signal given by

$$v(t) = \cos \left[\omega_c t + \sum_k \sum_r a_k^r g_r(t - kT)\right] \tag{1}$$

can also be written in the case of non-overlapping pulses as

$$v(t) = \sum_k \sum_r a_k^r \gamma(t - kT) \cos \left[\omega_c t + P_r(t)\right]. \tag{2}$$

The $g_r(t)$ in (1) define the pulse shapes of the different levels. The discrete variables a_k^r can take the values 0 or 1 and are mutually exclusive in the same time-slot T. In the equivalent expression (2), $\gamma(t)$ is a unity rectangular pulse time-limited to one signaling interval T. The $P_r(t)$ are periodic functions of period T and are respectively equal to the pulse shape functions $g_r(t)$ in the interval T.

The spectral formula which is obtained has the same form as the expression found by H. C. Van Den Elzen[4] for data signals which can be written as a random process:

$$v(t) = \sum_k \sum_r g_{k,r}(t - kT). \tag{3}$$

Digital phase-modulated signals written as (2) can be seen as part of the class of signals given by (3). The same spectral distribution may therefore be expected, once the $g_{k,r}(t)$ are made explicit.

The spectral analysis made in this paper follows the method developed by Bennett and Rice.[1] A simple compact formula is obtained which depends on the pulse shape functions $g_r(t)$ and their probability distributions. The power spectra are found to be generally the sum of a continuous spectrum and of a spectrum made of discrete lines. These discrete lines occur at the carrier frequency ω_c and at frequencies shifted from ω_c by multiples of the timing frequency. Several examples of application are given for

(*i*) rectangular pulses of duration $t \leqq T$
(*ii*) rectangular pulses with finite rise time and decay time
(*iii*) raised-cosine pulses.

The asymptotic limit of the spectrum as $f \to \infty$ is also given for these pulse shapes.

II. LINEAR FORMULATION OF DIGITAL PHASE-MODULATED SIGNALS

Digital phase-modulation presents some similarity with amplitude modulation. For example, binary phase shift-keying by $x(t)(\pi/2)$ radians is identical to amplitude modulation by $x(t)$ if $x(t)$ is a rectangular wave which takes the values 1 or -1, thus

$$v(t) = \cos\left(\omega_c t + x(t)\frac{\pi}{2}\right)$$
$$= x(t)\sin\omega_c t. \tag{4}$$

The same signal can also be written with $\Gamma(t) = [1 + x(t)/2]$ as

$$v(t) = \Gamma(t)\cos\left(\omega_c t + \frac{\pi}{2}\right) + [1 - \Gamma(t)]\cos\left(\omega_c t - \frac{\pi}{2}\right). \tag{5}$$

In a previous paper,[5] it has been shown that this expression can be generalized for any pulse shape which is time-limited to one timing period T. Equation (5) yields in the case of a binary encoding by two pulse shapes $g_1(t)$ and $g_2(t)$:

$$v(t) = \Gamma(t)\cos[\omega_c t + P_1(t)] + [1 - \Gamma(t)]\cos[\omega_c t + P_2(t)]. \tag{6}$$

$P_1(t)$ and $P_2(t)$ are two periodic functions equal respectively to $g_1(t)$ and $g_2(t)$ in a timing period T.

In order to extend this formulation to multilevel digital phase-modulated signals, let us consider the multilevel baseband-encoded signal

$$\phi(t) = \sum_{k=-\infty}^{+\infty}\sum_{r=1}^{n} a_k^r g_r(t - kT). \tag{7}$$

The pulse shape functions $g_r(t)$ associated to the different levels can be arbitrary providing that

$$g_r(t) = 0 \text{ for } 0 > t > T \quad r = 1, 2, \cdots, n. \tag{8}$$

The a_k^r are discrete variables which take the values 1 or 0 at each time slot T. The a_k^r relative to any time slot T are mutually exclusive since only one pulse can exist for each time interval T. These conditions lead to

$$\sum_{r=1}^{n} a_k^r = 1 \quad \text{for} \quad kT \leqq t \leqq (k+1)T$$

$$k = -\infty, \cdots 0, \cdots +\infty. \tag{9}$$

For a signal made of independent pulses, the a_k^r relative to different time slots form a discrete stationary random process.

The baseband signal (7) can be seen as n mutually exclusive signals, each one related to a particular level as shown in Fig. 1. For example

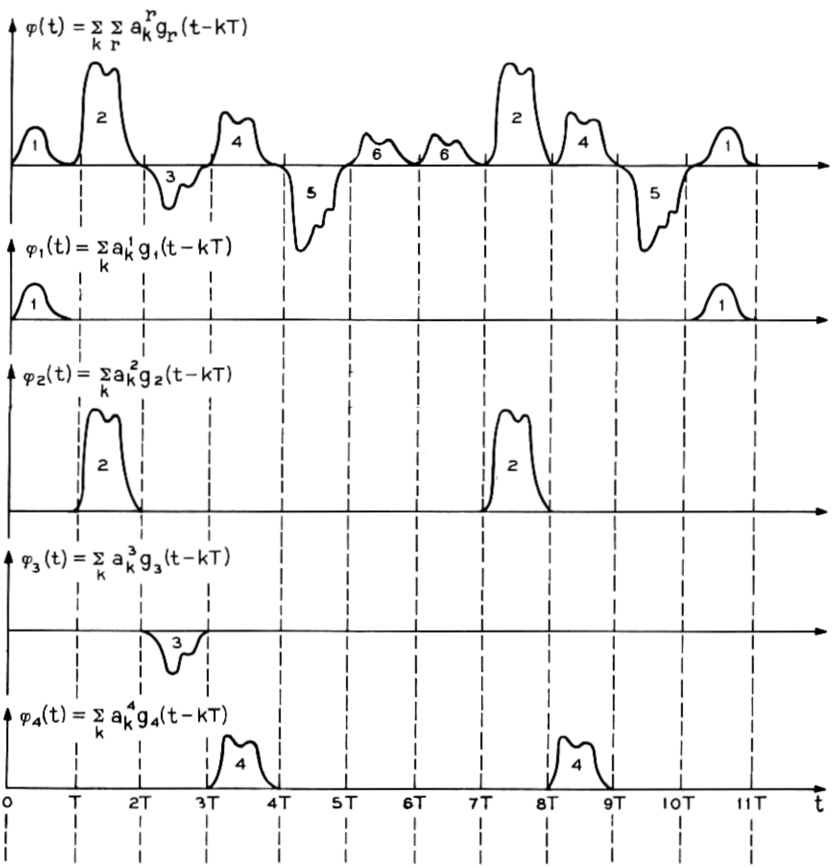

Fig. 1—Multilevel baseband-encoded signal.

