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Stochastic Stability of Delta Modulation

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The discrete-time model of delta modulation is considered for a stationary random input process with a rational spectral density, and an autocovariance that goes to zero as the lag approaches infinity. For leaky integration, the joint distribution of input and decoded approximation processes is shown to approach a unique stationary distribution from any initial condition. Under the stationary distribution, the decoded process may take on all values in a bounded interval that is independent of the input process. For the often-studied ideal integration model of delta modulation, it is shown that the successive distributions at even parity time instants converge to a limiting stationary distribution, while at odd parity time instants the distributions converge to a different limiting distribution. Under these limiting distributions, the decoded process is assigned a positive probability for each level of a (discrete) lattice of amplitudes. The mean-absolute approximation error and mean-absolute amplitude of the decoded process are shown to be finite under the limiting distributions. For both ideal and leaky integration cases, an explicit upper bound on mean-absolute approximation error is given, which is independent of the spectral density of the input process.

I. INTRODUCTION

In spite of the great simplicity of delta modulation as an analog-to-digital encoding technique, it has not yet succumbed to an adequate mathematical analysis. Although realistic inputs such as speech are extremely difficult to characterize, considerable insight could be obtained from a thorough analysis for the case of a stationary random input process with a prescribed spectral density. Yet no such results have been obtained because of the mathematical complexity of the nonlinear feedback loop. In fact, the presence of a feedback loop raises the possibility that instability in some sense could arise. The possibility that the decoded signal could "run away" or become unbounded, failing to track the original signal, has never been theoretically excluded.

Although experience with delta modulation shows that such an extreme form of instability never arises, it has never been shown analytically that the mean-square or mean-absolute quantizing noise has a finite upper bound. Another possibility which has never been theoretically excluded is erratic operation, where the statistical average of the quantizing noise magnitude continues to vary with time. In other words, although the input process is stationary, the decoded approximation process would be nonstationary with a time-varying probability distribution even after low-pass filtering. If this were the case, the decoded process would not be replicating the original process very effectively.

Recently, D. Slepian[1] has developed an exact computational approach for finding the joint probability distribution of the original and encoded processes. These results make it possible to accurately compute such curves as the mean-square quantizing noise versus step size for particular spectral densities of the input process. Slepian's results are based on the initial assumption that, for a stationary input process with rational spectral density, the joint probability distribution will approach a unique stationary distribution from any starting condition. (For delta modulation with ideal integration, the stationary distribution actually refers to half the sum of the distributions at two successive time instants to account for the well-known parity change between even and odd amplitude levels.) On a practical level, this stationarity assumption seems to be entirely reasonable; yet it has never been theoretically justified.

Other authors have also assumed stationarity. In particular, for ideal integration H. van de Weg[2] assumed implicitly that the decoded process had two different stationary distributions for the even and odd parity time instants. D. J. Goodman[3] assumed a random phase initial condition so that only one stationary distribution, half the sum of the even and odd parity distributions, need be considered. In both cases the assumption is implicitly made that for any initial condition the delta modulation process will approach a steady-state mode of operation with a separate stationary distribution for even and odd time instants.

As a final argument to point out the need for an analysis of stochastic stability properties of delta modulation, consider the fact that most heuristic and semianalytical approximate considerations of delta modulation are based on the model of an ideal integrator in the feedback path, while most physical realizations involve a leaky integrator. There is a basic qualitative difference between these two cases, even for extremely wide-band integrators. This is because ideal integration gives equal weight to a current input sample and an arbitrarily remote past input sample, while leaky integration forgets remote past samples. Hence, it is not clear that the ideal integration model is meaningful, even if it is known that the leaky integration system is well behaved. To justify the validity of using an ideal integration model, a rigorous demonstration of the stability of this model is needed.

This paper demonstrates the stochastic stability of delta modulation for both ideal and leaky integration by giving a mathematical proof that the joint distribution of the input and decoded processes approaches a unique stationary distribution from an arbitrary starting point and by deriving an explicit, finite upper bound on the mean-absolute quantizing noise. The input process is assumed to be stationary and continuously distributed, with finite variance, a rational spectral density, and an autocovariance that approaches zero as the lag goes to infinity. A shaping filter with white noise input is assumed as the generating mechanism of the input process.

