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Controllability and Observability in Linear Time-Variable Networks With Arbitrary Symmetry Groups

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This paper presents a unified treatment of linear time-variable networks displaying arbitrary geometrical symmetries by incorporating group theory into an analysis scheme. Symmetric networks have their elements arranged so that certain permutations of the network edges result in a configuration which is identical with the original. These permutations lead to a group of monomial matrices which are shown to commute with the network A-matrix and the state transition matrix of the normal form equation. The representation theory of groups facilitates the study of those network properties which are determined solely by symmetry. By using group theory, a simple arithmetic condition is derived which, when satisfied, implies that the network is noncontrollable or nonobservable because of symmetry alone. The results allow the determination by inspection of linear combinations of the original state variables which result in noncontrollable variables. It is shown that networks displaying axial point group symmetry are generally only weakly controllable.

I. INTRODUCTION

In the past two decades, engineers and applied mathematicians have devoted a great deal of attention to diverse aspects of linear time-variable networks and systems. However, one problem that has not been treated in depth is that of analyzing time-varying networks displaying arbitrary geometrical symmetries. A symmetric network may be regarded as a set of identical subnetworks connected in a symmetric pattern. Such a circuit may be more easily implemented in an integrated form than is a nonsymmetric network, especially when the circuit is time-variable and the construction and synchronization of the variable elements are major technical problems. Since the trend in integrated circuit technology is towards large-scale integration, it may soon become practically important to consider large networks displaying arbitrary geometrical symmetries. The present research was undertaken partly as a possible first step toward developing a modular approach to linear network design.

While it has long been known that network symmetries can be used to facilitate analysis, previous work on symmetric networks dealt mainly with bisection techniques for networks with mirror-plane symmetry and has not incorporated general types of symmetries into an analysis scheme. The present work treats arbitrary symmetries by utilizing the mathematics of group theory, a natural tool for studying symmetry.

Network controllability and observability are important concepts in analysis and synthesis, and group theory may be employed in determining symmetry-constrained noncontrollability and nonobservability of the network. Furthermore, the determination of these properties may be made by inspection without writing network equations. The group-theoretic approach enables us to prove several theorems concerning controllability and observability of a wide class of symmetric networks. The theorems would be difficult or impossible to prove, or to state precisely, without the use of group theory.

The reader who is unfamiliar with the results and notation used in both the abstract and representation theories of groups can find this material in Appendixes A and B in a form consistent with that used in the remainder of the text. The reader may wish to study the appendixes before continuing to Section II.

II. GROUP THEORY AND NETWORK EQUATIONS

Symmetric networks have their elements arranged so that certain permutations of the network edges result in a configuration which is identical with the original. For example, the geometrically symmetric network shown in Fig. 1 is invariant under permutations of the network edges which result from a rotation of the network structure by π radians about an axis perpendicular to the plane of the paper or from reflections in the planes σ_1 and σ_2.

Definition 1: A *covering operation* or *symmetry operation* is a transformation (rotation, reflection, etc.) which will bring the symmetric object (network) into a form indistinguishable from the original one. The following is well-known and shown in Ref. 1.

Theorem 1: The set of symmetry operations of an object constitutes a group.‡

‡ See Appendix A for definitions of pertinent group theoretic terms.

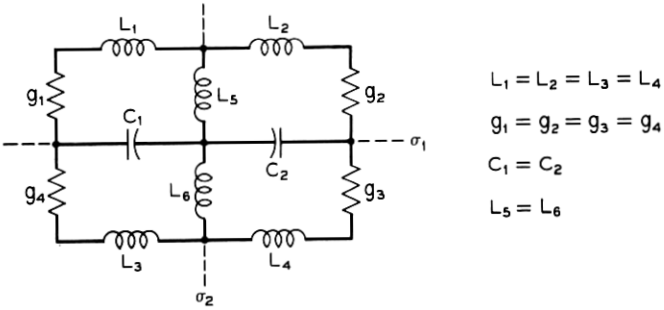


Fig. 1—Network with C_{2v} symmetry.

The effect of each symmetry operation is to permute the network edges. Specifically, only resistive edges are permuted among themselves, capacitive edges are permuted among themselves, etc., i.e., only edges of like type and equal element value or variation may be permuted. The letter R is used to denote the general symmetry operation of the symmetry group. Thus, R denotes either E, C_2, σ_1, or σ_2 for the network of Fig. 1, where E denotes the identity, C_2 denotes rotation by π radians, and σ denotes reflection in the plane σ. Thus, the operations $\{R\}$ form a group G_S which describes the symmetry of the network structure. The following exposition shows how group theory may be incorporated into the network analysis scheme.

Analysis in the time-domain can proceed from the normal form equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{1}$$

where $x(t)$ is an n-vector of state variables, $u(t)$ is a k-vector of inputs, and $A(t)$ and $B(t)$ are time-variable matrices conformable with x and u. In the context of our analysis, it is sufficient to consider $A(t)$, T. R. Bashkow's A-matrix.[2] The A-matrix contains some information about the network topology and also determines the natural response of the network. B. K. Kinariwala[3] showed that the A-matrix description is valid for time-varying as well as for fixed networks. The explicit form of the A-matrix given by P. R. Bryant[4] may be found in textbooks such as Refs. 5 and 6.

The A-matrix is derived with respect to a normal tree, i.e., a tree containing a maximum number of capacitive edges and a minimum number of inductive edges. It is assumed that the reader is familiar with the procedures needed to obtain the A-matrix, and the form of the network equilibrium equations is therefore given below:‡

‡ A superscript t appearing with a matrix denotes the transpose of that matrix; a superscript * denotes the complex conjugate of a scalar quantity.

$$\begin{bmatrix} pC_t & 0 & 0 & H_0 & H_1 & H_2 \\ 0 & G_t & 0 & 0 & H_3 & H_4 \\ 0 & 0 & \Gamma_t p^{-1} & 0 & 0 & H_5 \\ -H_0^t & 0 & 0 & D_c p^{-1} & 0 & 0 \\ -H_1^t & -H_3^t & 0 & 0 & R_c & 0 \\ -H_2^t & -H_4^t & -H_5^t & 0 & 0 & pL_c \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ i_3 \\ i_2 \\ i_1 \end{bmatrix} = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \\ e_3 \\ e_2 \\ e_1 \end{bmatrix}. \tag{2}$$

In (2), the *subscript* t denotes elements in the tree, the *subscript* c denotes elements in the cotree, and p is differentiation with respect to time. Submatrices H_0 through H_5 express the topological relationship between elements in the tree and elements in the cotree. The letters e and j refer to voltage and current sources, respectively, while V and i denote branch variables. In order to obtain the A-matrix from equation (2), the nondynamic vector variables V_2, V_3, i_2, i_3 are eliminated algebraically, thus yielding the equation

$$\begin{bmatrix} pC + G & T \\ -T^t & pL + R \end{bmatrix} \begin{bmatrix} V \\ i \end{bmatrix} = \begin{bmatrix} j \\ e \end{bmatrix}, \tag{3}$$

where j and e are regarded as inputs. For a complete interpretation of submatrices in the above equations, see Ref. 5. Equation (3) may be put into the form of equation (1) by choosing capacitor charges ($q = CV$) and inductor fluxes ($\phi = Li$) as state variables and writing

$$\frac{d}{dt}\begin{bmatrix} q \\ \phi \end{bmatrix} = -\begin{bmatrix} G & T \\ -T^t & R \end{bmatrix} \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} q \\ \phi \end{bmatrix} + \begin{bmatrix} j \\ e \end{bmatrix}. \tag{4}$$

With reference to equation (2), observe that tree edges are used to define basic cutsets[7] of the network graph and hence current-law equations, while cotree edges are used to define basic loopsets[7] of the network graph and hence voltage-law equations. Thus, if a symmetry operation of the network structure permutes an edge in the tree with one of the cotree, the equilibrium equation (2) will be in a form different from that of the original equations; such an operation is not a symmetry operation of the network equilibrium equations. Those covering operations of the network structure which do not permute tree edges with cotree edges form a subgroup G_N of the group G_S (if two operations R_1 and R_2 do not permute tree edges with cotree edges, then the compound operation R_1R_2 also possesses that property), and the group G_N thus contains the symmetry operations of the equilibrium equations. Since the network equilibrium equations are being considered, the transformations of edge voltages and currents are of importance rather than merely the permutations of network edges. The operations R of the group G_N may transform a voltage (or current) into the negative of another voltage (or current). If the network contains b edges, a b-dimensional monomial matrix‡ $\hat{D}(R)$ may be formed which represents the transformation of the b voltages and currents under the symmetry operation R. The rows and columns of $\hat{D}(R)$ correspond to edge voltages and currents, and the matrix entries show how these quantities transform under the symmetry operation. Matrices $\hat{D}(R)$ form a reducible representation of the group G_N.

2.1 *Commutativity Relations*

In (2), denote the column vector of edge currents and voltages by $\hat{f}$, the column vector of current sources and voltage sources by $\hat{g}$, and the coefficient matrix by $\hat{N}$. Thus, equation (2) becomes

$$\hat{N}\hat{f} = \hat{g}. \tag{5}$$

Consider the new arrangement of sources and edges obtained by operating on the network with symmetry operation R, i.e., consider the equilibrium equations for the case

$$\begin{aligned} \hat{\hat{g}} &= \hat{D}(R)\hat{g} \\ \hat{\hat{f}} &= \hat{D}(R)\hat{f}. \end{aligned} \tag{6}$$

Since the operation R yields a network configuration, including the choice of tree, which is identical to the original one, it must be that

$$\hat{N}\hat{\hat{f}} = \hat{\hat{g}}. \tag{7}$$

Hence,

$$\hat{N}\hat{D}(R)\hat{f} = \hat{D}(R)\hat{g} = \hat{D}(R)\hat{N}\hat{f}, \tag{8}$$

where use is made of equations (5) and (6). For a network of b edges, b linearly independent vectors $\hat{f}$ may be specified such that the current-law and voltage-law equations are satisfied. Furthermore, b values of $\hat{g}$ are then obtained such that for each $\hat{f}$ chosen, the terminal relations of the network elements are satisfied. Thus, equality of the first and last members of (8) implies that

$$\hat{N}\hat{D}(R) = \hat{D}(R)\hat{N},$$

or (9)

$$\hat{D}^{-1}(R)\hat{N}\hat{D}(R) = \hat{N}.$$

Thus the monomial representation $\hat{D}(R)$ commutes with $\hat{N}$.

‡ A monomial matrix has only one nonzero entry in any row or column. The nonzero entry is restricted here to the values $+1$ or -1.