the level r is given by

$$\phi_r(t) = \sum_k a_k^r g_r(t - kT). \tag{10}$$

Each $\phi_r(t)$ can itself be seen in Fig. 2 as the product of a unity rectangular wave by a periodic function, thus

$$\phi_r(t) = P_r(t) \sum_k a_k^r \gamma(t - kT). \tag{11}$$

$P_r(t)$ is a periodic function of period T equal to $g_r(t)$ in the interval T. $\gamma(t)$ is a unity rectangular pulse time-limited to one interval T. Let us set

$$\Gamma_r(t) = \sum_k a_k^r \gamma(t - kT), \tag{12}$$

the baseband signal becomes

$$\phi(t) = \sum_r \Gamma_r(t) \cdot P_r(t). \tag{13}$$

A signal phase-modulated by an n-level digital signal such as (7) can be written from (13) as follows:

$$v(t) = \cos\,[\omega_c t + \sum_r \Gamma_r(t) \cdot P_r(t)]. \tag{14}$$

The n unity rectangular signals $\Gamma_r(t)$ as given by (9) are mutually exclusive for all values of t:

$$\sum_r \Gamma_r(t) = 1. \tag{15}$$

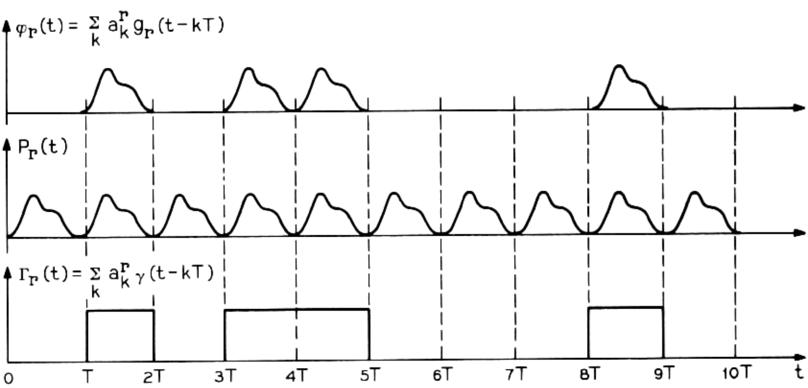

Fig. 2—Baseband signal corresponding to the level r.

As a result, (14) is also equal to

$$v(t) = \sum_r \Gamma_r(t) \cos [\omega_c t + P_r(t)]. \tag{16}$$

The modulated signal written in this form presents some similarity with amplitude modulation. Note that the argument is independent of the random variables a_k^r . This equivalent formulation simplifies greatly the spectral analysis of multilevel phase-modulated signals.

III. POWER SPECTRA OF MULTILEVEL PHASE-MODULATED SIGNALS

3.1 *General Considerations*

The spectrum calculations are made under the following conditions:

(*i*) The baseband-encoded signal is made of independent pulses.
(*ii*) The levels are defined by an ensemble of functions $g_r(t)$ which are time-limited to an interval shorter than or equal to the timing period T.

Otherwise the $g_r(t)$ are arbitrary and can have different probability distributions. Let p_r be the probability of occurrence of the level r. From (10) and (12)

$$p_r = \langle a_k^r \rangle, \tag{17}$$

and from (15)

$$\sum_r p_r = 1. \tag{18}$$

The power spectra, calculated by the Fourier transform method, are given by

$$G(f) = \lim_{NT \to \infty} \frac{1}{NT} \langle | X(f, NT) |^2 \rangle. \tag{19}$$

$X(f, NT)$ is the Fourier transform of a sample of the signal in the interval NT. $\langle | X(f, NT) |^2 \rangle$ is the expectation taken over all possible values of the signal in the interval NT.

The Fourier transform $X(f, NT)$ is calculated by using the linear formulation (16), thus

$$X(f, NT) = \sum_r \int_0^{NT} \Gamma_r(t) \cos [\omega_c t + P_r(t)] e^{-i\omega t} \, dt. \tag{20}$$

$X(f, NT)$ can be separated in two parts which correspond respectively to the positive and negative frequencies:

$$X(f_+ , NT) = \tfrac{1}{2} \sum_r \int_0^{NT} \Gamma_r(t) e^{j[(\omega_c - \omega)t + P_r(t)]} \, dt \tag{21}$$

and

$$X(f_- , NT) = \tfrac{1}{2} \sum_r \int_0^{NT} \Gamma_r(t) e^{-j[(\omega_c + \omega)t + P_r(t)]} \, dt. \tag{22}$$

Note that $X(f_- , NT) = X * (-f_+ , NT)$ [* indicates that j is made $-j$ in (22)].

The Fourier transform (22) relative to the positive frequencies can be written from (12) as

$$X(f_+ , NT) = \tfrac{1}{2} \sum_{k=0}^{N-1} \sum_{r=1}^{n} a_k^r \int_{kT}^{(k+1)T} e^{j[(\omega_c - \omega)t + P_r(t)]} \, dt. \tag{23}$$

Let us set the new variable of integration $y = t - kT$. From the periodicity of the functions $P_r(t)$

$$P_r(y + kT) = P(y), \tag{24}$$

which gives

$$X(f_+ , NT) = \tfrac{1}{2} \sum_{k=0}^{N-1} \sum_{r=1}^{n} a_k^r e^{jkT(\omega_c - \omega)} \int_0^T e^{j[(\omega_c - \omega)y + P_r(y)]} \, dy. \tag{25}$$

In the limit of integration $P_r(y) = g_r(y)$; $g_r(y)$ can then be substituted to $P_r(y)$, thus

$$X(f_+ , NT) = \tfrac{1}{2} \sum_{k=0}^{N-1} \sum_{r=1}^{n} a_k^r e^{jkT(\omega_c - \omega)} \int_0^T e^{j[(\omega_c - \omega)y + g_r(y)]} \, dy. \tag{26}$$