II. PROBLEM FORMULATION AND SUMMARY OF RESULTS

For most purposes the following discrete-time model of delta modulation is an acceptable description of the actual continuous-time operation.[4] Let u_t denote the sampled analog values of the *input* process at successive time instants $t = 0, 1, 2, \cdots$. The time scale is normalized for convenience without loss of generality. The delta modulator shown in Fig. 1 generates a binary-valued process b_k according to

$$b_k = \text{sgn}\,(u_k - x_k) \tag{1}$$

where $\text{sgn}\, y = +1$ if $y \geqq 0$ and -1 if $y < 0$, and x_k is the *decoded process* which approximates u_k . The decoded process x_k is given recursively by

$$x_{k+1} = x_k + \Delta\, \text{sgn}\,(u_k - x_k) \tag{2}$$

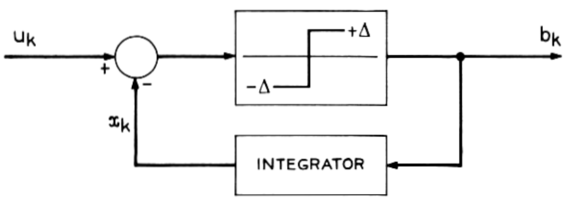

Fig. 1—Delta modulator.

for the *ideal integration* case, and by

$$x_{k+1} = \alpha x_k + \Delta \operatorname{sgn} (u_k - x_k) \tag{3}$$

for the *leaky integration* case with a simple RC integrator, where $\Delta > 0$ is the *step size* and $\frac{1}{2} \leqq \alpha < 1$. (In practice, α is very close to one because the integrator time constant is much longer than the sampling period.) The *quantizing noise,*

$$e_k = u_k - x_k ,$$

is the error at time k due to the analog-to-digital-to-analog processing of the delta modulation system.

Assume the input process u_k is stationary, continuously distributed with finite variance, and has autocovariance approaching zero as the time lag goes to infinity. For convenience, assume also that the probability density of u_k is everywhere positive so that, as in the Gaussian case, there is a positive probability of u_k lying in any open interval. Note that u_k is not assumed to have zero mean value.

A summary of the results obtained in this paper follows. Let $\hat{F}_k$ denote the joint distribution of the input and decoded processes u_k and x_k at time k.

2.1 *Ideal Integration*

(*i*) For any initial condition of the form $x_1 = m\Delta + \theta$ with m an integer and $|\theta| \leqq \Delta/2$, the two sequences of distributions $\{\hat{F}_{2k}\}$ and $\{\hat{F}_{2k+1}\}$ separately converge to unique stationary distributions $\hat{G}_0$ and $\hat{G}_1$, respectively. One distribution assigns positive probability for the process $x_k - \theta$ to even integer multiples of Δ, the other distribution to odd integer multiples, depending on the even-odd parity of the initial integer m.

(*ii*) With these stationary distributions the mean-absolute quantizing noise, averaged over two successive time instants, is bounded according to

$$\tfrac{1}{2}[E\,|x_k - u_k| + E\,|x_{k+1} - u_{k+1}|] \leqq E\,|u_i| + \Delta/2 + 2\,|\theta|. \tag{4}$$

2.2 *Leaky Integration*

(*i*) For any initial value of x_1, the distributions $\{\hat{F}_k\}$ converge to the unique stationary distribution $\hat{G}$, under which x_k may take on all values in the range

$$|x_k| < \frac{\Delta}{1 - \alpha}. \tag{5}$$

(*ii*) Under this stationary distribution, the mean-absolute quantizing noise is bounded according to:

$$E \mid x_k - u_k \mid \leqq E \mid u_i \mid + \Delta/2\alpha. \tag{6}$$

Note the qualitative difference between the leaky and ideal integration cases. In the leaky case, x_k is distributed over a finite interval; in the ideal case, x_k is discretely distributed on a lattice.