Equation (3) shows how elimination of nondynamic variables in equation (2) reduces $\hat{N}$ to a new matrix

$$N = \begin{bmatrix} pC + G & T \\ -T^t & pL + R \end{bmatrix},$$

and reduces $\hat{f}$ and $\hat{g}$ to the vectors

$$f = \begin{bmatrix} V \\ i \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} j \\ e \end{bmatrix}.$$

The algebraic operations have eliminated from $\hat{N}$ the rows and columns corresponding to submatrices G_t , R_c , Γ_t , and D_c in equation (2). The elimination of rows and columns of $\hat{D}(R)$ which correspond to G_t , R_c , Γ_t , and D_c results in a group of matrices $D(R)$ which show only how tree capacitive voltages and cotree inductive currents are transformed under the operation R. By expanding equation (9) in terms of the submatrices of equation (2), it is possible to show[1] that $D(R)$ satisfies the following commutativity relation:

$$D^{-1}(R)ND(R) = N,$$

or (10)

$$D^{-1}(R)\begin{bmatrix} pC + G & T \\ -T^t & pL + R \end{bmatrix}D(R) = \begin{bmatrix} pC + G & T \\ -T^t & pL + R \end{bmatrix},$$

where $D(R)$ commutes with $[\begin{smallmatrix} C & 0 \\ 0 & L \end{smallmatrix}]$ and with $[\begin{smallmatrix} G & T \\ -T^t & R \end{smallmatrix}]$. Thus, the following may be stated.

Theorem 2: For a symmetric network, construct the monomial representation $D(R)$ of the symmetry group G_N , where $D(R)$ shows how the tree capacitive voltages and cotree inductive currents are transformed under the symmetry operation R. The monomial representation $D(R)$ commutes with the network A-matrix based either on voltages and currents or fluxes and charges as state variables. That is,

$$D^{-1}(R)A(t)D(R) = A(t), \quad \textit{for all} \quad R \in G_N .$$

The commutativity relation given in Theorem 2 establishes a basic connection between group theory and the network analysis problem, and allows group theoretic methods to be applied to linear networks displaying arbitrary geometrical symmetries.

The state transition matrix $\Phi(t, \tau)$ is the matric solution to the homogeneous part of equation (1) which satisfies

$$\Phi(\tau, \tau) = I, \tag{11}$$

where I is the unit matrix of appropriate order. $\Phi(t, \tau)$ is given in series form[8] as

$$\Phi(t, \tau) = \sum_{k=0}^{\infty} \Phi_k(t, \tau), \tag{12}$$

where

$$\Phi_{k+1}(t, \tau) = \int_\tau^t A(\rho)\Phi_k(\rho, \tau)\, d\rho \tag{13}$$

$$\Phi_0 = I.$$

Theorem 3: For a symmetric network, the monomial representation $D(R)$ of the symmetry group G_N commutes with $\Phi(t, \tau)$, i.e.,

$$D^{-1}(R)\Phi(t, \tau)D(R) = \Phi(t, \tau).$$

Proof: From (12) and (13),

$$D^{-1}(R)\Phi(t, \tau)\, D(R) = \sum_{k=0}^{\infty} D^{-1}(R)\Phi_k(t, \tau)\, D(R).$$

An induction procedure shows that $D(R)$ commutes with each term $\Phi_k(t, \tau)$ in the above sum.

$$D^{-1}(R)\Phi_1(t, \tau)\, D(R) = \int_\tau^t [D^{-1}(R)A(\rho)\, D(R)]I\, d\rho$$

$$= \int_\tau^t A(\rho)\, d\rho = \Phi_1(t, \tau).$$

Assume that $D(R)$ commutes with $\Phi_k(t, \tau)$. Hence,

$$D^{-1}(R)\Phi_{k+1}(t, \tau)\, D(R) = \int_\tau^t [D^{-1}(R)A(\rho)\, D(R)][D^{-1}(R)\Phi_k(\rho, \tau)\, D(R)]\, d\rho$$

$$= \int_\tau^t A(\rho)\Phi_k(\rho, \tau)\, d\rho = \Phi_{k+1}(t, \tau).$$

Thus, the theorem is proved.

III. EXPLICIT FORM OF TRANSFORMATION α TO REDUCE $A(t)$

In Appendix B, a procedure is given for the construction of a unitary matrix α from the irreducible representations of symmetry group G_N and representation $D(R)$. The important property possessed by the transformation α is that it transforms the state space to a new basis in which $D(R)$ appears in block diagonal form and in which $A(t)$ appears in block diagonal form.[9] For the remainder of this paper, it is important to determine the positions of zero elements in the matrix α. Thus, the characterization of α in an explicit form is undertaken at this point. The following definition is adapted from group theory in a way useful to network analysis.

Definition 2: A symmetric network is said to be *transitive* if there is at least one group operation which transforms a given state variable into any other state variable (with plus or minus sign). The network is *intransitive* if it is not transitive.

Since an inductor and a capacitor cannot be permuted by any symmetry operation, general *RLC* symmetric networks are intransitive. The state variables can be partitioned into sets such that the group operations permute among themselves only those variables in the same set. Hence, each set is transitive, and the state variables are said to be partitioned into transitive sets.

Theorem 4: For a symmetric network, the number of transitive sets into which the state variables may be partitioned is equal to the number of times the totally symmetric irreducible representation [i.e., $D^{(1)}(R)$ *having all group operations represented by unity] appears in* $D_p(R)$*, the permutation representation obtained from* $D(R)$ *by replacing each −1 entry in* $D(R)$ *by +1.*

Proof: This result follows from a theorem given by W. Burnside (Ref. 10, p. 191) which states that

$$gs = \sum_{r=1}^{n} r\nu_r \,, \tag{14}$$

where g is the order of the group, n the number of symbols (state variables) operated on by the group, s the number of transitive sets in which the n symbols are permuted, and ν_r the number of group operators which leave exactly r symbols unchanged.

Let c_1^p denote the number of times that $D^{(1)}(R)$ appears in $D_p(R)$ and χ the trace of D. From (52) in Appendix A,

$$\begin{aligned} c_1^p &= \frac{1}{g} \sum_R \chi^{(1)}(R)^* \chi_p(R) \\ &= \frac{1}{g} \sum_R \chi_p(R) \end{aligned} \tag{15}$$

since $\chi^{(1)}(R) = 1$ for all R. Because $D_p(R)$ is a permutation representation, $\chi_p(R)$ is precisely the number of state variables left unchanged by operation R, and hence is an integer from zero to n. The group operators can be partitioned such that all operations in a given set leave unchanged the same number of state variables. It is now evident that (14) and (15) are identical sums, and c_1^p is equal to s.

The column vectors $\alpha_{p\pi a}$, $a = 1, \cdots, c_p$, of the matrix α are given in (56) in Appendix B and repeated here for convenience. They are c_p linearly independent (and orthonormal) columns of

$$G_\pi^{(p)} = \sum_R D^{(p)}(R)^*_{\pi\pi} \, D(R)I, \tag{16}$$

where I is the unit matrix and $p\pi a$ are indices defined as follows. The index p denotes one of the distinct irreducible representations of the symmetry group, the index π runs from 1 to l_p and denotes a row of the matrix $D^{(p)}(R)$ [so that the dimension of $D^{(p)}(R)$ is l_p], and the index a denotes one of the c_p appearances of $D^{(p)}(R)$ in $D(R)$. Thus, α has the form

$$\alpha = [\alpha_{111}, \cdots, \alpha_{\mu 11}, \cdots, \alpha_{\mu 1 c_\mu}, \cdots, \alpha_{\mu\pi 1}, \cdots, \alpha_{\mu\pi c_\mu}, \cdots]. \tag{17}$$

We consider a typical column vector $\alpha_{\mu\pi a}$ corresponding to the πth row of $D^{(\mu)}(R)$. Let e_m be the vector containing all zeros except for unity in the mth row. From (16), $\alpha_{\mu\pi m}$ may be considered to result from (we delete the normalization factor)

$$\alpha_{\mu\pi m} = \sum_R D^{(\mu)}(R)^*_{\pi\pi} \, D(R)e_m \, . \tag{18}$$

If c_μ is less than or equal to c_1^p , the number of transitive sets into which the state variables may be partitioned (Theorem 4), the c_μ values of the index m can be chosen such that each vector e_m corresponds to a different transitive set. The operation $D(R)e_m$ results in a new vector e_k where m and k are in the same transitive set. Thus, the c_μ vectors $\alpha_{\mu\pi m}$ chosen above are necessarily linearly independent if they are not zero. If any choice of m yields a zero result in (18), merely choose a value of m corresponding to a different transitive set; c_μ linearly independent $\alpha_{\mu\pi m}$ must be obtained in this way since the matrix $G_\pi^{(\mu)}$ has rank c_μ.[11]

Lemma 1: If the vectors $\alpha_{\mu\pi m}$ are chosen as outlined above, only one of the vectors having indices μ and π can possibly have a nonzero result in row r, namely, $\alpha_{\mu\pi\rho}$ where r and ρ are in the same transitive set.

The rth component of $\alpha_{\mu\pi m}$ is denoted by $\alpha_{\mu\pi m}^r$. The group operators may be partitioned into the set $\{R_\rho^r\}$ and its complement $\{\overline{R_\rho^r}\}$, where $\{R_\rho^r\}$ consists of all group operations which take the ρth state variable into the rth state variable. Thus,

$$\begin{aligned} \alpha_{\mu\pi\rho} &= \sum_{R} D^{(\mu)}(R)^*_{\pi\pi}\, D(R)e_\rho \\ &= \sum_{R_\rho{}^r} D^{(\mu)}(R^r_\rho)^*_{\pi\pi}\, D(R^r_\rho)e_\rho + \sum_{\bar{R}_\rho{}^r} D^{(\mu)}(\bar{R}^r_\rho)^*_{\pi\pi}\, D(\bar{R}^r_\rho)e_\rho \\ &= \sum_{R_\rho{}^r} s^r_\rho\, D^{(\mu)}(R^r_\rho)^*_{\pi\pi} e_r + \sum_{\bar{R}_\rho{}^r} D^{(\mu)}(\bar{R}^r_\rho)^*_{\pi\pi}\, D(\bar{R}^r_\rho)e_\rho\ , \end{aligned} \tag{19}$$

where s^r_ρ is $+1$ or -1 as R^r_ρ transforms state variable x_ρ into state variable x_r with positive or negative sign, respectively. Hence, except for a scale factor,

$$\alpha^r_{\mu\pi\rho} = \sum_{R_\rho{}^r} s^r_\rho\, D^{(\mu)}(R^r_\rho)^*_{\pi\pi}\ . \tag{20}$$

In determining whether the rth component of vectors $\alpha_{\mu\pi\rho}$ is zero, there may be some ambiguity in choosing the index ρ in the same transitive set as r. The following lemma eliminates any ambiguity in this choice.