Let us set

$$F_r(f_+) = \frac{1}{2} \int_0^T e^{j[(\omega_c - \omega)y + g_r(y)]} \, dy \tag{27}$$

and

$$F_r(f_-) = \frac{1}{2} \int_0^T e^{-j[(\omega_c + \omega)y + g_r(y)]} \, dy. \tag{28}$$

The Fourier transform (20) takes the form

$$X(f, NT) = \sum_{k=0}^{N-1} \sum_{r=1}^{n} a_k^r \{ e^{jkT(\omega_c - \omega)} F_r(f_+) + e^{-jkT(\omega_c + \omega)} F_r(f_-) \}. \tag{29}$$

3.2 *Power Spectrum Calculations*

The power spectrum calculated from (19) is the sum of four terms given by

$$G(f) = \operatorname*{Lim}_{NT\to\infty} \frac{1}{NT}\left\langle \sum_{\substack{k\\l}=0}^{N-1} \sum_{\substack{r\\s}=1}^{n} a_k^r a_l^s \begin{Bmatrix} F_r(f_+)\cdot F_s^*(f_+)e^{jT(k-l)(\omega_c-\omega)} \\ +F_r(f_-)\cdot F_s^*(f_-)e^{-jT(k-l)(\omega_c+\omega)} \\ +F_r(f_+)\cdot F_s^*(f_-)e^{j[kT(\omega_c-\omega)+lT(\omega_c+\omega)]} \\ +F_r(f_-)\cdot F_s^*(f_+)e^{-j[kT(\omega_c+\omega)+lT(\omega_c-\omega)]} \end{Bmatrix}\right\rangle. \tag{30}$$

The third and fourth terms give the foldover between the positive and negative frequency parts of the spectrum. This effect is negligible when the bandwidth spectrum is much smaller than the carrier frequency which is the case in most applications. The spectral distributions are then accurately described by the first and the second terms of (30) which give respectively the power spectra relative to the positive and negative frequencies. The power spectrum relative to the positive frequencies yields in that case

$$G(f_+) = \operatorname*{Lim}_{NT\to\infty} \frac{1}{NT}\left\langle \sum_{\substack{k\\l}=0}^{N-1} \sum_{\substack{r\\s}=1}^{n} a_k^r a_l^s e^{jT(k-l)(\omega_c-\omega)} F_r(f_+)F_s^*(f_+)\right\rangle. \tag{31}$$

The ensemble average is taken on the random variables $a_k^r a_l^s$. The bracket sign can therefore be introduced on the summation as

$$G(f_+) = \operatorname*{Lim}_{NT\to\infty} \frac{1}{NT}\left\{ \sum_{\substack{k\\l}=0}^{N-1} \sum_{\substack{r\\s}=1}^{n} \langle a_k^r a_l^s\rangle e^{jT(k-l)(\omega_c-\omega)} F_r(f_+)\cdot F_s^*(f_+)\right\}. \tag{32}$$

From the definition given in Section II:

$$\langle a_k^r a_l^s\rangle = \begin{cases} p_r p_s & \text{if} \quad k \neq l \quad \text{and} \quad r \neq s \\ p_r^2 & \text{if} \quad k \neq l \quad \text{and} \quad r = s \\ p_r & \text{if} \quad k = l \quad \text{and} \quad r = s \\ 0 & \text{if} \quad k = l \quad \text{and} \quad r \neq s \end{cases} \tag{33}$$

where p_r and p_s are the probability distributions of the levels r and s.

After substitution of the terms $\langle a_k^r a_l^s\rangle$ by their values given by (33), a summation is made on the variable l. The result gives

$$G(f_+) = \operatorname*{Lim}_{NT\to\infty}\left[\sum_{\substack{r\\s}=1}^{n}\left\{\frac{1}{T}\sqrt{p_r p_s}\,\delta_r^s + \frac{2}{T}p_r p_s \right.\right.$$
$$\left.\left.\cdot\sum_{k=1}^{N-1}\frac{(N-k)}{N}\cos kT(\omega_c-\omega)\right\}F_r(f_+)F_s^*(f_+)\right], \tag{34}$$

δ_r^s is the Kronecker symbol given by

$$\delta_r^s = \begin{cases} 1 & \text{if} \quad r = s \\ 0 & \text{if} \quad r \neq s \end{cases}.$$

Since the limit operation acts only on the second term of (34), $G(f_+)$ can be rewritten as

$$G(f_+) = \Bigg[\sum_{\substack{s \\ r}}^{n}{}_{=1} \bigg\{ \frac{1}{T} \sqrt{p_r p_s}\ \delta_r^s + 2 p_r p_s \lim_{NT \to \infty} \frac{1}{NT}$$

$$\cdot \sum_{k=1}^{N-1} (N - k) \cos kT(\omega_c - \omega) \bigg\} F_r(f_+) F_s(f_+) \Bigg] \cdot \qquad (35)$$

Summation on the discrete variable k yields for the series on k

$$\lim_{NT \to \infty} \frac{2}{NT} \sum_{k=1}^{N-1} (N - k) \cos kT(\omega_c - \omega)$$

$$= \lim_{NT \to \infty} \frac{1}{NT} \left[-N + \frac{\sin^2 (\omega_c - \omega) \dfrac{NT}{2}}{\sin^2 (\omega_c - \omega) \dfrac{T}{2}} \right]$$

$$= -\frac{1}{T} + \frac{1}{T^2} \sum_{m=-\infty}^{+\infty} \delta\left(f_c - f + \frac{m}{T} \right) \cdot \qquad (36)$$

$$\sum_{m=-\infty}^{+\infty} \delta\left(f_c - f + \frac{m}{T} \right)$$

forms a set of Dirac functions which give a series of spikes:

$$\sum_{m=-\infty}^{+\infty} \delta\left(f_c - f + \frac{m}{T} \right) = \begin{cases} +\infty & \text{if} \quad f = f_c + \dfrac{m}{T} \\ 0 & \text{if} \quad f \neq f_c + \dfrac{m}{T} \end{cases}. \qquad (37)$$

The substitution of (36) into (35) yields finally for the power spectrum relative to the positive frequencies

$$G(f_+) = \frac{1}{T} \left\{ \sum_{r=1}^{n} p_r \mid F_r(f_+) \mid^2 - \left| \sum_{r=1}^{n} p_r F_r(f_+) \right|^2 \right\}$$

$$+ \frac{1}{T^2} \sum_{m=-\infty}^{+\infty} \left| \sum_{r=1}^{n} p_r F_r(f_+) \right|^2 \delta\left(f_c - f + \frac{m}{T} \right) \cdot \qquad (38)$$

The first term of (38) gives the continuous power spectrum. The second part gives the spectral discrete lines. These lines occur at the frequencies

$$f = f_c + \frac{m}{T} \quad (m = 0, \pm 1, \pm 2, \cdots)$$

$$\text{if} \quad \left| \sum_{r=1}^{n} p_r F_r(f_+) \right|^2 \neq 0 \quad \text{at} \quad f = f_c + \frac{m}{T}.$$

The final formula (37) depends on two sets of parameters:

(*i*) the probability distributions p_r of each level

(*ii*) the pulse shape functions $g_r(t)$ of each level from the Fourier transforms (27).