III. MARKOVIAN MODELING

Since the input process u_k is stationary with finite variance and rational spectrum, then $\tilde{u}_k = u_k - \mu$ (where μ is the mean value of u_k) can be modeled[5] as the response of a stable discrete-time shaping filter to a zero mean, finite variance "white noise" process w_k (with w_k independent of w_{k-i} for $i = 1, 2, 3, \cdots$). More precisely, a white process w_k and a stable rational shaping filter $H(z)$ can always be specified in such a way that the response $\hat{u}_k$ of the filter to the excitation w_k will be a stationary random process with spectral density identical to that of $\tilde{u}_k$. If w_k is also chosen to be continuously distributed with a positive density, then $\hat{u}_k$ will also satisfy this property. Thus, all the assumptions made in Section II about the process u_k are possessed by the process $\hat{u}_k + \mu$. It is therefore reasonable to study the effect of the delta modulation system for the input $\hat{u}_k + \mu$, whose structure or generating mechanism is known. For the remainder of this paper, no distinction will be made between $\tilde{u}_k$ and $\hat{u}_k$.

Using this model and the assumption that the autocovariance of u_k goes to zero as the lag approaches infinity, Appendix A shows that u_k can be imbedded in a vector Markov process, $\mathbf{d}_k$, with

$$\mathbf{d}_k = (d_{k1}, d_{k2}, \cdots, d_{kn})'$$

and

$$u_k = d_{k1} + \mu \tag{7}$$

where μ denotes the dc value of the input process and n is the number of poles in the shaping filter. The vector $\mathbf{d}_k$ characterizes the state of the filter at time k and is generated by the recursion

$$\mathbf{d}_{k+1} = A\mathbf{d}_k + \mathbf{b}w_k \tag{8}$$

where A is an $n \times n$ matrix with eigenvalues within the unit circle, and $\mathbf{b}$ is a fixed vector. The process $\mathbf{d}_k$ is Markovian, since the conditional distribution of $\mathbf{d}_{k+1}$ given all past states $\mathbf{d}_k, \mathbf{d}_{k-1}, \cdots$, depends only on the given value of $\mathbf{d}_k$. Appendix B shows that for any initial state $\mathbf{d}_0$, the distribution of $\mathbf{d}_k$ approaches a unique stationary distribution. Equations (2), (7), and (8) for the ideal integrator case, or eqs. (3), (7), and (8) for the leaky integrator case, jointly characterize the evolution in time of a Markovian process whose state $\mathbf{s}_k$ at time k is given by the $n + 1$ component vector

$$\mathbf{s}_k = (x_k , u_k , d_{k2} , d_{k3} , \cdots , d_{kn})'.$$

Then, given the value of $\mathbf{s}_k$, the distribution of $\mathbf{s}_{k+1}$ is completely determined. Henceforth, a *distribution* F_k , describing the joint distribution of the $n + 1$ components of the vector $\mathbf{s}_k$, will be regarded as a set function which assigns a probability $F_k(A)$ to the region A of the $n + 1$ dimensional space of possible values of the state vector $\mathbf{s}_k$. The *probability transition function*[6] characterizing the Markov process is defined as

$$p(\mathbf{s}, A) = P\{\mathbf{s}_{k+1} \in A \mid \mathbf{s}_k = \mathbf{s}\}$$

which is independent of k. By averaging this conditional probability over a distribution F_k assigned to $\mathbf{s}_k$, the unconditioned distribution F_{k+1} of $\mathbf{s}_{k+1}$ is obtained:

$$F_{k+1}(A) = \int p(\mathbf{s}, A)F_k(d\mathbf{s}) = E_{F_k}p(\mathbf{s}, A). \tag{9}$$

Thus, F_{k+1} is related to F_k by a linear mapping T, so that in operator notation

$$F_{k+1} = TF_k . \tag{10}$$

Note that T plays the same role as the probability transition matrix in Markov chains. The process has a *stationary distribution* G, if $G = TG$, so that G is self-reproducing. If any state vector has distribution G, all subsequent state vectors will have this distribution. The existence of a stationary distribution is a necessary but not sufficient condition for the convergence of the distributions F_{k+1} to a limiting distribution.

The Markovian model will be used in Sections V and VI to obtain the convergence properties of the distributions $\{F_k\}$. But first, it is necessary to obtain a bound on the time and ensemble average of the quantizing noise.