Lemma 2: A necessary and sufficient condition for $\alpha^r_{\mu\pi\rho}$ to be zero for all ρ in the same transitive set as r is that $\alpha^r_{\mu\pi r}$ be equal to zero, i.e., that

$$\sum_{R_r{}^r} s^r_r\, D^{(\mu)}(R^r_r)^*_{\pi\pi} = 0. \tag{21}$$

Proof: Consider the subgroup $\mathcal{K}$ of the group G_N , where $\mathcal{K}$ consists of those group operations which transform the rth state variable into itself with either positive or negative sign. The subset H of $\mathcal{K}$ which transform the rth state variable into itself with positive sign forms a subgroup of index two in $\mathcal{K}$.[12] Thus, $\mathcal{K}$ may be partitioned into cosets with respect to H as

$$\mathcal{K} = H,\ PH, \tag{22}$$

where P is an operation of $\mathcal{K}$ not contained in H, and thus transforms the rth state variable into itself with minus sign. The group G_N may be partitioned into cosets with respect to $\mathcal{K}$ as

$$G_N = H,\ PH,\ R^\rho_r H,\ R^\rho_r PH,\ \cdots\ ,\ R^\eta_r H,\ R^\eta_r PH,$$

where R^i_r denotes a group operation which transforms the rth state variable into the ith with plus sign. It is clear from the above that if any group operations transform any symbols (state variables) with negative sign, there must exist an equal number of group operations which transform the symbols with positive sign. Hence, for each transitive set of symbols (state variables) operated on by G_N , the subset of group operations which permutes the symbols with positive sign forms a subgroup of index two. Therefore, there exists a one-dimensional irreducible representation $D^{(\bar{\mu})}(R)$ of G_N in which each group operation which transforms the symbols with plus sign is represented by $+1$, while each operation which transforms the symbols with minus sign is represented by -1.[12] Thus, in equation (20),

$$s_\rho^r = D^{(\bar{\mu})}(R_\rho^r). \tag{23}$$

The orthogonality relation (51) for irreducible representations requires that

$$\sum_R D^{(\bar{\mu})}(R)\, D^{(\mu)}(R)_{\pi\pi}^* = \frac{g}{l_{\bar{\mu}}}\, \delta_{\bar{\mu}\mu} = 0 \tag{24}$$

if $\bar{\mu} \neq \mu$. Notice that if $D(R)$ is a permutation representation, then $\bar{\mu} = 1$, and $D^{(1)}(R)$ is the totally symmetric irreducible representation.

Let the transitive set to which r belongs be denoted by M_r consisting of $\{r, \rho, \cdots, \eta\}$. Hence, the set of group operations may be partitioned into $\{R_r^r\}$, $\{R_r^\rho\}$, $\cdots$, $\{R_r^\eta\}$, and from equations (23) and (24),

$$\sum_R D^{(\bar{\mu})}(R)\, D^{(\mu)}(R)_{\pi\pi}^* = \sum_{R_r{}^r} s_r^r\, D^{(\mu)}(R_r^r)_{\pi\pi}^* + \sum_{R_r{}^\rho} s_r^\rho\, D^{(\mu)}(R_r^\rho)_{\pi\pi}^*$$
$$+ \cdots + \sum_{R_r{}^\eta} s_r^\eta\, D^{(\mu)}(R_r^\eta)_{\pi\pi}^* = 0. \tag{25}$$

Now, the matrices $D^{(\mu)}(R)$ are unitary, and $\{R_r^\rho\} = \{[R_\rho^r]^{-1}\}, \cdots, \{R_r^\eta\} = \{[R_\eta^r]^{-1}\}$. Hence, if $\alpha_{\mu\pi\rho}^r = \cdots = \alpha_{\mu\pi\eta}^r = 0$, then by virtue of (20), equation (25) implies that $\alpha_{\mu\pi r}^r = 0$ as well. This proves necessity.

There are two cases to consider in proving sufficiency, namely, $c_\mu \leqq c_1^p$ and $c_\mu > c_1^p$.

Case (a) $c_\mu \leqq c_1^p$:

In this case c_μ linearly independent vectors $\alpha_{\mu\pi a}$ may be obtained by choosing vectors e_m in (18) so that each m corresponds to a different transitive set. Suppose e_r is chosen corresponding to the set M_r. The addition of $\alpha_{\mu\pi\rho}$ to the set thus results in a dependent set. Furthermore, by construction, all vectors except $\alpha_{\mu\pi r}$ are zero in positions where $\alpha_{\mu\pi\rho}$ is nonzero. Thus, $\alpha_{\mu\pi r}$ and $\alpha_{\mu\pi\rho}$ are proportional, i.e., if $\alpha_{\mu\pi r}^r = 0$, then $\alpha_{\mu\pi\rho}^r = 0$. The last result holds true for all $\rho \in M_r$, and sufficiency is established for Case (a).

Case (b) $c_\mu > c_1^p$:

In this case, it may be possible to choose more than one index in the transitive set M_r such that the $\alpha_{\mu\pi a}$ vectors obtained from equation (18) are linearly independent. A direct argument shows that a contradiction results if $\alpha_{\mu\pi r}^r = 0$ while $\alpha_{\mu\pi\rho}^r \neq 0$; namely, more linearly independent vectors than is actually possible can be obtained from equation (18) by using indices m corresponding to the transitive set M_r. Thus, sufficiency for Case (b) is proved.

Hence, it has been established that for any indices μ and π, the vanishing of the rth component of $\alpha_{\mu\pi a}$ is completely determined by the matrices $D^{(\mu)}(R_r^r)$, where the only group operations involved are those which transform the rth state variable into itself. This result will be used in the next section.

IV. CONTROLLABILITY OF SYMMETRIC NETWORKS

The concept of controllability relates to the degree to which the state of a system is affected by the application of some input. The following definition may be found in Ref. 13.

Definition 3: The system (1) is *completely controllable on an interval* (t_o, t_1) if for any state x_o at t_o and any desired final state x_1 at t_1, there exists an input $u(t)$ defined on (t_o, t_1) such that $x(t_1) = x_1$.

The system (1) is *totally controllable on an interval* (t_o, t_1) if it is completely controllable on every subinterval of (t_o, t_1).

For networks with sufficiently smooth time-variations, controllability of the linear time-varying system (1) may be characterized by the controllability matrix[13]

$$\left.\begin{aligned} Q_c &= [P_0 P_1 \cdots P_{n-1}], \\ P_k &= -A(t)P_{k-1} + \dot{P}_{k-1},\ P_0 = B \end{aligned}\right\} \tag{26}$$

where

and n is the order of the system.

The following theorem is a paraphrase of Theorem 4 of Ref. 13.

Theorem 5: For the system (1) assume that $A(t)$ and $B(t)$ together with their first $n - 2$ and $n - 1$ derivatives, respectively, are continuous functions. System (1) is totally controllable on the interval (t_o, t_1) if and only if Q_c does not have rank less than n on any subinterval of (t_o, t_1).

Lemma 3: The system described in partitioned form by

$$\frac{d}{dt}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \tag{27}$$

is noncontrollable (i.e., not completely controllable).

A sufficient condition for noncontrollability is given in the above lemma. This section is concerned with determining conditions in which symmetry alone is sufficient to cause the network to be noncontrollable.

Definition 4: A symmetric network is said to be NCS (possess the NCS property) if it is noncontrollable because of symmetry alone.

In (4), if only k of the inputs $[{}^{i}_{e}]$ are nonzero, the equation can be rewritten using the k-vector of inputs, $u(t)$. Thus,

$$\dot{x} = A(t)x + Bu(t) \tag{28}$$

where B is an $n \times k$ constant matrix and $x = [{}^{q}_{\phi}]$. By making the unitary change of variable‡ $Z = \alpha^{\dagger}x$, we arrive at the block diagonal system of equations

$$\frac{d}{dt}\begin{bmatrix} Z_1^1 \\ \vdots \\ Z_\pi^\mu \\ \vdots \\ Z_{l_\beta}^\beta \end{bmatrix} = \begin{bmatrix} \bar{A}_1^1 & & & & \\ & \ddots & & & \\ & & \bar{A}_\pi^\mu & & \\ & & & \ddots & \\ & & & & \bar{A}_{l_\beta}^\beta \end{bmatrix}\begin{bmatrix} Z_1^1 \\ \vdots \\ Z_\pi^\mu \\ \vdots \\ Z_{l_\beta}^\beta \end{bmatrix} + \alpha^{\dagger}Bu(t), \tag{29}$$

where Z is shown partitioned according to the submatrices $\bar{A}_\pi^\mu$. Controllability of the network reduces to that of all of the subsystems corresponding to the $\bar{A}_\pi^\mu$. From Lemma 3, the network is noncontrollable if a submatrix of $\alpha^{\dagger}B$ corresponding in its partition location to one of the $\bar{A}_\pi^\mu$ is zero. This occurrence is due solely to symmetry; we now investigate this condition more closely.

First consider the case where a single input is coupled only to the rth state variable, i.e., in (28), $B = e_r$ and $u(t)$ is a scalar. From α given in (17), it is evident that the submatrix of $\alpha^{\dagger}e_r$ corresponding to $\bar{A}_\pi^\mu$ is simply the $c_\mu \times 1$ partition consisting of the rth components of the vectors $\alpha_{\mu\pi 1}, \cdots, \alpha_{\mu\pi c_\mu}$. As mentioned in Lemma 1, at most one of these vectors can have nonzero entry in row r. Using the notation developed previously and Lemma 2, the following theorem has therefore been established.

Theorem 6: A symmetric time-varying network having a single input coupled only to the rth state variable is noncontrollable by virtue of its symmetry (i.e., is NCS) if and only if there is a μ *such that* $D^{(\mu)}(R)$ *appears in* $D(R)$ *and*

$$\sum_{R_r^r} s_r^r \, D^{(\mu)}(R_r^r)^*_{\pi\pi} = 0$$

for some value of π.

It is clear that if a table of irreducible representations is available, the arithmetic computation involved in the above theorem is quite simple. For any μ, all values of $\pi = 1, \cdots, l_\mu$ should be checked to determine

‡ The complex conjugate transpose of the matrix α is denoted by $\alpha^{\dagger}$.

which state variables are uncontrollable. For most cases of interest, $l_\mu = 1, 2,$ or 3; the point group of the regular icosahedron has irreducible representations of order five.[12]

If the irreducible representation in (21) is one-dimensional, the quantities $D^{(\mu)}(R)_{\pi\pi}$ are unambiguous. However, for irreducible representations whose dimension exceeds unity, any representation which is equivalent to $D^{(\mu)}(R)$ may be used to form the matrix α which reduces the system of (28) to that of (29). Clearly, although the block diagonal form of (29) will be essentially the same under transformations produced from equivalent irreducible representations, the matrix α will be different depending on which irreducible representation is used to construct it. Hence, for multidimensional irreducible representations, it is possible for $\alpha^\dagger B$ in equation (29) to have a zero submatrix if $D^{(\mu)}(R)$ is used to construct α, whereas nonzero submatrices may result if a representation equivalent to $D^{(\mu)}(R)$ is used to construct the transformation α. The above discussion shows that for multidimensional irreducible representations,

$$\sum_{R_r{}^r} s_r^r \, D^{(\mu)}(R_r^r)^*_{\pi\pi} \neq 0$$

is not sufficient to conclude that the network is *not* NCS. The inequality to zero of the sum in (21) must be shown for all representations equivalent to $D^{(\mu)}(R)$. In most cases of interest, the set $\{R_r^r\}$ consists of very few elements, and it may be quite easy to determine an irreducible representation which satisfies (21). The points mentioned in the above discussion will be illustrated by example in the sequel.