When the n levels have identical probability distributions, (38) becomes

$$G(f_+) = \frac{1}{T}\left\{\frac{1}{n}\sum_{r=1}^{n} | F_r(f_+) |^2 - \frac{1}{n^2}\left|\sum_{r=1}^{n} F_r(f_+)\right|^2\right\} + \frac{1}{T^2}\sum_{m=-\infty}^{+\infty}\frac{1}{n^2}\left|\sum_{r=1}^{n} F_r(f_+)\right|^2 \delta\left(f_c - f + \frac{m}{T}\right). \tag{39}$$

The pulse shape, in most cases, is given by a unique function $g(t)$ for all the levels. Each level is then defined by a different pulse amplitude. In this case, $g(t)$ can only have one maximum value which is assumed to be equal to one. The levels are then given by the ensemble $\alpha_r g(t)$, $r = 1, 2, \cdots, n$, where the α_r are equal to the peak phase deviations. Several examples of power spectra will be calculated for this important case.

IV. EXAMPLE OF POWER SPECTRA

4.1 *Rectangular Pulse Shape of Duration T*

The application of (38) to a signal phase-modulated by rectangular pulses of duration equal to the timing period T is straightforward. The Fourier transforms (27) are given in this case by

$$F_r(f_+) = \frac{1}{2}\int_0^T e^{j[(\omega_c-\omega)y+\alpha_r]}\,dy = \frac{T}{2} e^{j\alpha_r}\cdot e^{j(\omega_c-\omega)(T/2)} \frac{\sin(\omega_c-\omega)\dfrac{T}{2}}{(\omega_c-\omega)\dfrac{T}{2}}, \tag{40}$$

where the α_r are the phase deviations of the different levels. Equation (40) applied to (38) gives

$$G(f_+) = \frac{T}{4}\left[\frac{\sin(\omega_c - \omega)\frac{T}{2}}{(\omega_c - \omega)\frac{T}{2}}\right]^2$$

$$\cdot\left\{1 - \left|\sum_{r=1}^{n} p_r e^{i\alpha_r}\right|^2\right\} + \frac{1}{4}\left|\sum_{r=1}^{n} p_r e^{i\alpha_r}\right|^2 \delta(f_c - f). \qquad (41)$$

The spectrum has only one line at the carrier frequency.

If one assumes that the levels are equidistant and equiprobable such that

$$\left.\begin{aligned} \alpha_r &= \alpha_0 + \frac{r-1}{n}\alpha \\ p_r &= \frac{1}{n} \end{aligned}\right\} \quad r = 1, 2, \cdots, n \qquad (42)$$

one obtains

$$\left|\sum_{1}^{n} p_r e^{i\alpha_r}\right|^2 = \frac{1}{n^2}\frac{\sin\frac{\alpha}{2}\,2}{\sin\frac{\alpha}{2n}}, \qquad (43)$$

and the spectrum is given by

$$G(f_+) = \frac{T}{4}\left[\frac{\sin(\omega_c - \omega)\frac{T}{2}}{(\omega_c - \omega)\frac{T}{2}}\right]^2$$

$$\cdot\left\{1 - \frac{1}{n^2}\left[\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2n}}\right]^2\right\} + \frac{1}{4n^2}\left[\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2n}}\right]^2 \delta(f_c - f). \qquad (44)$$

In polar modulation, the α_r can take positive and negative values. The term $\sum_{r=1}^{n} p_r e^{i\alpha_r}$ can then become equal to zero for an infinite number of solutions. All these solutions give the same power spectrum, but without the discrete line:

$$G(f_+) = \frac{T}{4}\left[\frac{\sin(\omega_c - \omega)\frac{T}{2}}{(\omega_c - \omega)\frac{T}{2}}\right]^2. \qquad (45)$$

For example, the polar systems made of q pairs of equidistant and equiprobable levels such that

$$\sum_{r=1}^{2q} e^{i\alpha_r} = \sum_{r=1}^{q} \cos \alpha_r = 0$$

are given by

$$\text{2 levels polar modulation} \quad \left\{ \alpha_{\pm 1} = \pm \frac{\pi}{2} \right.$$

$$\text{4 levels polar modulation} \quad \begin{cases} \alpha_{\pm 1} = \pm \dfrac{\pi}{4} \\ \alpha_{\pm 2} = \pm \dfrac{3\pi}{4} \end{cases}$$

$$\text{6 levels polar modulation} \quad \begin{cases} \alpha_{\pm 1} = \pm \dfrac{\pi}{4} \\ \alpha_{\pm 2} = \pm \dfrac{\pi}{2} \\ \alpha_{\pm 3} = \pm \dfrac{3\pi}{4} \end{cases}$$

· · · etc.

All these modes have the same spectrum given by (45).

4.2 *Rectangular Pulse of Duration* $\tau < T$

Let us now consider a rectangular pulse shape of duration $\tau < T$. Assume the pulses are located symmetrically in the interval T as shown in Fig. 3. The Fourier transforms $F_r(f_+)$ are equal in this case to

$$F_r(f_+) = \frac{1}{2} \left\{ \int_0^{(T-\tau)/2} e^{i(\omega_c - \omega)y} \, dy \right.$$
$$\left. + e^{i\alpha_r} \int_{(T-\tau)/2}^{(T+\tau)/2} e^{i(\omega_c - \omega)y} \, dy + \int_{(T+\tau)/2}^{T} e^{i(\omega_c - \omega)y} \, dy \right\}. \tag{46}$$

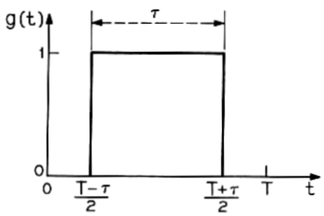

Fig. 3—Rectangular pulse with $\tau < T$.

