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Capacitances of a Shielded Balanced-Pair Transmission Line

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The exact formulae (calculable to any accuracy) for mutual and conductor-to-ground capacitance (C_m and C_g) for a shielded balanced pair are expressed as infinite determinants. Convergence of these determinants is rapid except as the conductors of the pair approach each other or the shield. Approximate expressions developed by Philips Research, though not extremely accurate, are simple and in closed form thereby allowing capacitance surfaces to be plotted. These surfaces show qualitatively the variation of capacitance with dimensions.

I. INTRODUCTION

The shielded balanced pair consists of two straight cylindrical conductors immersed in a homogeneous dielectric, surrounded by an electrically thick tubular shield. Figure 1 is a cross section of the shielded-pair structure.

Dimensional restrictions are imposed which serve to keep the three conductors of the structure from touching. These restrictions are $S > d$ and $D > S + d$. In terms of the traditional dimensional ratios u and V, $u < 0.5$ and $V < 1/(1 + 2u)$. Thus, the variables u and V are contained in the shaded area in Fig. 2.

The exact capacitance expression for a pair in free space (unshielded) is derived in most texts.

$$C_m = \pi\epsilon_0/\cosh^{-1}(0.5/u) \qquad (1)$$

where $\epsilon_0 = (c^2\mu_0)^{-1} = 8.8541853 \times 10^{-12}$ farads/meter.

As the spacing between the conductors approaches zero, $u \to 0.5$ and $C_m \to \infty$. The effect of placing a shield around the pair at a large relative distance cannot alter the limiting values of C_m as $u \to 0.5$. Also, intuitively, as the inner conductors of the shielded pair structure approach the outer conductor, $V \to 1/(1 + 2u)$ and $C_g \to \infty$. The capacitance values at the limiting dimensions appear in Table I.

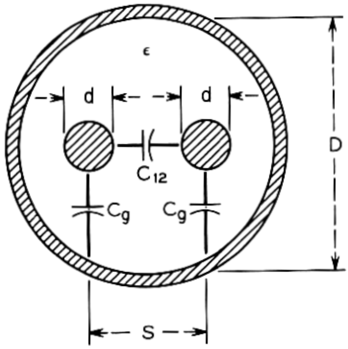

Fig. 1—The shielded-pair transmission line.

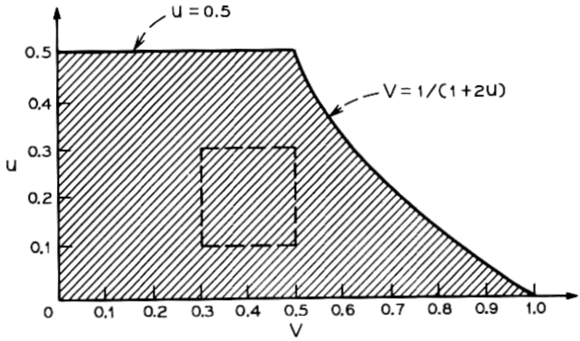

Fig. 2—Dimensional restrictions for the shielded pair. Typical values of u and V for existing shielded balanced-pair cables and equivalent u and V for multipair cables generally lie within the dashed square.

II. EXACT CAPACITANCE EXPRESSIONS

J. W. Craggs and C. J. Tranter[1] derived the exact expression for the mutual capacitance of the shielded balanced pair. The method assumed a Fourier surface charge density on the conductors, the coefficients being determined from the constancy of the potential over the surfaces. The factor δ_{12} is expressed in an infinite determinant set equal to zero.

$$\frac{C_m}{\epsilon} = \frac{0.044766}{\log_e \left(\frac{1}{u} \cdot \frac{1 - V^2}{1 + V^2}\right) - \delta_{12}} \ \mu\text{F/mile} \tag{2}$$

$$\begin{vmatrix} -\delta_{12} & \alpha_1(2u) & \alpha_2(2u)^2 & \cdots \\ -\frac{\alpha_1}{1}(2u) & 1 + A_{11}(2u)^2 & A_{21}(2u)^3 & \cdots \\ -\frac{\alpha_2}{2}(2u)^2 & A_{12}(2u)^3 & 1 + A_{22}(2u)^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0, \qquad (3)$$

where

$$\alpha_m = \left(-\frac{1}{2}\right)^m + \left(\frac{V^2}{1 - V^2}\right)^m - \left(\frac{-V^2}{1 + V^2}\right)^m$$

and

$$A_{mn} = -{}^mB_n\left(-\frac{1}{2}\right)^{m+n} - 2\sum_q^{\infty} {}^{2q+1}C_n \; {}^mB_{2q+1-m} V^{2(2q+1)},$$

and q equals $(n - 1)/2$ or $(m - 1)/2$, whichever is greater. B represents binomial coefficients and C represents combinatorial coefficients.

TABLE I—LIMITING CAPACITANCE VALUES

Dimensional Condition		Capacitance Value
$D = \infty$,	$V = 0$	$C_m = 0.044765\epsilon_r/\cosh^{-1}(0.5/u)$ uF/mile
$d \to S$,	$u \to 0.5$	$C_m \to \infty$
$S + d \to D$,	$V \to 1/(1 + 2u)$	$C_g \to \infty$

This solution lends itself well to computer calculation and for small values of u and V, a 2×2 determinant will give accurate answers. As u and V increase, more terms must be included until at $u = 0.45$, a 10×10 determinant is required to give five-digit accuracy. The convergence of this determinant is very slow as the conductors approach each other or the shield (u and V approach the limiting values).