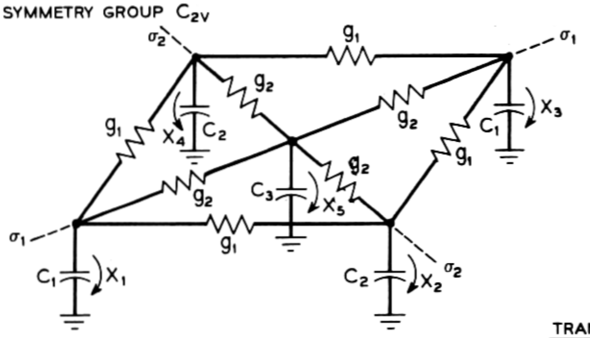
As an example illustrating the use of Theorem 6, consider the network of Fig. 2. The network has C_{2v} symmetry, and a table of irreducible representations of the group is given in Fig. 2. By utilizing (52), it is determined that the monomial‡ representation $D(R)$ contains $D^{(1)}(R)$ three times, $D^{(2)}(R)$ zero times, and $D^{(3)}(R)$ and $D^{(4)}(R)$ each one time. If a current source $I(t)$ is placed in parallel with the capacitor associated with state variable x_1, the following calculations can be made with regard to Theorem 6. The group operations which leave x_1 invariant are E and σ_1. Thus,

$$\sum_{R_r{}^r} D^{(1)}(R_r^r)^*_{\pi\pi} = 1 + 1 \neq 0$$

$$\sum_{R_r{}^r} D^{(3)}(R_r^r)^*_{\pi\pi} = 1 + 1 \neq 0$$

$$\sum_{R_r{}^r} D^{(4)}(R_r^r)^*_{\pi\pi} = 1 - 1 = 0.$$

‡ $D(R)$ is a permutation representation in this case, so that $s_r{}^r = +1$.



SYMMETRY OPERATIONS

$E = (1)(2)(3)(4)(5)$

$C_2 = (13)(24)$

$\sigma_1 = (24)$

$\sigma_2 = (13)$

IRREDUCIBLE REPRESENTATIONS

R	E	C_2	σ_1	σ_2
$D^{(1)}(R)$	1	1	1	1
$D^{(2)}(R)$	1	1	−1	−1
$D^{(3)}(R)$	1	−1	1	−1
$D^{(4)}(R)$	1	−1	−1	1

TRANSFORMATION

$$\alpha = \frac{1}{\sqrt{2}} \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & \sqrt{2} & 0 & 0 \end{array}\right]$$

$D^{(1)}(R)$ $D^{(3)}(R)$ $D^{(4)}(R)$

Fig. 2—Network with C_{2v} symmetry, including symmetry operations, irreducible representations, and transformation matrix α.

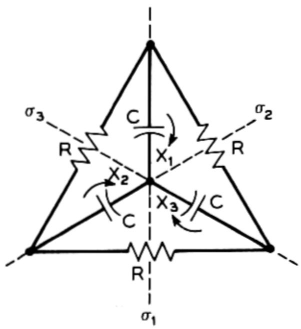
Hence, if the excitation is coupled solely to state variable x_1, the basis function corresponding to $D^{(4)}(R)$ will be uncontrollable. Indeed, the block-diagonal system has the form

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} I(t). \tag{30}$$

It is seen from the matrix α in Fig. 2 that $z_5 = x_2 - x_4$, and it is this linear combination of the original state variables which is uncontrollable in the present example. Note that since $D^{(2)}(R)$ does not appear in $D(R)$ in the above example, no basis functions are associated with it, and hence a computation is not made for this irreducible representation.

The next example serves to illustrate some complications that arise when the symmetry group possesses irreducible representations of dimension greater than unity. The network shown in Fig. 3 possesses symmetry C_{3v}. Two equivalent two-dimensional irreducible representations are given, and two transformation matrices α_1 and α_2 are shown

SYMMETRY GROUP C_{3V}



SYMMETRY OPERATIONS

$E = (1)(2)(3)$ $\quad C_3^2 = (123)$ $\quad \sigma_2 = (13)$

$C_3 = (132)$ $\quad \sigma_1 = (23)$ $\quad \sigma_3 = (12)$

IRREDUCIBLE REPRESENTATIONS

R	E	C_3	C_3^2	σ_1	σ_2	σ_3
$D^{(1)}(R)$	1	1	1	1	1	1
$D^{(2)}(R)$	1	1	1	−1	−1	−1
$D^{(3)}(R)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$
$\bar{D}^{(3)}(R)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon^* \end{bmatrix}$	$\begin{bmatrix} \epsilon^* & 0 \\ 0 & \epsilon \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \epsilon^* \\ \epsilon & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \epsilon \\ \epsilon^* & 0 \end{bmatrix}$

TRANSFORMATION α_1 USING $D^{(3)}(R)$

$$\alpha_1 = \left[\begin{array}{c|cc} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{array}\right]$$

$D^{(1)}(R)$ $\quad D^{(3)}(R)$

TRANSFORMATION α_2 USING $D^{(3)}(R)$

$$\alpha_2 = \frac{1}{\sqrt{3}} \left[\begin{array}{c|cc} 1 & 1 & 1 \\ 1 & \epsilon^* & \epsilon \\ 1 & \epsilon & \epsilon^* \end{array}\right]$$

$D^{(1)}(R)$ $\quad \bar{D}^{(3)}(R)$

NOTE:

① $\epsilon = e^{j\frac{2\pi}{3}}$ $\quad \epsilon^* = e^{-j\frac{2\pi}{3}}$

② $D^{(3)}(R)$ AND $\bar{D}^{(3)}(R)$ ARE EQUIVALENT

Fig. 3—Network with C_{3v} symmetry, including symmetry operations, irreducible representations, and transformation matrix α.

which will transform the differential equations to block diagonal form. It is determined that the irreducible representations $D^{(1)}(R)$ and $D^{(2)}(R)$ are contained one time and zero times, respectively, in the permutation representation $D(R)$, while the two-dimensional irreducible representation is contained once in $D(R)$. The transformation matrix α_1 is constructed using the real two-dimensional representation while α_2 is constructed using the complex two-dimensional representation. Both α_1 and α_2 are given in Fig. 3.

If a current source $I(t)$ is placed in parallel with the capacitor associated with state variable x_1, the block-diagonal system has the form (using the transformation α_1)

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1/(3)^{\frac{1}{2}} \\ 2/(6)^{\frac{1}{2}} \\ 0 \end{bmatrix} I(t),$$

while if α_2 is used as the transformation matrix, the block-diagonal system has the form

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} I(t).$$

The network is uncontrollable as shown with α_1 above. Uncontrollability of the network may be determined by inspection by using the real two-dimensional representation in Theorem 6. The set $\{R_r^r\}$ consists of $\{E, \sigma_1\}$, where $r = 1$.

The following corollary results from a trivial application of Theorem 6, but is by no means obvious without the use of the theorem.

Corollary 1: Given the assumptions of Theorem 6, if there is just one group operation that leaves the rth state variable invariant, then the network cannot be NCS.

Proof: The lone group operation must be the identity, and $D^{(\mu)}(E)_{\pi\pi} = 1$ for all μ and π.

An interesting and important result of Corollary 1 is that a network whose only symmetry operations are rotations (i.e., C_n groups) cannot be NCS except in the special case treated in Corollary 2 which follows.

Corollary 2: If the symmetric network contains a state variable which is invariant under all the group operations, and if the single input is coupled solely to this state variable, the network is NCS.

Proof: Since $\{R_r^r\}$ is the entire group, equation (51) yields

$$\sum_{R_r{}^r} s_r^r\, D^{(\mu)}(R_r^r)_{\pi\pi}^* = \sum_{R} D^{(\bar{\mu})}(R)\, D^{(\mu)}(R)_{\pi\pi}^* = \frac{g}{l_{\bar{\mu}}}\,\delta_{\mu\bar{\mu}} = 0, \qquad \mu \neq \bar{\mu}.$$

The network of Fig. 2 may also serve to illustrate Corollary 2. State variable x_5 is invariant under all the group operations. If an excitation $I(t)$ is coupled only to x_5, the block-diagonal system has the form ($\bar{\mu} = 1$ since $D(R)$ is a permutation representation)

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} I(t). \tag{31}$$

Corollary 3: The state variables associated with $D^{(\bar{\mu})}(R)$ represent an always excitable portion of the network, i.e., these variables are never NCS. If $\bar{\mu} \neq 1$, basis functions corresponding to $D^{(1)}(R)$ are always NCS.

Proof: From (22), the subset of group operations which transform the rth state variable into itself with plus sign forms a subgroup of index two in the group $\mathcal{K}$ of operations which transforms the rth state variable into itself with either plus or minus sign. Hence, the quantities s_r^r in equation (21) form an irreducible representation $D^{(\bar{\mu})}(R)$ of the group $\mathcal{K}$. Thus, for the basis functions corresponding to $D^{(\bar{\mu})}(R)$, equation (21) becomes

$$\sum_{R_r{}^r} s_r^r \, D^{(\bar{\mu})}(R_r^r)^* = \sum_{R_r{}^r} D^{(\bar{\mu})}(R) \, D^{(\bar{\mu})}(R)^* = k \neq 0,$$

where k denotes the order of the group $\mathcal{K}$. Likewise, $D^{(1)}(R)$ forms an irreducible representation of $\mathcal{K}$ since the totally symmetric representation is an irreducible representation of any abstract group. Thus, for the basis functions corresponding to $D^{(1)}(R)$, equation (21) becomes

$$\sum_{R_r{}^r} s_r^r \, D^{(1)}(R_r^r)^* = \sum_{R_r{}^r} D^{(\bar{\mu})} R_r^r \, D^{(1)}(R_r^r)^* = 0.$$

Thus, the corollary is proved.

Corollary 3 is illustrated in (30) and (31) where the excitation is coupled to the state variables associated with $D^{(1)}(R)$. (Matrix $D(R)$ is a permutation representation for the example of Fig. 2, so that $\bar{\mu} = 1$.)

The applicability of Theorem 6 is extended somewhat by considering the case where the single input is coupled to more than one state variable. For the moment, it is assumed that only two state variables are coupled to the input so that in equation (28)

$$B(t) = h_1(t)e_r + h_2(t)e_j \,, \tag{32}$$

where $h_1(t)$ and $h_2(t)$ are scalars.