The result of (46) applied to (38) yields for the continuous part of the spectrum

$$G_c(f_+) = \frac{T}{4}\left\{\sum_{r=1}^{n} p_r \left| \frac{\sin(\omega_c - \omega)\frac{T}{2}}{(\omega_c - \omega)\frac{T}{2}} - \frac{\tau}{T}\frac{\sin(\omega_c - \omega)\frac{\tau}{2}}{(\omega_c - \omega)\frac{\tau}{2}}(1 - e^{j\alpha_r})\right|^2 - \left| \frac{\sin(\omega_c - \omega)\frac{T}{2}}{(\omega_c - \omega)\frac{T}{2}} - \frac{\tau}{T}\frac{\sin(\omega_c - \omega)\frac{\tau}{2}}{(\omega_c - \omega)\frac{\tau}{2}}\left(1 - \sum_{r=1}^{n} p_r e^{j\alpha_r}\right)\right|^2\right\}. \tag{47}$$

The discrete lines are given by

$$G_d(f_+) = \frac{1}{4}\sum_{m=-\infty}^{+\infty}\left| \frac{\sin m\pi}{m\pi} - \frac{\tau}{T}\frac{\sin m\pi\frac{\tau}{T}}{m\pi\frac{\tau}{T}}\left(1 - \sum_{r=1}^{n} p_r e^{j\alpha_r}\right)\right|^2 \delta\left(f_c - f + \frac{m}{T}\right). \tag{48}$$

There are always discrete lines in this case. As an example let us consider the case of a polar phase modulation with two equiprobable levels given by $\alpha_r = \pm\pi/2$ radians. Equations (47) and (48) give in this case:

$$G(f_+) = \frac{T}{4}\left(\frac{\tau}{T}\right)^2\left[\frac{\sin(\omega_c - \omega)\frac{\tau}{2}}{(\omega_c - \omega)\frac{\tau}{2}}\right]^2 + \frac{1}{4}\sum_{m=-\infty}^{+\infty}\left[\frac{\sin m\pi}{m\pi} - \frac{\tau}{T}\frac{\sin m\pi\frac{\tau}{T}}{m\pi\frac{\tau}{T}}\right]^2 \delta\left(f_c - f + \frac{m}{T}\right). \tag{49}$$

For $T - \tau \ll T$ the continuous spectrum is practically the same as the one obtained for pulses of duration T, except for the discrete lines at $f = f_c + m/T$ ($m = \pm 1, \pm 2, \cdots$). In both cases the spectrum decreases as $1/f^2$ as $f \to \infty$.

4.3 *Rectangular Pulse Shape with Finite Rise Time and Decay Time*

We consider in this example a rectangular pulse shape of duration T

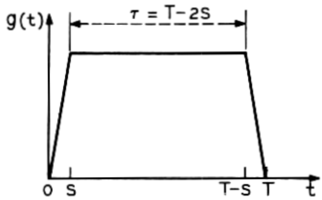

Fig. 4—Rectangular pulse with finite rise time and decay time.

with a finite rise time and decay time as shown in Fig. 4. The calculations are made for the case of two equiprobable levels with peak phase deviations equal to $\pm\pi/2$ radians. We assume that the rise time and decay time are both equal to $s = (T - \tau)/2$; τ is the top pulse length The Fourier transforms $F_r(f_+)$ are given in this case by

$$F_1(f_+) = \frac{1}{2}\left\{\int_0^s e^{j[(\omega_c-\omega)+(\pi/2s)]y}\,dy + e^{j(\pi/2)}\int_s^{T-s} e^{j(\omega_c-\omega)y}\,dy + e^{j(\pi/2)(T/s)}\int_{T-s}^{T} e^{[j(\omega_c-\omega)-(\pi/2s)]y}\,dy\right\}. \tag{50}$$

$F_2(f_+)$ can be obtained from (50) by changing the sign of $\pi/2$. The calculations yield for the continuous power spectrum:

$$G_c(f_+) = \frac{T}{4}\left(\frac{\pi}{4}\right)^2\left\{\frac{\dfrac{\tau}{T}\dfrac{\sin(\omega_c-\omega)\frac{\tau}{2}}{(\omega_c-\omega)\frac{\tau}{2}} + \dfrac{s}{T}\cos(\omega_c-\omega)\dfrac{T}{2}}{\left[(\omega_c-\omega)\frac{s}{2}\right]^2 - \left(\frac{\pi}{4}\right)^2}\right\}^2. \tag{51}$$

The discrete lines are given by

$$G_d(f_+) = \frac{1}{4}\left(\frac{\pi}{4}\right)^2\left(\frac{s}{T}\right)^2\sum_{m=-\infty}^{+\infty}\frac{\cos^2\left(2\pi m\frac{s}{T}\right)}{\left[\left(\pi m\frac{s}{T}\right)^2 - \left(\frac{\pi}{4}\right)^2\right]^2}\,\delta\left(f_c - f + \frac{m}{T}\right). \tag{52}$$

The power spectrum decreases in this case as $1/f^4$ as $f \to \infty$. Figure 5 shows the spectrum given by (51) and (52) which is calculated for $s/T = 0.1$.

4.4 *Raised-Cosine Pulse Shape*

In the case of raised-cosine pulse shapes, the spectra are computed for polar modulation. We assume that the pulse duration is equal to the timing period T. The pulse, of maximum height 1, is then given by $g(t) = \sin^2 \Omega t/2$ with $\Omega = 2\pi/T$. A first example is calculated for a two-level system with different probability distributions. The second example is calculated for a multilevel system with equiprobable level distributions. The two-level and four-level cases are then compared for peak phase distributions which give $\sum_1^n e^{j\alpha_r} = 0$.

4.4.1 *Two-Level Polar Phase Modulation with Raised-Cosine Pulses*

Let $\pm 2\alpha$ radians be the peak phase deviations of the two levels. The Fourier transforms $F_r(f_+)$ are given in this case by

$$F_1(f_+) = \frac{1}{2} \int_0^T e^{j[(\omega_c - \omega)y + 2\alpha \sin^2 \Omega y/2]} \, dy \,. \tag{53}$$

$F_2(f_+)$ can be obtained from (53) by changing the sign of α.