The exact expression for C_g, capacitance to ground of one conductor, was derived by the author using the method of Craggs and Tranter. This derivation yields another infinite determinant equal to zero. With polar coordinates (r, θ) let the surface charge density on $r = r_1$ be

$$f(\theta) = \frac{Q}{2\pi r_1}\left(a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta\right).$$

An even function is selected since the conductors will be equipotential circles. The potential due to this charge density is then

$$^\dagger \; V = -2Qa_0 \log r_1 + Q \sum_{n=1}^{\infty} \left(\frac{r}{r_1}\right)^n \frac{a_n \cos n\theta}{n} \quad \text{for} \quad r \leqq r_1 \tag{4}$$

or

$$^\dagger \; V = -2Qa_0 \log r + Q \sum_{n=1}^{\infty} \left(\frac{r_1}{r}\right)^n \frac{a_n \cos n\theta}{n} \quad \text{for} \quad r \geqq r_1 \, . \tag{5}$$

Since three conductors, each with different polar origins, are involved, several coordinate transformations are required. For polar coordinates (ρ, ω) with origin at $r = c$, $\theta = 0$, and $c > r_1$ and $\rho < c - r_1$ use equation (5). Using

$$\left(\frac{1}{1-x}\right)^n = \sum_{m=0}^{\infty} {}^nB_m x^m, \qquad {}^nB_m = \frac{(n+m-1)!}{m!\,(n-1)!}$$

and

$$\log\left(1 + 2\frac{\rho}{c}\log\omega + \frac{\rho^2}{c^2}\right) = -2\sum_{n=1}^{\infty}\frac{\cos n\omega}{n}\left(-\frac{\rho}{c}\right)^n,$$

then

$$^\dagger \; V = -2Qa_0 \log c + 2Qa_0 \sum_{n=1}^{\infty} \frac{\cos n\omega}{n}\left(-\frac{\rho}{c}\right)^n + Q\sum_{n=1}^{\infty}\frac{a_n}{n}\left(\frac{r_1}{c}\right)^n {}^nB_m\left(\frac{-\rho}{c}\right)^m \cos m\omega. \tag{6}$$

For $c < r_1$ and $\rho < r_1 - c$ and $\rho < c$ use equation (4). Using

$$(1+x)^n = \sum_{m=0}^{n} {}^nC_m x^m, \qquad {}^nC_m = \frac{n!}{m!\,(n-m)!},$$

then

$$^\dagger \; V = -2Qa_0 \log r_1 + Q\sum_{n=1}^{\infty}\frac{a_n}{n}\left(\frac{c}{r_1}\right)^n \sum_{m=0}^{n} {}^nC_m\left(\frac{\rho}{c}\right)^m \cos m\omega. \tag{7}$$

For $c > r_1$ and $\rho > c + r_1$ use equation (5). Using previously stated identities,

$$^\dagger \; V = -2Qa_0 \log \rho + 2Qa_0 \sum_{n=1}^{\infty} \frac{\cos n\omega}{n}\left(-\frac{c}{\rho}\right)^n + Q\sum_{n=1}^{\infty}\left(\frac{r_1}{\rho}\right)^n \frac{a_n}{n}\sum_{m=0}^{\infty} {}^nB_m\left(-\frac{c}{\rho}\right)^m \cos (m+n)\omega. \tag{8}$$

† This result is stated in Ref. 1.

For polar coordinates (ρ, ω) with origin at $r = c$, $\theta = \pi$, and $c > r_1$ and $\rho > c + r_1$ use equation (5).

$$V = -2Qa_0 \log \rho + 2Qa_0 \sum_{n=1}^{\infty} \frac{\cos n\omega}{n} \left(\frac{c}{\rho}\right)^n + Q \sum_{n=1}^{\infty} \left(\frac{r_1}{\rho}\right)^n \frac{a_n}{n} \sum_{m=0}^{\infty} {}^nB_m \left(\frac{c}{\rho}\right)^m \cos (m + n)\omega. \tag{9}$$

For the shielded balanced pair structure let the inner and outer conductors be numbered as shown in Fig. 3.

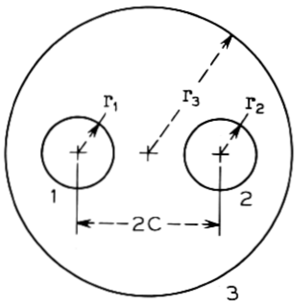

Fig. 3—Polar coordinates for the three conductors of the shielded-pair structure.