Corollary 4: A symmetric network having a single input coupled only to the rth and jth state variables is NCS if and only if a proper value‡ *of μ exists such that for some value of π,*

$$\sum_{R_r{}^r} s_r^r \, D^{(\mu)}(R_r^r)^*_{\pi\pi} = 0 \quad \textit{and} \quad \sum_{R_j{}^j} s_j^j \, D^{(\mu)}(R_j^j)^*_{\pi\pi} = 0. \tag{33}$$

Proof: For r and j in the same transitive set and $c_\mu \leqq c_1^p$, the partition of $\alpha^\dagger B$ in equation (29) corresponding to $\bar{A}_\pi^\mu$ can have nonzero terms only from§

‡ A proper value of μ is one for which $D^{(\mu)}(R)$ is contained in $D(R)$.

§ A normalization factor is not included in the vector $\alpha_{\mu\pi r}$.

$$(\alpha_{\mu\pi r})^{\dagger} B = h_1(t) \sum_{R_r{}^r} s_r^r D^{(\mu)}(R_r^r)^*_{\pi\pi} + h_2(t) \sum_{R_r{}^j} s_r^j D^{(\mu)}(R_r^j)^*_{\pi\pi} . \tag{34}$$

Provided that $h_1(t)$ and $h_2(t)$ are not specially chosen to cause (34) to vanish, Lemma 2 implies that (34) vanishes if and only if

$$\sum_{R_r{}^r} s_r^r D^{(\mu)}(R_r^r)^*_{\pi\pi} = 0$$

(only one of the sums in (33) need be computed in this case).

For the case where $c_\mu > c_1^p$, the possibility exists that $\alpha_{\mu\pi r}$ and $\alpha_{\mu\pi j}$ are linearly independent. This linear independence also occurs when r and j are in different transitive sets. Thus, for these cases, the possible nonzero terms in the partition of $\alpha^\dagger B$ corresponding to $\bar{A}_\pi^\mu$ in (29) arise from $(\alpha_{\mu\pi r})^\dagger h_1(t) e_r$ and from $(\alpha_{\mu\pi j})^\dagger h_2(t) e_j$. From Lemma 2, $(\alpha_{\mu\pi k})^\dagger e_k$ is zero if and only if

$$\sum_{R_k{}^k} s_k^k D^{(\mu)}(R_k^k)^*_{\pi\pi} = 0, \qquad k = r, j.$$

Thus, the proof is complete.

The above method may be extended in a fairly obvious manner to treat the case where any number of state variables are coupled to the single input. A separate statement is required for each set of variables in a given transitive set.

At this point, we consider the problem of determining general conditions which guarantee that (21) will or will not be satisfied for some proper value of μ. Thus, the summations for all values of μ need not be computed. A partial solution to this problem is offered in Theorems 7, 8, and 9 below. It is assumed that the single input is coupled only to the rth state variable; the results can be extended to the case of multiple couplings by utilizing the reasoning in Corollary 4 above.

Use is made of the following well-known properties of finite groups[10].

(*i*) The order of a group G which is transitive on k symbols is mk where m is an integer giving the number of group operations which leave any given symbol unchanged.

(*ii*) If a group G is intransitive on k symbols, the symbols may be partitioned into transitive sets $M_r, M_s, \cdots$. If the operations of G are allowed to operate only on symbols in the transitive set M_r (i.e., permutations of symbols not in M_r are simply ignored), G reduces to a new group G_r. The result is that

$$g = g_r g_{\bar{r}} , \tag{35}$$

where the lower-case letters indicate the orders of the appropriate groups, and $G_{\bar{r}}$ is the invariant subgroup leaving fixed all symbols in M_r.

A general intransitive symmetric network is considered in which the state variables are partitioned into the transitive sets M_r, M_s, $\cdots$. The set M_r contains the rth state variable, x_r, and the number of state variables in M_r is denoted by k_r. The following theorem is a simple application of property (*i*) above; it guarantees that the rth state variable is left unchanged by only one operation of the network symmetry group, G.

Theorem 7: If G is isomorphic with G_r and if k_r is equal to the order of G, the network is not NCS.

Proof: Since G_r is necessarily transitive on the k_r symbols of M_r, property (*i*) above implies that $g_r = mk_r$. However, $g_r = g$ since G and G_r are isomorphic. Thus, $g = mk_r$. By hypothesis, k_r equals the order of G; m must be unity. Hence, only one group operation of G leaves the rth state variable invariant, and the NCS property is impossible as shown in Corollary 1 to Theorem 6.

Theorem 8: Let G be an axial point group and let G_r be a proper subgroup of G. The symmetric network with symmetry group G is NCS.

Proof: Since G_r is a proper subgroup of G, equation (35) implies that $g_{\bar{r}}$ is greater than unity. For the axial point groups excluding D_{nh} groups, only the identity and a reflection plane σ_v (a rotation C_2 about a two-fold axis perpendicular to the principal axis may be included instead of a reflection plane) can have an invariant effect on any given state variable. For D_{nh} groups, in addition to E and σ_v, a C_2 operation and a σ_h operation can have an invariant effect on a given state variable. Furthermore, at most only one symmetry plane σ_v (rotation C_2) can leave a given state variable unchanged. Hence $g_{\bar{r}} = 2$, or possibly $g_{\bar{r}} = 4$, for a D_{nh} group. Thus, the subgroup $G_{\bar{r}}$ in property (*ii*) above is either $\{E, \sigma_v\}$, $\{E, C_2\}$, or $\{E, \sigma_v, C_2, \sigma_h\}$, and these operations leave invariant all variables in the transitive set M_r.

To show that the networks considered in this theorem are NCS, a proper value of μ is determined for use in (21); we compute c_2, the number of times $D^{(2)}(R)$ is contained in $D(R)$, using (52). To facilitate the computation of c_2, the n state variables are partitioned into h_2 transitive sets of two variables each, h_3 transitive sets of three variables each, $\cdots$, h_{k_r} transitive sets of k_r variables each, etc. Thus,

$$n = 2h_2 + 3h_3 + \cdots + h_{k_r}k_r + \cdots . \tag{36}$$

In $D^{(2)}(R)$, all σ_v and C_2 operations are represented by -1 while E and σ_h are represented by $+1$.[12] Hence, c_2 is at least

$$c_2 = \frac{1}{g} \sum_R \chi^{(2)}(R)^* \chi_p(R)$$

$$= \begin{cases} \dfrac{1}{g}\,[n - h_{k_r}k_r], & \text{for} \quad C_{nv}\,,\ D_n\,,\ D_{nd} \quad \text{symmetry} \\ \text{or} \\ \dfrac{1}{g}\,[n - h_{k_r}k_r + h_{k_r}k_r - h_{k_r}k_r], & \text{for} \quad D_{nh} \quad \text{symmetry.} \end{cases} \tag{37}$$

In general, c_2 is not zero, and (21) is satisfied for $\mu = 2$ since $\{R_r^r\} = \{E, \sigma_v\}$ or $\{E, C_2\}$, or $\{E, \sigma_v\,, C_2\,, \sigma_h\}$, and

$$\sum_{R_r{}^r} s_r^r\, D^{(2)}(R_r^r)^*_{\pi\pi} = 1 - 1 = 0, \quad \text{for} \quad C_{nv}\,,\ D_n\,, \quad \text{or} \quad D_{nd} \quad \text{groups}$$

and

$$\sum_{R_r{}^r} s_r^r\, D^{(2)}(R_r^r)^*_{\pi\pi} = 1 - 1 + 1 - 1 = 0, \quad \text{for } D_{nh} \text{ groups.}$$

Thus, these networks are NCS.

As an example illustrating the use of Theorems 7 and 8, consider the network of Fig. 4 which includes the operations of the symmetry group C_{2v} for this case. A table of irreducible representations of C_{2v} may be found in Fig. 2. There are two transitive sets, namely, $M_1 = \{x_1\,, x_4\}$ and $M_2 = \{x_2\,, x_3\,, x_5\,, x_6\}$. By allowing the permutations of G to operate only on state variables in M_1, G reduces to

$$G_1 = \{E, \sigma\}$$

where

$$E = (1)(4)$$

$$\sigma = (14).$$

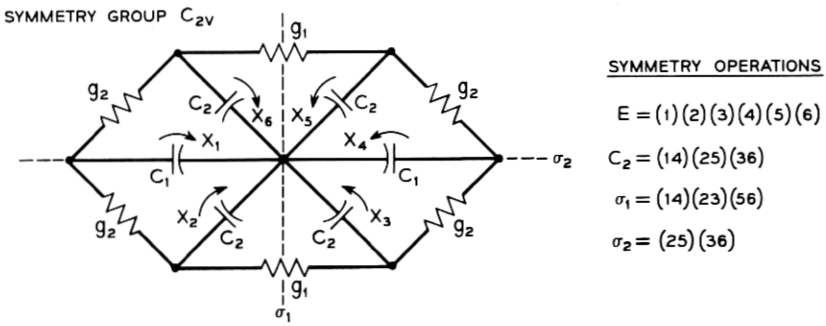


Fig. 4—Network with C_{2v} symmetry and symmetry operations.

By allowing the permutations of G to operate only on state variables in M_2, G reduces to $G_2 \sim G$ (i.e., G_2 and G are isomorphic). Furthermore, the number of state variables in M_2 is equal to the order of G. Thus, by Theorem 7, no symmetry constraints are placed on controllability if the input is coupled to one of the variables in the set M_2. However, from Theorem 8, if the input is coupled either to x_1 or to x_4, the network has the NCS property. The above statements are verified by computing the sum in equation (21) for each case.

Theorem 9: Let G be an axial point group having at least one irreducible representation of dimension two. A network possessing symmetry group G is NCS.

Proof: Let $D^{(\mu)}(R)$ be an irreducible representation of G of dimension two, and let c_μ be the number of times that $D^{(\mu)}(R)$ appears in the monomial representation $D(R)$. Since the character $\chi^{(\mu)}(R)$ of all group operators, excluding E, that can possibly have an invariant effect on any state variable is zero (see tables of irreducible representations in Ref. 12),

$$c_\mu = \frac{1}{g} \sum_R \chi^{(\mu)}(R)^*\chi(R) = \frac{1}{g} \chi^{(\mu)}(E)^*\chi(E) = \frac{1}{g} 2 \cdot n.$$

Hence, $D^{(\mu)}(R)$ is contained in $D(R)$. From the proof of the previous theorem, $\{R_r^r\} = \{E, \sigma_v\}$ or $\{E, C_2\}$ or $\{E, \sigma_v, C_2, \sigma_h\}$. A table of irreducible representations[12] shows that $D^{(\mu)}(R)$ is equivalent to a representation in which

$$D^{(\mu)}(E) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D^{(\mu)}(\sigma_v) \quad \text{or} \quad D^{(\mu)}(C_2) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D^{(\mu)}(\sigma_h) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, equation (21) is satisfied for $\pi = 1$ (or $\pi = 2$ if $s_r^r = -1$), and the network is NCS. The theorem is proved.