The term $2j\alpha \sin^2 \Omega y/2$ of (53) expanded in Bessel series gives

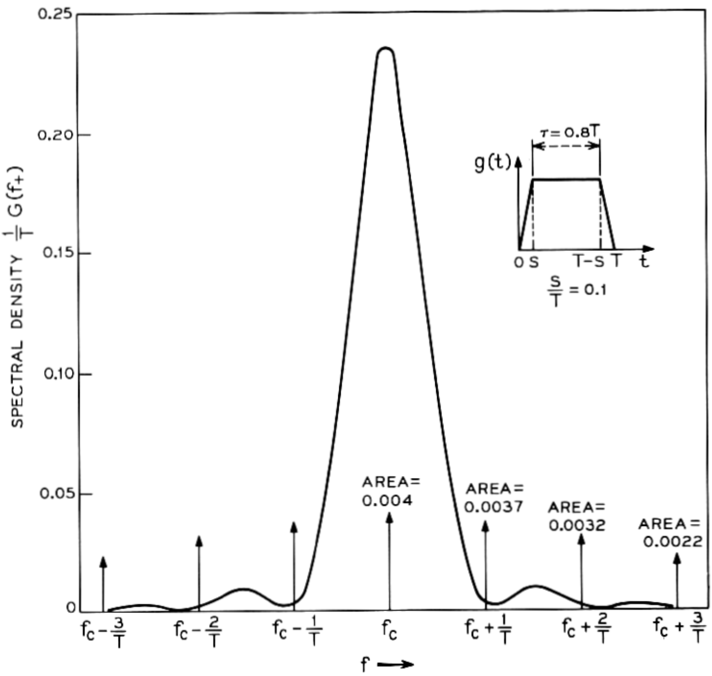

Fig. 5—Power spectrum for two-level polar phase modulation with rectangular pulses with finite rise time and decay time.

$$e^{2j\alpha \sin^2 \Omega y/2} = e^{j\alpha}\left\{J_o(\alpha) + 2\sum_{n=1}^{+\infty}(-1)^n J_{2n}(\alpha)\cos 2n\Omega y + 2j \cdot \sum_{n=1}^{+\infty}(-1)^n J_{2n-1}(\alpha)\cos(2n-1)\Omega y\right\}. \tag{54}$$

Substitution of $e^{2j\alpha \sin^2 \Omega y/2}$ by its Bessel series expansion (54) into (53) yields

$$F(f_+, \pm\alpha) = \frac{T}{2} e^{j[(\omega_c-\omega)(T/2)\pm\alpha]}\left[\frac{\sin(\omega_c-\omega)\frac{T}{2}}{(\omega_c-\omega)\frac{T}{2}}\right] \cdot\left\{J_0(\alpha) + 2\sum_1^\infty(-1)^n J_{2n}(\alpha)\frac{(\omega_c-\omega)^2}{(\omega_c-\omega)^2-(2n\Omega)^2} \pm 2j\sum_1^\infty(-1)^n J_{2n-1}(\alpha)\frac{(\omega_c-\omega)^2}{(\omega_c-\omega)^2-(2n-1)^2\Omega^2}\right\}. \tag{55}$$

Equation (55) applied to (38) yields for the continuous part of the power spectrum:

$$G_c(f_+) = p_1p_2T\left[\frac{\sin(\omega_c-\omega)\frac{T}{2}}{(\omega_c-\omega)\frac{T}{2}}\right]^2 \cdot\left\{\left[J_0(\alpha) + 2\sum_1^\infty(-1)^n J_{2n}(\alpha)\frac{(\omega_c-\omega)^2}{(\omega_c-\omega)^2-(2n\ \Omega)^2}\right]\sin\alpha + 2\sum_1^\infty(-1)^n J_{2n-1}(\alpha)\frac{(\omega_c-\omega)^2}{(\omega_c-\omega)^2-[(2n-1)\Omega]^2}\cos\alpha\right\}^2. \tag{56}$$

Noting that

$$J_0(\alpha) + 2\sum_1^\infty(-1)^n J_{2n}(\alpha) = \cos\alpha$$

and

$$2\sum_1^\infty(-1)^n J_{2n-1}(\alpha) = -\sin\alpha, \tag{57}$$

(56) can be rewritten as

$$G_c(f_+) = 4p_1p_2T\left[\frac{\sin(\omega_c-\omega)\frac{T}{2}}{(\omega_c-\omega)\frac{T}{2}}\right]^2\left\{\sum_{m=1}^\infty(-1)^m J_m(\alpha)\frac{\sin\left(\alpha+\frac{m\pi}{2}\right)}{1-\left(\frac{\omega_c-\omega}{m\Omega}\right)^2}\right\}^2. \tag{58}$$

The discrete lines are given by

$$G_d(f_+) = \frac{1}{4} \sum_{m=-\infty}^{+\infty} J_m^2(\alpha)\left[1 - 4p_1p_2 \sin^2\left(\alpha + m\frac{\pi}{2}\right)\right] \delta\left(f_c - f + \frac{m}{T}\right). \tag{59}$$

The asymptotic limit of $G_c(f_+)$ as $f \to \infty$ is obtained by summing (58) for $| \omega_c - \omega | > m\Omega$, thus

$$\operatorname*{Lim}_{f\to\infty} G_c(f_+) = 4p_1p_2T\left\{\frac{\pi\alpha \sin (\omega_c - \omega)\dfrac{T}{2}}{\left[(\omega_c - \omega)\dfrac{T}{2}\right]^3}\right\}^2, \tag{60}$$

which shows that $G_o(f_+) \sim 1/f^6$ as $f \to \infty$.

The power sharing between $G_c(f_+)$ and $G_d(f_+)$ is a function of α and p_1p_2. Since $p_1 + p_2 = 1$, $G_c(f_+) = 0$ if p_1 or $p_2 = 1$ as expected. All the power is then contained in the discrete lines (case of a signal phase-modulated by a periodic function).