Assuming a charge distribution on conductor 1

$$f(\theta) = \frac{Q}{2\pi r_1}\left(1 + \sum_{n=1}^{\infty} a_n \cos n\theta\right),$$

and on conductor 2

$$f(\pi - \theta) = \frac{Q}{2\pi r_1}\left[1 + \sum_{n=1}^{\infty} a_n \cos n(\pi - \theta)\right] = \frac{Q}{2\pi r_1}\left[1 + \sum_{n=1}^{\infty} (-1)^n a_n \cos n\theta\right],$$

and on conductor 3

$$f(\theta) = \frac{Q}{2\pi r_3}\left[b_0 + \sum_{n=1}^{\infty} b_n \cos n\theta\right].$$

Writing the equation for the potential on conductor 3 with both inner conductors at $+V_0$ potential and conductor 3 at zero potential yields

$$0 = V_{31} + V_{32} + V_{33},$$

where V_{ab} is the potential on (a) due to a charge distribution on (b). From (8) with $\rho = r_3$ and $2c$ separation between conductors

$$\frac{V_{31}}{Q} = -2\log r_3 + 2\sum_{n=1}^{\infty}\left(-\frac{c}{r_3}\right)^n \frac{\cos n\omega}{n} + \sum_{n=1}^{\infty}\left(\frac{r_1}{r_3}\right)^n \frac{a_n}{n}\sum_{m=0}^{\infty} {}^nB_m\left(-\frac{c}{r_3}\right)^m \cos(m+n)\omega.$$

From (9) with $\rho = r_3$,

$$\frac{V_{32}}{Q} = -2\log r_3 + 2\sum_{n=1}^{\infty}\left(\frac{c}{r_3}\right)^n \frac{\cos n\omega}{n} + \sum_{n=1}^{\infty}\left(\frac{r_1}{r_3}\right)^n \frac{a_n}{n}(-1)^n\sum_{m=0}^{\infty} {}^nB_m\left(\frac{c}{r_3}\right)^m \cos(m+n)\omega.$$

From (4) or (5) with $r = r_1 = r_3$,

$$\frac{V_{33}}{Q} = -2b_0\log r_3 + \sum_{n=1}^{\infty}\frac{b_n\cos n\omega}{n}.$$

Substituting $k = m + n$ yields the total potential on conductor 3,

$$0 = -2(2+b_0)\log r_3 + 2\sum_{n=1}^{\infty}\left(\frac{c}{r_3}\right)^n\frac{\cos n\omega}{n}[1+(-1)^n] + \sum_{n=1}^{\infty}\left(\frac{r_1}{r_3}\right)^n\frac{a_n}{n}\sum_{k=n}^{\infty}{}^nB_{k-n}\left(\frac{c}{r_3}\right)^{k-n}(-1)^n[1+(-1)^k]\cos k\omega + \sum_{n=1}^{\infty}\frac{b_n\cos n\omega}{n}.$$

From this relationship we select the charge distribution on conductor 3 so as to eliminate the angle (ω) dependence.

$$f(\theta) = \frac{Q}{2\pi r_3}\left(-2 + \sum_{n=1}^{\infty} b_{2n}\cos 2n\theta\right) \tag{10}$$

Using

$${}^nB_{k-n} = 0 \quad \text{for} \quad k < n$$

and

$$\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}{}^nB_{k-n}f(n,k) = \sum_{k=1}^{\infty}\sum_{n=1}^{k}{}^nB_{k-n}f(n,k),$$

then

$$b_{2n} = -4\left(\frac{c}{r_3}\right)^{2n}\left[1 + n\sum_{k=1}^{2n}\left(\frac{r_1}{c}\right)^k \frac{a_k}{k}\,{}^kB_{2n-k}\right]. \tag{11}$$

Equation (11) gives a functional relationship between the Fourier coefficients for the assumed charge distributions. For conductor 2 the total potential $V_0 = V_{21} + V_{22} + V_{23}$. From (6) with $\rho = r_1$ and $2c$ separation between conductors,

$$V_{21} = -2Q\log 2c + 2Q\sum_{n=1}^{\infty}\frac{\cos n\omega}{n}\left(-\frac{r_1}{2c}\right)^n + Q\sum_{n=1}^{\infty}\frac{a_n}{n}\left(\frac{r_1}{2c}\right)^n\sum_{m=0}^{\infty}{}^nB_m\left(-\frac{r_1}{2c}\right)^m\cos m\omega.$$

From (4) or (5) with $r = r_1$,

$$V_{22} = -2Q\log r_1 + \sum_{n=1}^{\infty}\frac{a_n}{n}(-1)^n\cos n\omega.$$

From (7) and (10) with $\rho = r_1$ and $r_1 = r_3$,

$$V_{23} = +4Q\log r_3 + Q\sum_{n=1}^{\infty}\frac{b_{2n}}{2n}\left(\frac{c}{r_3}\right)^{2n}\sum_{m=0}^{2n}{}^{2n}C_m\left(\frac{r_1}{c}\right)^m\cos m\omega.$$

The total potential on conductor 2 is then,

$$\frac{V_0}{Q} = -2\log 2cr_1 + 2\sum_{n=1}^{\infty}\frac{\cos n\omega}{n}\left(-\frac{r_1}{2c}\right)^n + \sum_{n=1}^{\infty}\frac{a_n}{n}(-1)^n\cos n\omega + \sum_{m=1}^{\infty}\frac{a_m}{m}\left(\frac{r_1}{2c}\right)^m\sum_{n=0}^{\infty}{}^mB_n\left(-\frac{r_1}{2c}\right)^n\cos n\omega + 4\log r_3 + \sum_{m=1}^{\infty}\frac{b_{2m}}{2m}\left(\frac{c}{r_3}\right)^{2m}\sum_{n=0}^{2m}{}^{2m}C_n\left(\frac{r_1}{c}\right)^n\cos n\omega.$$