An example illustrating Theorem 9 has already been given in the discussion following Theorem 6. The example concerned the network displaying C_{3v} symmetry shown in Fig. 3. Since all C_{nv}, D_n, D_{nd}, and D_{nh} groups of complexity C_{3v} or greater possess two-dimensional irreducible representations, Theorem 9 shows that the general symmetric network displaying axial point group symmetry is NCS.

The single-input case, considered extensively here, assumes an added significance in view of the following definition found in Ref. 14.

Definition 5: A k-input system is said to be *strongly controllable* if it is controllable by each input separately while all others are zero; otherwise it is *weakly controllable.*
From the discussion of the present section, it is observed that a symmetric network is generally only weakly controllable. Thus, several inputs are required to control the state of a symmetric network in general. The results of the present section can be used to determine the number and placement of inputs to insure that the network is not NCS.

The previous results of this section may be applied to the multiple-input case by means of the following.

Theorem 10: The k-input system of equation (28) *with symmetry group G_N is NCS if and only if a proper value of μ exists such that for some value of π,*

$$\sum_{R_\phi{}^\phi} s_\phi^\phi \, D^{(\mu)}(R_\phi^\phi)^*_{\pi\pi} = 0, \qquad \phi = r, \cdots, j,$$

where ϕ is an index denoting all nonzero couplings of the inputs to the state variables in the k columns of B.

Proof: It follows from Theorem 6 and its Corollary 4 that if the above conditions hold, the submatrix of $\alpha^\dagger B$ in (29) corresponding to $\bar{A}_\pi^\mu$ is identically zero. Thus, the network is noncontrollable due to symmetry constraints. From Theorem 6, the above conditions are also necessary for the NCS property.

The discussion just concluded shows that simple arithmetic computations involving the irreducible representations of the network symmetry group can be used to detect noncontrollability which is due solely to symmetry. For an input coupled to a given state variable, the NCS property is determined completely by those group operations that leave the given state variable unchanged. The interpretation of (21) is obtained from (52), in which the generating matrix $G_\pi^{(\mu)}$ is obtained by analogy with the projection operation $P_\pi^{(\mu)}$. If the input is coupled to the rth state variable, then, by Lemma 2, condition (21) is equivalent to the statement that the projection of the input onto the invariant subspace associated with the πth row of $D^{(\mu)}(R)$ is zero.

In the application of Theorem 6, if equation (21) is not satisfied, the network *may* be controllable. The nonsatisfaction of equation (21) amounts to a necessary condition for controllability of a symmetric network. The transformation α obtained using group theory then enables us to test controllability of several smaller subsystems [equation (29)] rather than that of system (1).

V. OBSERVABILITY OF SYMMETRIC NETWORKS

The concept of observability relates to the degree to which the past state of a system may be determined from knowledge of the system outputs. The following definition may be found in Ref. 13.

Definition 6: The system (1) is said to be *completely observable on an interval* (t_o, t_1) if any initial state x_o at t_o can be determined from knowledge of the system output over (t_o, t_1).

The system (1) is said to be *totally observable on an interval* (t_o, t_1) if it is completely observable on every subinterval of (t_o, t_1).

For networks with sufficiently smooth time variations, observability of the linear time-varying system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u(t) \\ y(t) &= C(t)x \end{aligned} \tag{38}$$

may be characterized by the observability matrix[13]

$$Q_0 = [S_0 S_1 \cdots S_{n-1}] \tag{39}$$

where

$$S_k = A^t S_{k-1} + \dot{S}_{k-1}, \qquad S_0 = C^t$$

and n is the order of the system.

The following theorem is a paraphrase of Theorem 5 of Ref. 13.

Theorem 11: For the system (38), *assume that* $A(t)$ *and* $C(t)$ *and their first* $n - 2$ *and* $n - 1$ *derivatives, respectively, are continuous functions. System* (38) *is totally observable on the interval* (t_o, t_1) *if and only if* Q_0 *does not have rank less than* n *on any subinterval of* (t_o, t_1).

The results of this section are completely analogous to those obtained for controllability. Hence, only some theorems will be presented; their proofs follow exactly from their counterparts in the previous sections.

Lemma 4: The system described in partitioned form by

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + Bu(t) \\ y(t) &= [C_1 \quad 0]\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \end{aligned} \tag{40}$$

is unobservable (*i.e., not completely observable*).

Definition 7: A symmetric network is said to be NOS (possess the NOS property) if it is nonobservable because of symmetry alone.

To examine the NOS property, we reduce $A(t)$ to block-diagonal form; for $Z = \alpha^\dagger x$, equation (38) becomes

$$\frac{d}{dt}\begin{bmatrix} Z_1^1 \\ \vdots \\ Z_\pi^\mu \\ \vdots \\ Z_{l_\beta}^\beta \end{bmatrix} = \begin{bmatrix} \bar{A}_1^1 & & & & \\ & \ddots & & 0 & \\ & & \bar{A}_\pi^\mu & & \\ & 0 & & \ddots & \\ & & & & \bar{A}_{l_\beta}^\beta \end{bmatrix} \begin{bmatrix} Z_1^1 \\ \vdots \\ Z_\pi^\mu \\ \vdots \\ Z_{l_\beta}^\beta \end{bmatrix} + \alpha^\dagger Bu(t) \tag{41}$$

$$y(t) = C(t)\alpha \begin{bmatrix} Z_1^1 \\ \vdots \\ Z_\pi^\mu \\ \vdots \\ Z_{l_\beta}^\beta \end{bmatrix}.$$

Hence, if a submatrix of $C(t)\alpha$ corresponding to Z_π^μ is zero, these variables will not be observed in the output. In analogy to Section IV, first consider the output in (38) to be a function of a single state variable, x_r. Thus, $C(t)$ is $[e_r]^t$, and $C(t)\alpha$ in (39) is the rth row of α.

Theorem 12: A symmetric time-varying network whose output is a function of the rth state variable only is NOS if and only if there exists a proper value of μ and a value of π such that

$$\sum_{R_r{}^r} s_r^r \, D^{(\mu)}(R_r^r)^* = 0. \tag{42}$$

Corollaries 1–4 of Theorem 6 carry through directly for this case with slight and obvious changes of wording (i.e., "NOS" replaces "NCS", etc.), and they are not repeated here. Of course the other results of Section IV follow for observability with slight modification of the wording.

VI. APPEARANCE OF BASIS FUNCTIONS IN $\Phi(t, \tau)$

It may happen that one or more basis functions of the normal form differential equation do not appear in the expression for component $\phi_{ij}(t, \tau)$ of the state-transition matrix $\Phi(t, \tau)$. In the case of fixed systems, this condition corresponds to one in which certain modes are cancelled.

The present section investigates the use of symmetry in predicting such cancellation of basis functions in a symmetric network.

In Section II, it was shown that the monomial representation $D(R)$ of the group G_N commutes with $\Phi(t, \tau)$ for a symmetric network. Hence, $\Phi(t, \tau)$ is reduced to block-diagonal form by the same transformation α which reduces $D(R)$ and $A(t)$, and we can write

$$\Phi(t, \tau) = \alpha\bar{\Phi}(t, \tau)\alpha^{\dagger}$$
$$= \alpha \begin{bmatrix} \bar{\phi}_1^1 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & \bar{\phi}_1^{\mu} & & & & & \\ & & & \ddots & & & & \\ & & & & \bar{\phi}_{\pi}^{\mu} & & & \\ & & & & & \ddots & & \\ & & & & & & \bar{\phi}_{l_\mu}^{\mu} & \\ & & & & & & & \ddots \\ 0 & & & & & & & & \bar{\phi}_{l_\beta}^{\beta} \end{bmatrix} \alpha^{\dagger}, \tag{43}$$

where $\bar{\phi}_{\pi}^{\mu}$ is a $c_\mu \times c_\mu$ matrix which does not depend on π.‡ Hence,

$$\phi_{ij}(t, \tau) = [\alpha_{111}^{i}, \cdots, \alpha_{\mu\pi 1}^{i}, \cdots, \alpha_{\mu\pi c_\mu}^{i}, \cdots, \alpha_{\beta l_\beta c_\beta}^{i}]$$
$$\times \begin{bmatrix} \bar{\phi}_1^1 & & & & & & & 0 \\ & \ddots & & & & & & \\ & & \bar{\phi}_1^{\mu} & & & & & \\ & & & \ddots & & & & \\ & & & & \bar{\phi}_{\pi}^{\mu} & & & \\ & & & & & \ddots & & \\ 0 & & & & & & \bar{\phi}_{l_\mu}^{\mu} & \\ & & & & & & & \ddots \\ & & & & & & & & \bar{\phi}_{l_\beta}^{\beta} \end{bmatrix} \begin{bmatrix} \alpha_{111}^{j} \\ \vdots \\ \alpha_{\mu\pi 1}^{j} \\ \vdots \\ \alpha_{\mu\pi c_\mu}^{j} \\ \vdots \\ \alpha_{\beta l_\beta c_\beta}^{j} \end{bmatrix}^{*} \tag{44}$$

It is evident from (44) that basis functions associated with $D^{(\mu)}(R)$

‡ In Ref. 2 it is shown that submatrices $\bar{A}_{\pi}{}^{\mu}$ of the A-matrix of (29) do not depend on π. Hence, submatrices $\bar{\phi}_{\pi}{}^{\mu}$ given above are independent of π.

will appear in $\phi_{ij}(t, \tau)$ if

$$K_\mu = [\alpha^i_{\mu 11}, \cdots, \alpha^i_{\mu 1 c_\mu}, \cdots, \alpha^i_{\mu\pi 1}, \cdots, \alpha^i_{\mu\pi c_\mu}, \cdots, \alpha^i_{\mu l_\mu 1}, \cdots, \alpha^i_{\mu l_\mu c_\mu}] \times \begin{bmatrix} \bar{\Phi}^\mu_1 & & & & \\ & \ddots & & & \\ & & \bar{\Phi}^\mu_\pi & & \\ & & & \ddots & \\ & & & & \bar{\Phi}^\mu_{l_\mu} \end{bmatrix} \begin{bmatrix} \alpha^j_{\mu 11} \\ \vdots \\ \alpha^j_{\mu 1 c_\mu} \\ \vdots \\ \alpha^j_{\mu\pi 1} \\ \vdots \\ \alpha^j_{\mu\pi c_\mu} \\ \vdots \\ \alpha^j_{\mu l_\mu 1} \\ \vdots \\ \alpha^j_{\mu l_\mu c_\mu} \end{bmatrix}^* \neq 0, \tag{45}$$

where K_μ is used to denote all terms corresponding to $D^{(\mu)}(R)$ in (44). From Lemma 1, only one of the c_μ terms $\alpha^i_{\mu\pi a}$ (or $\alpha^j_{\mu\pi a}$) can possibly be nonzero, $a = 1, \cdots, c_\mu$; $\pi = 1, \cdots, l_\mu$. Thus, using the results of Section II and excluding a scale factor, we have

$$K_\mu = (\bar{\Phi}^\mu_\pi)_{dd} \sum_{\pi=1}^{l_\mu} [\sum_{R_i{}^i} s^i_i\, D^{(\mu)}(R^i_i)^*_{\pi\pi}][\sum_{R_i{}^j} s^j_i\, D^{(\mu)}(R^j_i)_{\pi\pi}] \tag{46a}$$

if i and j are in the same transitive set, or

$$K_\mu = (\bar{\Phi}^\mu_\pi)_{de} \sum_{\pi=1}^{l_\mu} [\sum_{R_i{}^i} s^i_i\, D^{(\mu)}(R^i_i)^*_{\pi\pi}][\sum_{R_j{}^j} s^j_j\, D^{(\mu)}(R^j_j)_{\pi\pi}] \tag{46b}$$

if i and j are in different transitive sets.‡ In (46), $(\bar{\Phi}^\mu_\pi)_{dd}$ and $(\bar{\Phi}^\mu_\pi)_{de}$ are components of the matrix $\bar{\Phi}^\mu_\pi$, and are removed from the π-summation because they are independent of π. Thus, the following theorem has been established.