IV. BOUNDING THE TIME-AVERAGED MEAN-ABSOLUTE QUANTIZING NOISE

Suppose that at the initial time instant $k = 1$, the system has an arbitrary initial state $\mathbf{s}_1$. Appendix B shows that for any initial state

$\mathbf{d}_1$ of the shaping filter, the process $\mathbf{d}_k$, and hence u_k, converge in distribution and $E \mid u_k \mid$ converges to c, the mean-absolute value under the stationary distribution of the process u_t.

Squaring both sides of eq. (3) gives

$$\begin{aligned} x_{k+1}^2 &= \alpha^2 x_k^2 - 2\alpha\Delta \mid u_k - x_k \mid + 2\alpha\Delta u_k \operatorname{sgn}(u_k - x_k) + \Delta^2 \\ &\leqq x_k^2 - 2\alpha\Delta \mid u_k - x_k \mid + 2\alpha\Delta \mid u_k \mid + \Delta^2. \end{aligned}$$

Taking the expected value of both sides yields

$$Ex_{k+1}^2 \leqq Ex_k^2 - 2\alpha\Delta E \mid e_k \mid + 2\alpha\Delta E \mid u_k \mid + \Delta^2$$

and iterating backwards gives

$$Ex_{k+1}^2 \leqq x_1^2 + 2\alpha\Delta \sum_{i=1}^{k} (E \mid u_i \mid - E \mid e_i \mid) + \Delta^2 k.$$

Since the left side is nonnegative, it follows that

$$\frac{1}{k} \sum_{i=1}^{k} E \mid e_i \mid \leqq x_1^2/2\alpha\Delta k + \frac{1}{k} \sum_{i=1}^{k} E \mid u_i \mid + \Delta/2\alpha. \tag{11}$$

Hence, using the fact that $E \mid u_i \mid \to c$,

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} E \mid e_i \mid \leqq c + \Delta/2\alpha, \tag{12}$$

which shows that the long-term time average of the mean-absolute quantizing error is bounded. Since the preceding derivation holds for $\alpha \leqq 1$, it applies for both leaky and ideal integration, setting $\alpha = 1$ for ideal integration.

For leaky integration, the decoded process is, in fact, bounded deterministically. Integrating eq. (3) backwards yields

$$x_{k+1} = \alpha^k x_1 + \Delta \sum_{i=1}^{k} \alpha^{k-i} b_i \tag{13}$$

so that

$$\mid x_{k+1} \mid \leqq \alpha^k \mid x_1 \mid + \frac{\Delta}{1-\alpha}(1 - \alpha^k). \tag{14}$$

Hence,

$$\limsup_{k\to\infty} \mid x_k \mid \leqq \frac{\Delta}{1-\alpha}. \tag{15}$$

Thus, the decoded process is bounded with probability one in the case of leaky integration.

V. EXISTENCE OF A STATIONARY DISTRIBUTION

A sequence of random vectors and the corresponding sequence of distributions are said to be *stochastically bounded* if, for any probability ϵ, however small, there is a sufficiently large distance R such that each random vector of the sequence has probability less than ϵ of having length greater than R. Hence, the successive vectors cannot have a positive probability of moving out toward infinity.

For a sequence of distributions $\{F_k\}$, define the associated sequence of *averaged distribution* $\{G_k\}$ by

$$G_k(A) = \frac{1}{k} \sum_{i=1}^{k} F_i(A). \tag{16}$$

Thus, if I is a randomly selected time instant from the first k integers each having equal probability, then $G_k(A)$ is the probability that the random vector $\mathbf{s}_I$ lies in a region A, where $\mathbf{s}_i$ has distribution F_i. If the sequence $\{F_i\}$ is stochastically bounded, then the averaged distributions $\{G_k\}$ are also stochastically bounded; however, the converse is not always true.

To show that the Markov process $\mathbf{s}_k$ defined in Section III has a stationary distribution G, the following theorem, proved in Appendix C, may be used.