For $n = 0$,

$$\frac{V_0}{Q} = 2\log\frac{(r_3)^2}{2cr_1} + \sum_{m=1}^{\infty}\frac{a_m}{m}\left(\frac{r_1}{2c}\right)^m + \sum_{m=1}^{\infty}\frac{b_{2m}}{2m}\left(\frac{c}{r_3}\right)^{2m}. \tag{12}$$

Using ${}^{2m}C_n = 0$ for $n > 2m$,

$$\sum_{m=0}^{\infty}\sum_{n=0}^{2m}{}^{2m}C_nf(m, n) = \sum_{n=0}^{\infty}\sum_{2m=n}^{\infty}{}^{2m}C_nf(m, n).$$

For $n = 1, 2, 3 \cdots$

$$0 = +\frac{2}{n}\left(-\frac{r_1}{2c}\right)^n + (-1)^n\frac{a_n}{n} + \left(-\frac{r_1}{2c}\right)^n\sum_{m=1}^{\infty}{}^mB_n\left(\frac{r_1}{2c}\right)^m\frac{a_m}{m} + \left(\frac{r_1}{c}\right)^n\sum_{2m=n}^{\infty}{}^{2m}C_n\left(\frac{c}{r_3}\right)^{2m}\frac{b_{2m}}{2m}. \tag{13}$$

To obtain the final solution, substitute (11) into (12) and (13) to obtain a system of $n + 1$ homogeneous equations in $n + 1$ unknowns. The determinant of this system is then equal to zero.

$$\frac{V_0}{Q} + 2 \log \frac{2cr_1}{(r_3)^2} + 4 \sum_{k=1}^{\infty} \left(\frac{c}{r_3}\right)^{2(2k)} \frac{1}{2k} - \sum_{m=1}^{\infty} \frac{a_m}{m} \left(\frac{r_1}{c}\right)^m \alpha_m = 0.$$

$$\frac{2}{n}\left(\frac{r_1}{c}\right)^n \alpha_n + (-1)^n \frac{a_n}{n} + \left(\frac{r_1}{c}\right)^n \sum_{m=1}^{\infty} A_{mn}\left(-\frac{r_1}{c}\right)^m \left(-\frac{a_m}{m}\right) = 0,$$

where

$$\alpha_m = \left(-\frac{1}{2}\right)^m - \left(\frac{c^2}{r_3^2 - c^2}\right)^m - \left(\frac{-c^2}{r_3^2 + c^2}\right)^m,$$

$$A_{mn} = {}^mB_n\left(-\frac{1}{2}\right)^{m+n} - 2 \sum_{q}^{\infty} {}^{2q}C_n \, {}^mB_{2q-m}\left(\frac{c}{r_3}\right)^{2(2q)};$$

and

$$q = m/2 \text{ or } n/2, \quad \text{whichever is greater.}$$

Using

$$\frac{V_0}{Q} = \frac{\epsilon}{C_g},$$

and transferring to the traditional dimensional ratios, $u = r_1/2c$ and $V = c/r_3$, yields $n + 1$ equations in the variable $(-1)^m a_m/m$ so that

$$\frac{C_g}{\epsilon} = \frac{0.089532}{\log \frac{1}{4uV^2} \prod_{k=1}^{\infty} \left[\frac{1 - V^{2^{k+1}}}{1 + V^{2^{k+1}}}\right]^{(\frac{1}{2})^k} - \Delta_g} \ \mu\text{F/mile}, \tag{14}$$

where

$$\begin{vmatrix} -\Delta_g & -\frac{\alpha_1}{2}(2u) & -\frac{\alpha_2}{2}(2u)^2 & \cdots \\ \frac{2\alpha_1}{1}(2u) & 1 + A_{11}(2u)^2 & A_{21}(2u)^3 & \cdots \\ \frac{2\alpha_2}{2}(2u)^2 & A_{12}(2u)^3 & 1 + A_{22}(2u)^4 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0$$

The convergence characteristics of this solution are similar to the C_m solution with the added effect of the infinite product as $V \to 1$.

With these two exact solutions and the computer programs to calculate them, other closed-form approximate solutions can be evaluated.

III. APPROXIMATE CAPACITANCE EXPRESSIONS

Several approximate expressions (Refs. 2 through 6) using various methods and useful over limited ranges of u and V have been derived.

The Philips[3] approximate equations for C_m and C_g have several useful characteristics.

$$\frac{C_m}{\epsilon} = \frac{0.17906}{4 \cosh^{-1} \rho} \quad \frac{\mu\text{F}}{\text{mile}}, \tag{15}$$

where

$$\rho = \frac{1}{2u} \cdot \frac{1 - V^2(1 - 4u^2)}{1 + V^2(1 - 4u^2)}.$$

$$\frac{C_g}{\epsilon} = \frac{0.17906}{2 \cosh^{-1} q} \quad \frac{\mu\text{F}}{\text{mile}}, \tag{16}$$

where

$$q = \frac{1}{2u} \cdot \frac{1 - V^4(1 - 4u^2)}{4V^2}.$$

These equations are simple, easy to compute, and in closed form. Additionally, they give the values of capacitance shown in Table I at the dimensional limits. Tables II and III show that the overall accuracy of these relationships in the range of u and V corresponding to practical cables is not high.

IV. NUMERICAL TECHNIQUES

Several numerical techniques have been described to calculate capacitance by mapping equipotential lines. A grid is usually assumed to set values of x and y and known boundary values of potential are used. The value $V(x, y)$ is taken as the average of the four neighboring values, and when this is true for all points, Laplace's Equation, $\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$, has been solved. This relaxation method is used in a computer program[7] to calculate capacitance for a system of shielded circular conductors.