Theorem 13: Basis functions corresponding to $D^{(\mu)}(R)$ will not appear in $\phi_{ij}(t, \tau)$ as a result of the symmetry if and only if

$$\sum_\pi [\sum_{R_i{}^i} s^i_i\, D^{(\mu)}(R^i_i)^*_{\pi\pi}][\sum_{R_j{}^j} s^j_j\, D^{(\mu)}(R^j_j)_{\pi\pi}] = 0. \tag{47}$$

‡ If $c_\mu > c_1{}^p$, equation (46b) may apply even if i and j are in the same transitive set. See Lemma 2.

The following corollaries to the above theorem may be established; some of the corollaries are similar to those following Theorem 6.

Corollary 1: All basis functions corresponding to $D^{(\bar{\mu})}(R)$ appear in every $\phi_{ii}(t, \tau)$ (if all the s_i^i equal unity, $\bar{\mu} = 1$).

Corollary 2: If there is just one group operation taking the ith state variable into the jth state variable, basis functions corresponding to $D^{(\mu)}(R)$ appear in $\phi_{ij}(t, \tau)$ if $\chi^{(\mu)}(R_i^j) \neq 0$, where $\chi^{(\mu)}(R)$ denotes the trace of $D^{(\mu)}(R)$.

Proof: The hypothesis requires that only one group operator leave the ith state variable invariant.[10] This operator must be the identity, and $D^{(\mu)}(E)_{\pi\pi} = 1$ for all values of μ. Hence, equation (46a) becomes (s_r^r equals unity for the identity operation)

$$K_\mu = (\bar{\phi}_\tau^\mu)_{dd} \sum_\pi D^{(\mu)}(R_i^j)_{\pi\pi} = (\bar{\phi}_\pi^\mu)_{dd}\chi^{(\mu)}(R_i^j). \tag{48}$$

Thus, basis functions corresponding to $D^{(\mu)}(R)$ appear in $\phi_{ij}(t, \tau)$ if $\chi^{(\mu)}(R_i^j) \neq 0$.

Corollary 3: If no group operation transforms the ith state variable into the jth state variable, the types of basis functions which appear in $\phi_{ij}(t, \tau)$ are those which are common to $\phi_{ii}(t, \tau)$ and $\phi_{jj}(t, \tau)$.

The proof of Corollary 3 is a straightforward application of Theorem 13.

Corollary 4: If the kth state variable is invariant under all the group operators, the only basis functions appearing in $\phi_{ik}(t, \tau)$ are those which correspond to $D^{(\bar{\mu})}(R)$.

With regard to Section IV (Section V) the following statement can be made about noncontrollability (nonobservability) due to symmetry.

Theorem 14: A symmetric time-variable linear network with a single input coupled only to the rth state variable (output proportional only to the rth state variable) is NCS (NOS) if there exists a proper value of μ such that basis functions corresponding to $D^{(\mu)}(R)$ do not appear in $\phi_{rr}(t, \tau)$.

Proof: If the basis functions corresponding to $D^{(\mu)}(R)$ do not appear in $\phi_{rr}(t, \tau)$, Theorem 13 shows that

$$\sum_\pi \Big[\sum_{R_r{}^r} s_r^r\, D^{(\mu)}(R_r^r)^*_{\pi\pi}\Big]\Big[\sum_{R_r{}^r} s_r^r\, D^{(\mu)}(R_r^r)_{\pi\pi}\Big] = 0. \tag{49}$$

Since the squared magnitude of the bracketed term appears in the above equation, it is necessary that

$$\sum_{R_r^r} s_r^r \, D^{(\mu)}(R_r^r)^*_{\pi\pi} = 0. \tag{50}$$

Under the conditions of the present theorem, the network is NCS (NOS) by Theorem 6 (Theorem 12).

VII. CONCLUSION

We have presented a unified treatment of linear time-variable networks displaying arbitrary geometrical symmetries by incorporating group theory into the analysis scheme. Symmetric networks have their elements arranged so that certain permutations of the network edges result in a configuration identical with the original. The complete set of such permutations constitutes a group G_S, the symmetry group of the network structure. A group G_N of monomial matrices may then be determined, and it was shown that these matrices commute with the A-matrix and the state transition matrix of the normal form equation. The commutativity result establishes a basic connection between group theory and the network analysis problem and allows group theoretic methods to be employed in the study of networks with arbitrary symmetries. The group G_N may be a proper subgroup of G_S, since G_N contains those operations of G_S which do not permute edges in the tree with those in the cotree.

Group representation theory makes it possible to obtain information about properties of the network differential equations without writing or solving them. For the case of a network with a single input coupled to only one of the state variables, an extremely simple arithmetic condition is derived which determines whether symmetry alone causes the network to be noncontrollable. The condition involves only those group operators which transform the state variable in question into itself. It is equivalent to the algebraic statement that the projection of the input vector onto a subspace associated with an irreducible representation of the group be zero. The results allow a determination by inspection of linear combinations of the original state variables which result in noncontrollable variables. It was demonstrated that a network with axial point group symmetry is always noncontrollable if its symmetry group possesses an irreducible matrix representation of dimension two. This result agrees with intuition in that the network will be noncontrollable if the symmetry is high enough. Thus, networks with axial point group symmetry are generally noncontrollable because of symmetry alone. The case where the input is coupled to more than one state variable and the multiple input case were also treated. Furthermore, dual results were stated for network observability.

By utilizing the symmetry, a transformation may be constructed which transforms the A-matrix into block-diagonal form. The original differential equation is thereby resolved into several differential equations of relatively low order. Hence, there results an appreciable economy of effort in obtaining solutions for symmetric networks.

APPENDIX A

This appendix provides some basic definitions and results from the abstract theory of finite groups and the corresponding representation theory. A more detailed treatment of concepts mentioned here may be found in Refs. 10 and 15, 16.

Definition 8: A set of elements $G = \{A_1, A_2, A_3, \cdots\}$ is a *group* if

(*i*) for $A_p, A_q \in G$, $A_pA_q \in G$ (closure)
(*ii*) for $A_p, A_q, A_r \in G$, $(A_pA_q)A_r = A_p(A_qA_r)$ (associativity)
(*iii*) there exists $E \in G$ such that $A_pE = EA_p = A_p$ (identity element)
(*iv*) there exists $A_p^{-1} \in G$ such that $A_p^{-1}A_p = A_pA_p^{-1} = E$ (inverse element).

If the number of distinct elements of the group is finite, the group is said to be a *finite group*; the number of distinct elements in a finite group is called its *order*.

Definition 9: Two groups G and G' are said to be *isomorphic* if there exists a one-to-one correspondence (denoted $\sim$) between their elements such that products correspond to products, i.e., if $A \sim A'$ and $B \sim B'$, then $AB \sim A'B'$.

Definition 10: If among the elements of a group G there exists a subset H of elements satisfying the definition of a group, then H is said to be subgroup of the group G.

Consider a subgroup H of G, where the order of H is h while that of G is g. Now consider any element x_1 of G which is not contained in H, and form the product x_1H. That is, multiply every element of H on the left by x_1. Since x_1 is not in H, the resulting set of elements is different from H (H contains the identity, E, and hence x_1H contains x_1). The set of elements x_1H is called a *left coset* of G with respect to the subgroup H. A coset is not a subgroup since it does not contain the identity (H does not contain x_1^{-1}). If there are any elements of G not contained in H or x_1H, choose one of these elements, x_2 say, and form the coset x_2H.

Continue in this manner until all elements of G are exhausted. Thus, a partition has been effected of the group G into left cosets with respect to the subgroup H.

$$G = H, x_1H, x_2H, \cdots, x_{l-1}H.$$

A similar partition could be effected using right cosets, defined analogously. The quantity $l = g/h$ is an integer[10] called the *index* of H in G.

Definition 11: If H is a subgroup of G and $x \varepsilon G$, then $x^{-1}Hx$ is called a *conjugate subgroup* of H in G. If H coincides with all its conjugates (i.e., $x^{-1}Hx = H$, for all $x \varepsilon G$), then H is said to be an *invariant subgroup.*

Consider the set of n symbols $a_1, a_2, \cdots, a_n$. A rearrangement of these same symbols into the order $b_1, b_2, \cdots, b_n$ is called a *permutation.* Here, the symbol a_1 is replaced by b_1, a_2 by b_2, etc. One way of indicating this permutation is

$$\begin{bmatrix} a_1 a_2 & \cdots & a_n \\ b_1 b_2 & \cdots & b_n \end{bmatrix},$$

so that each symbol on the upper line is replaced by the symbol appearing below it. A more convenient notation is to write the permutation as a set of cycles. To do so, begin by choosing any symbol on the top line, say q, writing it followed by the symbol r on the bottom line which replaces it. Now find where r appears on the upper line, obtain the symbol which replaces r and write that. This procedure is continued until we arrive at the symbol which is replaced by q, the first symbol in the cycle. This step completes a cycle. If any symbols remain unused, a new cycle is written by choosing as the leading symbol any one of those which did not appear in the first cycle. This procedure is continued until all symbols are exhausted. The cycles are enclosed in parentheses. Thus,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 1 & 2 & 6 \end{bmatrix} = (14)(235)(6) = (14)(235),$$

where 1 is replaced by 4, 4 is replaced by 1, etc. Cycles composed of a single symbol, such as (6), need not be written. In examples of symmetric networks, the cycle notation may be used to make easy the identification of matrices $D(R)$ (Section II) by inspection.

Some important results in group representation theory are presented next.

Definition 12: A group of matrices $D(\cdot)$ is said to form a *representation* of a group $G = \{E, A, \cdots, R, \cdots\}$ if there exists a correspondence

(denoted $\sim$) between the matrices and the group elements such that products correspond to products, i.e., if $R_1 \sim D(R_1)$ and $R_2 \sim D(R_2)$, then $(R_1R_2) \sim D(R_1)D(R_2) = D(R_1R_2)$.
An example of a representation is the so-called *totally symmetric* representation in which each group element is represented by the scalar quantity unity.