The condition $p_1 = p_2 = \frac{1}{2}$ (identical probability distributions) gives a maximum for the continuous distributed power:

$$P_c(f_+) = \frac{1}{4} \sum_{m=-\infty}^{+\infty} J_m^2(\alpha) \sin^2\left(\alpha + m\frac{\pi}{2}\right)$$

and

$$P_d(f_+) = \frac{1}{4} \sum_{m=-\infty}^{+\infty} J_m^2(\alpha) \cos^2\left(\alpha + m\frac{\pi}{2}\right). \tag{61}$$

The power becomes equally divided between both parts of the spectrum for a peak phase deviation of $\pm\pi/2$ radians and $p_1 = p_2 = \frac{1}{2}$. For this particular case,[5] the series expansion of (55) and (59) can be limited to the first two terms. A good approximation for the power spectrum is then obtained by

$$G_c(f_+) \cong \frac{T}{8}\left[\frac{\sin (\omega_c - \omega)\dfrac{T}{2}}{(\omega_c - \omega)\dfrac{T}{2}}\right]^2$$

$$\cdot\left\{J_0\left(\frac{\pi}{4}\right) - 2(\omega_c - \omega)^2\left[\frac{J_1\left(\frac{\pi}{4}\right)}{(\omega_c - \omega)^2 - \Omega^2} + \frac{J_2\left(\frac{\pi}{4}\right)}{(\omega_c - \omega)^2 - (2\Omega)^2}\right]\right\}^2 \tag{62}$$

and

$$G_d(f_+) \cong \frac{1}{8} \sum_{m=-2}^{+2} J_m^2\left(\frac{\pi}{4}\right) \delta\left(f_c - f + \frac{m}{T}\right). \tag{63}$$

Figure 6 shows the power spectrum given by (62) and (63).

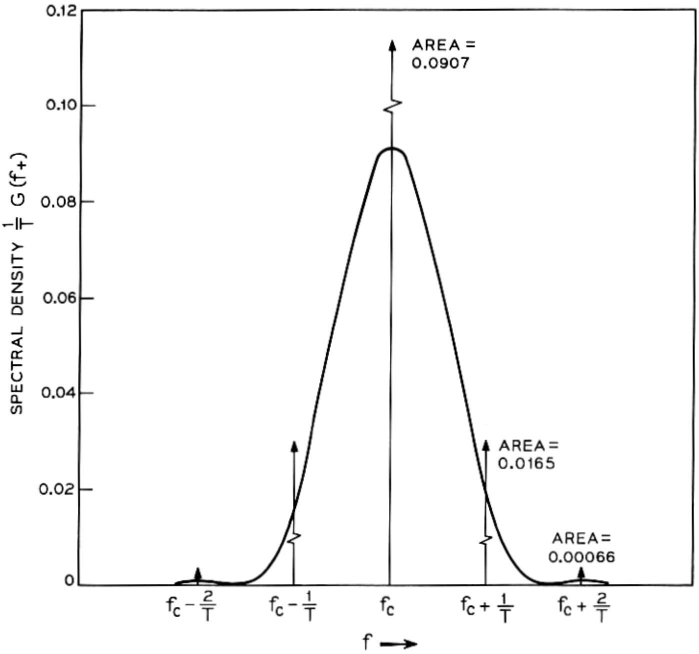

Fig. 6—Power spectrum for two-level polar phase modulation with raised-cosine pulses.

4.4.2 *Multilevel Polar Phase Modulation with Raised-Cosine Pulses*

The results obtained in the previous section can be generalized in the case of multilevel polar phase modulation with identical probability distributions. Let $\pm 2\alpha_r$ radians be the peak phase deviation of a multilevel system of q pairs of levels ($r = 1, 2, \cdots, q$). The Fourier transforms from (55) are equal to

$$F_r(f_+, \pm\alpha_r) = \frac{T}{2} e^{j[(\omega_c - \omega)(T/2) \pm \alpha_r]} \frac{\sin (\omega_c - \omega) \frac{T}{2}}{(\omega_c - \omega) \frac{T}{2}} \{A_r \pm jB_r\}, \qquad (64)$$

with

$$A_r = J_0(\alpha_r) + 2 \sum_1^\infty (-1)^n J_{2n}(\alpha_r) \frac{(\omega_c - \omega)^2}{(\omega_c - \omega)^2 - (2n\Omega)^2}$$

and \hfill (65)

$$B_r = 2 \sum_1^\infty (-1)^n J_{2n-1}(\alpha_r) \frac{(\omega_c - \omega)^2}{(\omega_c - \omega)^2 - [(2n - 1)\Omega]^2}.$$

The continuous power spectrum calculated from (38) with (64) and (65) yields

$$G_c(f_+) = \frac{T}{4}\left[\frac{\sin(\omega_c-\omega)\frac{T}{2}}{(\omega_c-\omega)\frac{T}{2}}\right]^2 \cdot\left\{\frac{1}{q}\sum_1^q[A_r^2+B_r^2]-\frac{1}{q^2}\left[\sum_1^{-q}(A_r\cos\alpha_r-B_r\sin\alpha_r)\right]^2\right\}. \quad (66)$$

Taking into account (57), A_r and B_r can be rewritten as

$$A_r = \cos\alpha_r + 2\sum_1^\infty(-1)^n\frac{J_{2n}(\alpha_r)}{1-\left(\frac{\omega_c-\omega}{2n\Omega}\right)^2}$$

and (67)

$$B_r = -\sin\alpha_r + 2\sum_1^\infty(-1)^n\frac{J_{2n-1}(\alpha_r)}{1-\left[\frac{\omega_c-\omega}{(2n-1)\Omega}\right]^2}.$$

Substituting of A_r and B_r by (67) into (66) yields

$$G_c(f_+) = T\left[\frac{\sin(\omega_c-\omega)\frac{T}{2}}{(\omega_c-\omega)\frac{T}{2}}\right]^2 \cdot\sum_{r=1}^q\left\{\frac{1}{q}\left[\left(\frac{\sum_1^\infty(-1)^nJ_{2n}(\alpha_r)}{1-\left[\frac{\omega_c-\omega}{2n\Omega}\right]^2}\right)^2+\left(\frac{\sum_1^\infty(-1)^nJ_{2n-1}(\alpha_r)}{1-\left[\frac{\omega_c-\omega}{(2n-1)\Omega}\right]^2}\right)^2\right] -\frac{1}{q^2}\left[\frac{\sum_1^\infty(-1)^nJ_{2n}(\alpha_r)\cos\alpha_r}{1-\left[\frac{\omega_c-\omega}{2n\Omega}\right]^2}-\frac{\sum_1^\infty(-1)^nJ_{2n-1}(\alpha_r)\sin\alpha_r}{1-\left[\frac{\omega_c-\omega}{(2n-1)\Omega}\right]^2}\right]^2\right\}. \quad (68)$$