It is difficult when using a relaxation method to determine the accuracy of the final solution (due to the finite grid and computer errors). The program[7] was run with the proper parameters to calculate

TABLE II—PERCENT ERROR IN PHILIPS C_m EQUATION
$(C_{\text{PHILIPS}} - C_{\text{EXACT}})^* \, 100/C_{\text{EXACT}}$

u

V	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0	0	0	0	0	0	0	0	0	0	0
0.05	0.001	0.004	0.012	0.024	0.044	0.073	0.114	0.174	0.267	0
0.1	0.003	0.017	0.047	0.097	0.176	0.292	0.458	0.700	1.07	0
0.15	0.007	0.038	0.104	0.218	0.394	0.657	1.04	1.59	2.47	0
0.2	0.013	0.066	0.182	0.383	0.699	1.17	1.86	2.89	4.52	0
0.25	0.019	0.101	0.279	0.592	1.09	1.84	2.96	4.64	7.39	0
0.3	0.026	0.142	0.395	0.843	1.56	2.67	4.35	6.94	11.3	0
0.35	0.035	0.189	0.528	1.14	2.13	3.68	6.09	9.93	16.7	0
0.4	0.44	0.240	0.677	1.47	2.79	4.90	8.28	13.9	24.4	0
0.45	0.054	0.295	0.843	1.86	3.58	6.41	11.1	19.4	36.5	0
0.5	0.064	0.355	1.03	2.30	4.53	8.33	15.0	28.1	61.7	0
0.55	0.075	0.422	1.24	2.84	5.73	11.0	21.5	51.0		
0.6	0.087	0.496	1.49	3.52	7.44	15.7				
0.65	0.099	0.584	1.82	4.49	10.8					
0.7	0.114	0.695	2.28	6.5						
0.75	0.132	0.854	3.22							
0.8	0.157	1.15								
0.85	0.200									
0.9	2.49									

Note: Typical values of u and V for shielded-pair and multipair cables generally lie within the enclosed area.

C_m and C_g of a shielded pair with $u = V$ between 0.05 and 0.45 in 0.05 increments. The finest grid (33 × 33 in each quadrant) was used and Table IV shows the error in the relaxation solution. The conductor diameter for $u = V = 0.05$ did not include enough of the grid to compute. Unlike the approximate equations, the errors in the relaxation method, for the most part, are not a strong function of u and V. For most values of u and V ($u = V > 0.2$) the relaxation method is more accurate than the Philips equations.

V. CAPACITANCE SURFACES

The complexity of the exact formulae for calculating the capacitance of a shielded balanced pair structure causes difficulty in visualizing how capacitance is affected by changes in dimensions. The Philip's

TABLE III—PERCENT ERROR IN PHILIPS C_g EQUATION $(C_{PHILIPS} - C_{EXACT})* 100/C_{EXACT}$

	u									
V	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0.05	−0.033	−0.144	−0.342	−0.630	−1.01	−1.47	−2.01	−2.62	−3.27	0
0.1	−0.040	−0.180	−0.435	−0.811	−1.31	−1.93	−2.66	−3.49	−4.40	0
0.15	−0.046	−0.211	−0.517	−0.974	−1.59	−2.36	−3.29	−4.34	−5.50	0
0.2	−0.052	−0.241	−0.596	−1.14	−1.87	−2.81	−3.94	−5.24	−6.68	0
0.25	−0.057	−0.270	−0.677	−1.31	−2.17	−3.28	−4.64	−6.22	−7.99	0
0.3	−0.062	−0.300	−0.762	−1.49	−2.50	−3.81	−5.42	−7.32	−9.47	0
0.35	−0.067	−0.330	−0.853	−1.68	−2.85	−4.40	−6.31	−8.58	−11.2	0
0.4	−0.073	−0.364	−0.952	−1.9	−3.26	−5.07	−7.35	−10.1	−13.2	0
0.45	−0.079	−0.400	−1.06	−2.15	−3.73	−5.87	−8.59	−11.9	−15.9	0
0.5	−0.085	−0.440	−1.19	−2.44	−4.30	−6.86	−10.2	−14.4	−19.8	0
0.55	−0.092	−0.487	−1.34	−0.280	−5.03	−8.20	−12.6	−19.1		
0.6	−0.099	−0.542	−1.53	−3.27	−6.06	−10.4				
0.65	−0.109	−0.611	−1.77	−3.96	−8.00					
0.7	−0.120	−0.703	−2.15	−5.38						
0.75	−0.135	−0.841	−2.91							
0.8	−0.159	−1.11								
0.85	−0.199									
0.9	−2.08									

Note: Typical values of u and V for shielded-pair and multipair cables generally lie within the enclosed area.