Definition 13: A representation is said to be *reducible* if it can be converted to block-diagonal form via a similarity transformation; i.e.,

$$D(R) = \begin{bmatrix} D_1(R) & 0 \\ 0 & D_2(R) \end{bmatrix}$$

is reducible. Otherwise, it is said to be irreducible. For a finite group, there can be only a finite number of distinct irreducible representations, and the irreducible representations may generally be specified to within a similarity transformation. The irreducible representations of a finite group satisfy the following important orthogonality relation.[15]

$$\sum_R D^{(i)}(R)^*_{\alpha\beta}\, D^{(j)}(R)_{\alpha'\beta'} = \frac{g}{l_i}\, \delta_{ij}\, \delta_{\alpha\alpha'}\, \delta_{\beta\beta'}\,, \tag{51}$$

where $D^{(k)}(R)_{\alpha\beta}$ denotes the $\alpha\beta$-element of irreducible representation $D^{(k)}(R)$, l_i denotes the dimension of $D^{(i)}(R)$, g denotes the order of the group, $\delta_{\mu\nu}$ is Kronecker's delta, and asterisk denotes the complex conjugate.

If a reducible representation $D(R)$ is reduced to block-diagonal form, the nonzero submatrices along the diagonal will be the irreducible representations of the group.[15] Some irreducible representations may appear more than once (i.e., several nonzero blocks may be identical) in $D(R)$ while others may not appear at all. The number of times that $D^{(k)}(R)$ appears in $D(R)$ is denoted by c_k and is given by[15]

$$c_k = \frac{1}{g} \sum_R \chi^{(k)}(R)^* \chi(R), \tag{52}$$

where $\chi(R)$ is the trace of $D(R)$ and $\chi^{(k)}(R)$ is the trace of $D^{(k)}(R)$.

A very brief account is now given of so-called axial point groups. Some important statements regarding networks with axial point group symmetry may be found in Theorems 7, 8, and 9. For a more complete treatment of these groups, see Ref. 17.

A point group is one whose symmetry operations leave fixed a point at the center of symmetry. Some symmetry operations contained in these groups are described in the following five definitions.

Definition 14: The *identity* is the trivial operation which does not transform the object at all. It is denoted by the letter E.

Definition 15: A rotation operation by $2\pi/n$ radians about an axis is denoted by C_n where $2\pi/n$ is the smallest angle for which the object may be rotated invariantly about this axis. The axis is said to be an *n-fold rotation axis.*

Definition 16: A reflection operation in a plane of symmetry is labelled σ. If the plane of symmetry is perpendicular to the principal rotation axis of symmetry, it is labelled σ_h; if it contains the principal axis, it is labelled either σ_v or σ_d.

Definition 17: The *rotation-reflection* operation S_n is a compound operation consisting of a rotation by $2\pi/n$ radians about an axis followed by a reflection in a plane perpendicular to the axis. Thus, $S_n = \sigma_h C_n$.

Definition 18: The *inversion,* denoted by i, is a reflection in the center of symmetry.

The distinguishing characteristic of axial point groups is their single n-fold axis of symmetry ($n > 2$), called the principal symmetry axis. A diagram, called an equivalent point diagram, often used to visualize the operations of an axial point group, is described below. The number of points in the diagram is equal to the order of the group;[12] the points transform into one another under the group operations. In the equivalent point diagram, a plus, +, and circle, ○, denote points above and below the plane of the paper, respectively. Reflection planes not in the plane of the paper are indicated by dotted lines while rotation axes are indicated by solid lines marked with one of the symbols 0, △, □, etc. to indicate a two-fold, three-fold, four-fold axis, etc. Reflection in the plane of the paper is indicated as a solid circle in the point diagram. In the equivalent point diagram, the principal symmetry axis is assumed to be perpendicular to the plane of the paper so that reflection in that plane is σ_h. See Fig. 5 for equivalent point diagrams of all the axial point groups mentioned below.

C_n *groups* have one n-fold rotation axis. The group operations consist of the rotations C_n^r of the object by $(2\pi r)/n$ radians ($r = 1, 2, \cdots, n$). These groups are cyclic.

S_n *groups* have one n-fold rotation-reflection axis.

C_{nv} *groups* have a symmetry axis C_n and n symmetry planes σ_v.

C_{nh} *groups* have a symmetry axis C_n and one symmetry plane σ_h perpendicular to it.

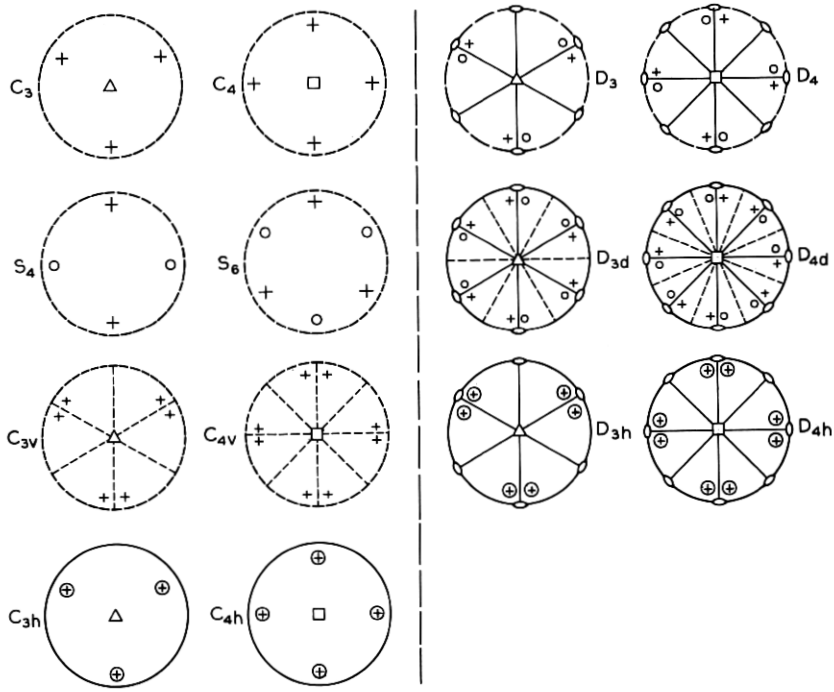


Fig. 5—Some axial point groups and their equivalent point diagrams.

D_n *groups* have an n-fold rotation axis and n two-fold rotation axes perpendicular to the principal axis. The angle between two adjacent two-fold axes is π/n radians.

D_{nd} *groups* contain all the symmetries of D_n and in addition contain n vertical symmetry planes σ_d which contain the principal axis and bisect the angles between the two-fold axes.

D_{nh} *groups* contain all the symmetries of D_n and in addition contain the symmetry plane σ_h perpendicular to the principal axis. These symmetries imply the existence of n symmetry planes σ_v containing both the principal axis and a two-fold rotation axis.

APPENDIX B

For a given reducible representation $D(R)$ of a group G_N, it is possible to construct a unitary matrix α such that the transformed representation‡ $\alpha^\dagger D(R)\alpha$ is in block-diagonal form. It is next shown how α is constructed.

‡ The complex conjugate transpose of matrix α is denoted by $\alpha^\dagger$.

Let $D^{(i)}(R)$ be an irreducible representation contained in the reducible representation $D(R)$ of the group G_N.

Definition 19: A set of k vectors $v_1^{(i)}, v_2^{(i)}, \cdots, v_k^{(i)}$ is said to form a *basis for an irreducible representation* $D^{(i)}(R)$ of dimension k if the effect of all group operators on these vectors is to produce a vector which is a linear combination of those already in the set. The set of vectors is said to *transform according to* $D^{(i)}(R)$.

Definition 20: The vector $v_x^{(i)}$ is said to *belong to* (or transform according to) the xth row of the irreducible representation $D^{(i)}(R)$ if it satisfies

$$\sum_R D^{(i)}(R)^*_{xx}\, D(R)v_x^{(i)} = \frac{g_N}{k} v_x^{(i)}, \tag{53}$$

where g_N is the order of G_N. The other vectors in the basis belong to other rows of $D^{(i)}(R)$ and are called *partners* of $v_x^{(i)}$. They satisfy

$$\frac{g_N}{k} v_m^{(i)} = \sum_R D^{(i)}(R)^*_{mx}\, D(R)v_x^{(i)}. \tag{54}$$

Let P_R be the operator which denotes the effect of operating on a vector with group operation R. Form the operator

$$P_k^{(i)} = \sum_R D^{(i)}(R)^*_{kk} P_R\,. \tag{55}$$

$P_k^{(i)}$ has the important property that its effect on any arbitrary vector v is to produce the component vector (which may be zero) which belongs to the kth row of $D^{(i)}(R)$.[17] Hence, $P_k^{(i)}$ is a projection operation.

The transformation α which places a reducible representation $D(R)$ in a block diagonal form may now be constructed. Let c_p be the number of times the irreducible representation $D^{(p)}(R)$ of dimension l_p appears in $D(R)$. Form the $n \times n$ generating matrix $G_\pi^{(p)}$ by analogy with the projection operator of equation (55), so that

$$G_\pi^{(p)} = \sum_R D^{(p)}(R)^*_{\pi\pi}\, D(R)I, \tag{56}$$

where I is the unit matrix. Some of the column vectors of $G_\pi^{(p)}$ may be zero and several may be identical; the number of linearly independent vectors among the columns of $G_\pi^{(p)}$ is c_p,[11] and each of these c_p vectors belongs to the πth row of $D^{(p)}(R)$. They are orthogonal and may be normalized to unity. Following Kerns' notation,[11] these c_p column vectors are labelled $\alpha_{p\pi 1}, \cdots, \alpha_{p\pi a}, \cdots, \alpha_{p\pi c_p}$, and are used as c_p column vectors of the matrix α. For every one of the vectors $\alpha_{p\pi a}$, $l_p - 1$ partner vectors must be constructed. The partner vectors are denoted by $\alpha_{p\mu a}$ where $\mu = 1, \cdots \pi - 1, \pi + 1, \cdots, l_p$ and $a = 1, \cdots, c_p$, and may be calculated as [using equation (54)]

$$\alpha_{p\mu a} = [\sum_R D^{(p)}(R)^*_{\mu\pi} D(R)]\alpha_{p\pi a} . \tag{57}$$

The index p runs over all distinct irreducible representations of the symmetry group G_N . Thus, if a table of irreducible representations is available, the matrix α may be computed relatively easily, and has the form

$$\alpha = [\underbrace{\alpha_{111} , \cdots , \alpha_{11c_1}}_{D^{(1)}(R)} \underbrace{\alpha_{211} , \cdots , \alpha_{2l_2c_2}}_{D^{(2)}(R)} , \cdots], \tag{58}$$

where the columns of α are shown associated with the appropriate irreducible representation in the above equation.

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