The discrete lines, obtained from the second term of (68), are given by

$$G_d(f_+) = \frac{1}{4q^2}\sum_{m=-\infty}^{+\infty}\left\{\sum_{r=1}^q J_m(\alpha_r)\cos\left(\alpha_r+m\frac{\pi}{2}\right)\right\}^2\delta\left(f_c-f+\frac{m}{T}\right). \quad (69)$$

For a four-level system given by

$$\begin{cases} 2\alpha_1 = \pm\frac{\pi}{4} \\ 2\alpha_2 = \pm\frac{3\pi}{4} \end{cases}$$

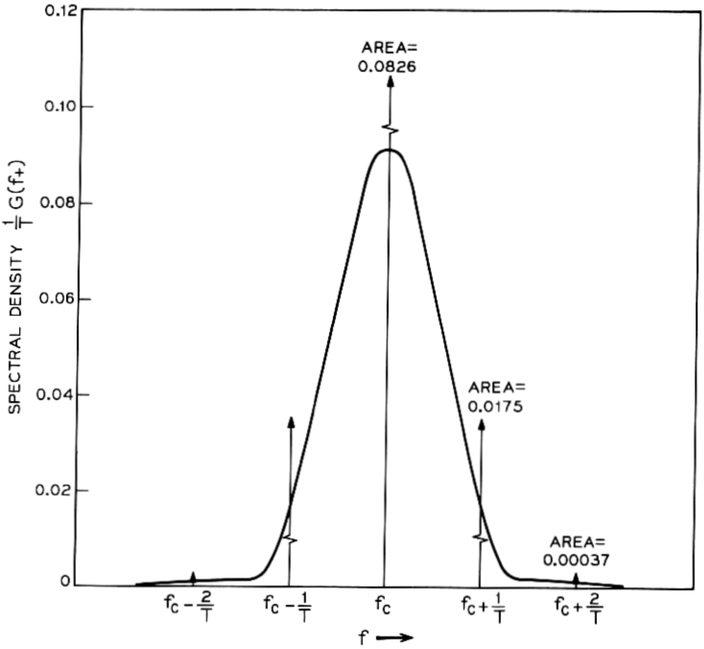

Fig. 7—Power spectrum for four-level polar phase modulation with raised-cosine pulses.

the series expansions in (68) and (69) can be limited to the second term. A good approximation of the power spectrum is then given by

$$G_c(f_+) \cong \frac{T}{2}\left[\frac{\sin\,(\omega_c - \omega)\,\dfrac{T}{2}}{(\omega_c - \omega)\,\dfrac{T}{2}}\right]^2$$

$$\cdot\left\{\frac{J_1^2\left(\dfrac{\pi}{8}\right) + J_1^2\left(\dfrac{3\pi}{8}\right)}{\left[1 - \left(\dfrac{\omega_c - \omega}{\Omega}\right)^2\right]^2} + \frac{J_2^2\left(\dfrac{\pi}{8}\right) + J_2^2\left(\dfrac{3\pi}{8}\right)}{\left[1 - \left(\dfrac{\omega_c - \omega}{2\Omega}\right)^2\right]^2}\right.$$

$$\left. - \frac{1}{2}\left[\frac{J_2\left(\dfrac{\pi}{8}\right)\cos\dfrac{\pi}{8} + J_2\left(\dfrac{3\pi}{8}\right)\cos\dfrac{3\pi}{8}}{1 - \left(\dfrac{\omega_c - \omega}{2\Omega}\right)^2} - \frac{J_1\left(\dfrac{\pi}{8}\right)\sin\dfrac{\pi}{8} + J_1\left(\dfrac{3\pi}{8}\right)\sin\dfrac{3\pi}{8}}{1 - \left(\dfrac{\omega_c - \omega}{\Omega}\right)^2}\right]^2\right\} \tag{70}$$

and

$$G_d(f_+) \cong \frac{1}{16} \sum_{m=-2}^{+2} \left\{ J_m\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{8} + m\,\frac{\pi}{2}\right) \right.$$

$$\left. + J_m\left(\frac{3\pi}{8}\right) \cos\left(\frac{3\pi}{8} + m\,\frac{\pi}{2}\right) \right\}^2 \delta\left(f_0 - f + \frac{m}{T}\right). \tag{71}$$

The spectrum given by (70) and (71) is shown in Fig. 7. Note that this spectrum is almost identical to the spectrum of the two-level system calculated with $2\alpha_r = \pm\pi/2$ and $p_1 = p_2$, shown in Fig. 6; in both cases $\sum_{r=1}^{q} e^{\pm 2j\alpha_r} = 0$. This result presents an analogy with the calculations made for rectangular pulses of duration T which satisfy the same condition.

V. CONCLUSION

A formula has been derived for computing the power spectrum of multilevel digital phase-modulated signals. The results apply to arbitrary pulse shapes and probability distributions providing that the pulses do not overlap and are independent.

This formula is easy to apply to various pulse shapes. Several examples are given such as:

(*i*) rectangular pulses of duration smaller than or equal to the timing period
(*ii*) rectangular pulses with finite rise time and decay time
(*iii*) raised-cosine pulses.

Approximate results can be obtained for more complex pulse shapes by the following method. The spectrum is a function of the pulse shapes $g_r(t)$ by the Fourier transforms

$$F_r(f) = \frac{1}{2} \int_0^T e^{+j[\omega_c + g_r(t)]} e^{-j2\pi f}\, df.$$

T can be segmented into a large number of intervals in which $g_r(t)$ is approximated to its average value in this interval. $F_r(f)$ is then given by the following series expansion

$$F_r(f) \cong \frac{T}{2n}\, e^{j(\omega_c - \omega)(T/2n)} \cdot \frac{\sin\,(\omega_c - \omega)\,\dfrac{T}{2n}}{(\omega_c - \omega)\,\dfrac{T}{2n}} \cdot \sum_{k=0}^{n} e^{jbk} \cdot e^{j(\omega_c - \omega)(kT/n)}$$

with

$$b_k = \frac{g_r\left(\frac{k+1}{n}T\right) + g_r\left(\frac{kT}{n}\right)}{2}.$$

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