TABLE IV—PERCENT ERROR USING A RELAXATION METHOD

(C_m, μF/mile)

u	V	C_{EXACT}	C_{RELAX}	Percent Error
0.05	0.05	0.01498054	0	−100.
0.1	0.1	0.01969278	0.01985524	0.825
0.15	0.15	0.02442631	0.02445576	0.121
0.2	0.2	0.02987059	0.02984397	−0.089
0.25	0.25	0.03669688	0.03665908	−0.103
0.3	0.3	0.04594544	0.04589750	−0.104
0.4	0.4	0.08314346	0.08311094	−0.039
0.45	0.45	0.13697544	0.13692383	−0.038

(C_g, μF/mile)

u	V	C_{EXACT}	C_{RELAX}	Percent Error
0.05	0.05	0.01178299	0	−100.
0.1	0.1	0.01624484	0.01363114	−16.1
0.15	0.15	0.02090778	0.02052972	−1.81
0.2	0.2	0.02632710	0.02623542	−0.348
0.25	0.25	0.03308222	0.03298556	−0.292
0.3	0.3	0.04210923	0.04203434	−0.178
0.35	0.35	0.05529502	0.05520572	−0.162
0.4	0.4	0.07744843	0.07732726	−0.156
0.45	0.45	0.12737824	0.12717648	−0.158

equations provide simple functions which can be plotted to give a qualitative picture of a capacitance surface. Computer plotting of a function of two variables[8] has been described and will be used to display capacitance surfaces along with scales, coordinate axes and a "cube of reference."

Figures 5, 6, and 7 are plots of capacitance surfaces as functions of d/D and S/D. The dimensional restrictions with these ratios, map into the region shown in Fig. 4.

These plots in the interval $0.05 \leqq d/D \leqq 0.3$ and $0.35 \leqq S/D \leqq 0.65$ can be interpreted as capacitance surfaces normalized to some constant outer conductor diameter (say $D = 1$). Figure 5 shows C_{12}/ϵ to be a monotonically increasing function of d/D and a monotonically decreasing function of S/D with a maximum slope as the inner conductors approach each other. Figure 6 shows C_g/ϵ is an increasing function of both d/D and S/D with a maximum slope as the inner conductors approach the shield. C_m/ϵ, plotted in Fig. 7, is seen to contain the sum of previous effects with approximately equal weighting.

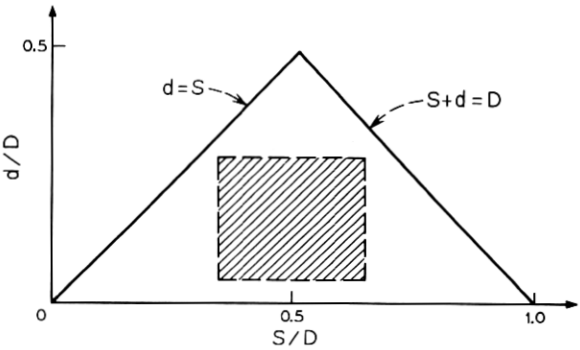

Fig. 4—Dimensional limits as functions of d/D and S/D. The shaded area defines the plotting interval in Figs. 5, 6, and 7.

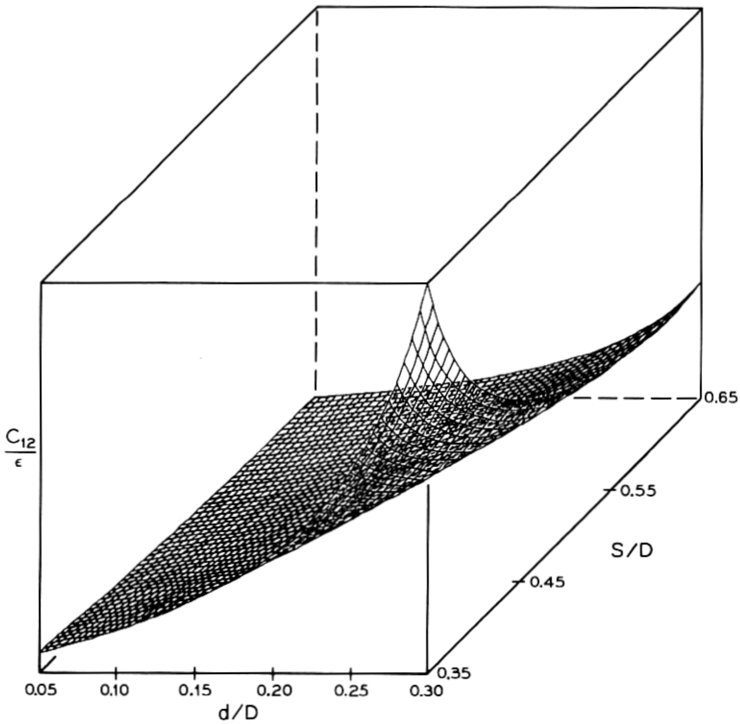

Fig. 5—C_{12}/ϵ as a function of d/D and S/D.

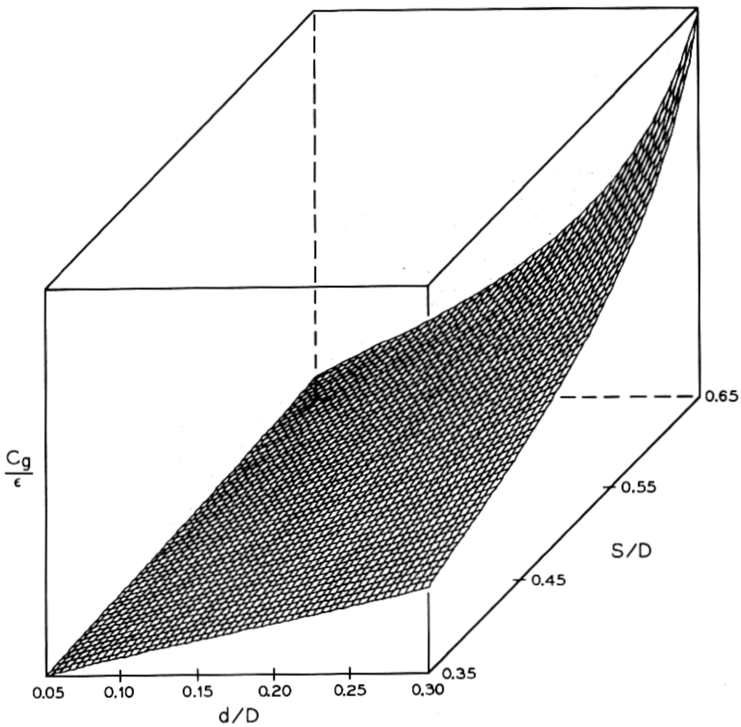

Fig. 6—C_g/ϵ as a function of d/D and S/D.

The previous three plots show qualitatively how C_m and its components vary with dimensions. They also reinforce the previous intuitive statements made regarding behavior at dimensional limits. Thus, for a given d/D, C_g/ϵ is seen to increase as S/D increases (wires approach shield) whereas C_{12}/ϵ is seen to increase as S/D decreases (wires approach each other). C_m/ϵ, the functional sum of C_g/ϵ and C_{12}/ϵ, therefore has a minimum with respect to S/D. The position of this minimum is only a slight function of d/D; for $0.05 \leqq d/D \leqq 0.3$, C_m/ϵ is minimum for $0.48 \leqq S/D \leqq 0.5$.

VI. INVERSE PLOTS OF PHILIPS EQUATION

With the Philips equations, values of u and V can be obtained for a given C_m/ϵ and C_g/ϵ (as could be measured in a cable). Thus from equations (15) and (16),

$$\rho + \sqrt{\rho^2 - 1} = \exp \frac{0.17906\epsilon}{4C_m} \tag{17}$$

$$q + \sqrt{q^2 - 1} = \exp \frac{0.17906\epsilon}{2C_g} \tag{18}$$

using $\cosh^{-1} a = \log_e (a + \sqrt{a^2 - 1})$ for $a \geqq 1$.

Solving (17) and (18) iteratively for ρ and q yields two simultaneous iterative equations in two unknowns.

$$\rho = \frac{1}{2u} \cdot \frac{1 - V^2(1 - 4u^2)}{1 + V^2(1 - 4u^2)}. \tag{19}$$

$$q = \frac{1}{2u} \cdot \frac{1 - V^2(1 - 4u^2)}{4V^2}. \tag{20}$$

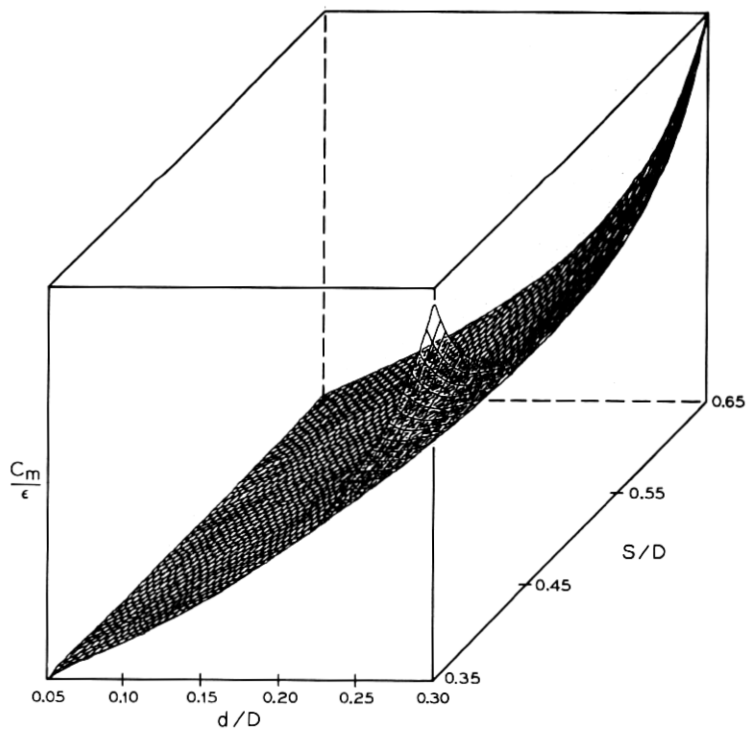

Fig. 7—C_m/ϵ as a function of d/D and S/D.

Solving (19) for V^2 and substituting in (20) yields,

$$V^2 = \frac{1 - 2u\rho}{1 - 4u^2 + 2u\rho - 8u^3\rho} \tag{21}$$

and

$$\frac{8u\rho(1 - 2u\rho)}{1 - 4u^2 + 2u\rho - 8u^3\rho} = 1 - \frac{(1 - 2u\rho)^2(1 - 4u^2)^2}{(1 - 4u^2 + 2u\rho - 8u^3\rho)^2}. \tag{22}$$

Equation (22) can now be iteratively solved for u and substituted in (21) to get V. The variables u and V are again inconvenient from the standpoint of intuitive visualization. In the manufacture of a shielded balanced-pair cable, D and S are difficult to control; however, d can be accurately controlled. Even if d varies down the length of the cable this can be related to die wear or to elongation, which can be modeled. Normalizing S and D with respect to d yields,

$$\frac{S}{d} = \frac{1}{2u} \tag{23}$$

and

$$\frac{D}{d} = \frac{1}{2uV}. \tag{24}$$

Figures 8 and 9 are surfaces of S/d and D/d as functions of C_m/ϵ and C_g/ϵ in the following regions.

$$0.02 \leqq C_m/\epsilon \leqq 0.07$$

$$0.015 \leqq C_g/\epsilon \leqq 0.035.$$

No attempt is made to define the bounding values for C_m/ϵ and C_g/ϵ; however, the above represents one of the largest rectangular regions the author could find by trial and error. These surfaces have the expected general shape ($S/d \to \infty$ as $C_m/\epsilon \to C_g/2\epsilon$ and $D/d \to \infty$ as $C_g/\epsilon \to 0$).

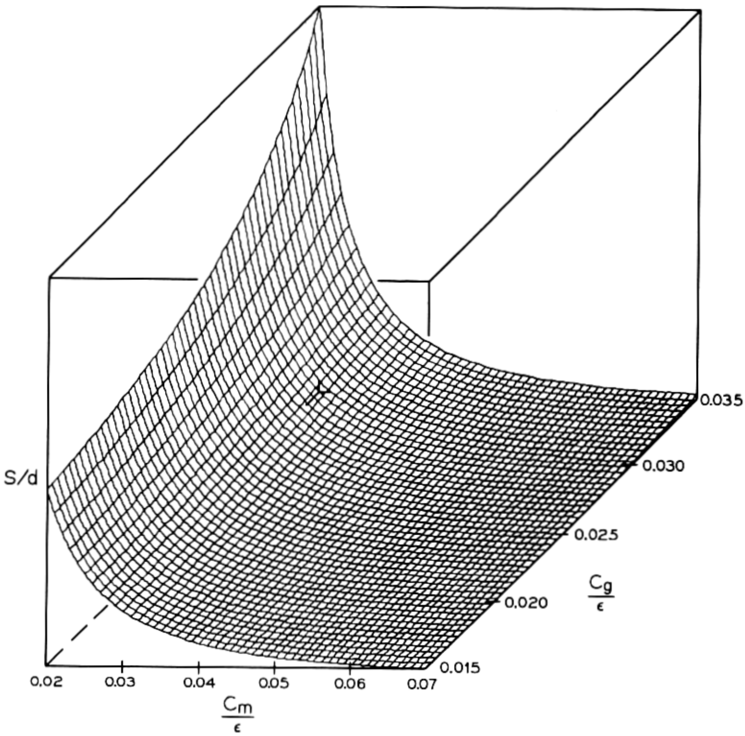

Fig. 8—S/d as a function of C_m/ϵ and C_g/ϵ.

VII. CONCLUSIONS

The exact expressions for C_m and C_g must be used if a high degree of accuracy is desired. As the conductors approach each other or the shield, the convergence of the exact solutions is slower and the accuracy of approximate expressions is less. For large values of u and V, the relaxation method yields a more accurate result than the Philips approximate equations; however, the error is difficult to ascertain. Capacitance surfaces using the Philips relationships show a great deal about the manner in which capacitance varies with dimensions and enables inverse plots of dimensions versus capacitance.

VIII. ACKNOWLEDGMENT

The author wishes to thank H. P-H Yuen for his help in deriving the exact expression for C_g .

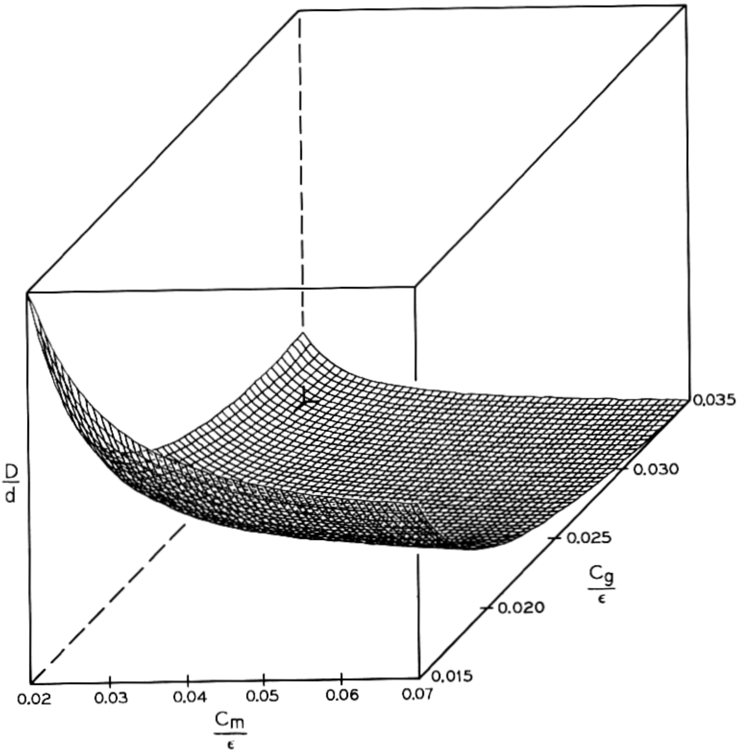

Fig. 9—D/d as a function of C_m/ϵ and C_g/ϵ.

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