Theorem: A Markov process has a stationary distribution if

(*i*) for any initial state $\mathbf{s}_1$, the averaged distributions G_k are stochastically bounded, and

(*ii*) for any region A, let D be the set of points $\mathbf{s}$ at which the transition probability function $p(\mathbf{s}, A)$ is discontinuous and let N_δ be the set of all points whose distance from D is less than δ; then there is a function $C(\delta)$ independent of $\mathbf{s}$ with $C(\delta) \rightarrow 0$ as $\delta \rightarrow 0$ and for all $\mathbf{s}$,

$$p(\mathbf{s}, N_\delta) \leqq C(\delta). \tag{17}$$

Condition (*i*) excludes the possibility that successive state vectors can move out toward infinity. Condition (*ii*) is concerned with the region of state space where an arbitrarily small perturbation of a given state vector can cause a substantial change in the induced distribution of the state vector at the next time instant. This region is contained within the set N_δ for each $\delta > 0$. The condition requires that this region be suitably unimportant.

To show that the delta modulation process satisfies condition (*i*), observe that

$$E \mid x_i \mid \leqq E \mid u_i - x_i \mid + E \mid u_i \mid$$

and since $E \mid u_i \mid$ is bounded, eq. (11) shows that for some constant K,

$$\frac{1}{k} \sum_{i=1}^{k} E \mid x_i \mid < \mathrm{K}. \tag{18}$$

Now Chebyshev's inequality,

$$MP \{\mid x_i \mid > M\} \leqq E \mid x_i \mid,$$

applied to eq. (18) yields

$$\frac{1}{k} \sum_{i=1}^{k} P\{\mid x_i \mid > M\} \leqq \mathrm{K}/M$$

for every $M > 0$, which shows that the averaged distributions of the decoded process x_k are stochastically bounded. But from Appendix B, the $\mathbf{d}_k$ process is stochastically bounded. Hence, the averaged distributions of $\mathbf{d}_k$ are also stochastically bounded. Thus, the marginal distributions of the joint distributions G_n are stochastically bounded so that G_n is itself a stochastically bounded sequence, and condition (*i*) holds for the ideal integration case. For the leaky integration case, condition (*i*) is satisfied since eq. (15) shows that the x components of the vectors $\mathbf{s}_k$ are uniformly bounded with probability one, so that the above argument shows that the joint distributions G_n are stochastically bounded.

To verify condition (*ii*), note from eqs. (2) or (3) that for any region A, $p(\mathbf{s}, A)$ is continuous, except in the set D of all points $\mathbf{s}$ the first two components of which, x and u, are equal. Appendix D shows that given $\mathbf{s}_k$, the variate u_{k+1} is continuously distributed and that this implies that there exists a function $C(\delta)$ which goes to zero as δ approaches zero and

$$P\{\mid x - u_k \mid \mid \mathbf{s}_k\} \leqq C(\delta) \tag{19}$$

where $C(\delta)$ is independent of x and $\mathbf{s}_k$. But since x_{k+1} is completely determined by $\mathbf{s}_k$, (19) implies that

$$P\{\mid x_{k+1} - u_{k+1} \mid < \delta \mid \mathbf{s}_k\} \leqq C(\delta). \tag{20}$$

Hence, eq. (17) is satisfied and condition (*ii*) holds for both ideal and leaky integration. Therefore, a stationary distribution exists.

VI. ALLOWABLE AMPLITUDE VALUES FOR THE DECODED PROCESS

For ideal integration, it is clear from eq. (2) that an initial condition of the form

$$x_1 = m\Delta + \theta$$

with m an integer and $|\theta| < \Delta$, implies that all subsequent amplitude values of the decoded process will be confined to the lattice

$$x_k = l\Delta + \theta \qquad l = 0, \pm 1, \pm 2, \cdots .$$

Since the preceding results did not specify a particular choice of initial condition, it follows that for each θ, a stationary distribution G_θ exists. For convenience, assume $\theta = 0$. No loss of generality will result because, for any θ, the problem can be converted to the $\theta = 0$ case by replacing the input process u_k by $u_k - \theta$ as can be seen from eq. (2). This simply changes the dc value of the input by, at most, one step size Δ.

Under the assumption that u_k has a positive density, it follows from eq. (2) that there is always a positive probability of either increasing or decreasing by Δ in going from x_k to x_{k+1}. This means that *every* integer multiple of Δ must have a positive probability under the stationary distribution, because each level can always be reached from any other level in a finite number of steps. (On the other hand, if u_k were bounded, then eq. (2) shows that the decoded process would get locked into a bounded set of levels from which it would never escape.)

For leaky integration, the situation is strikingly different. The decoded process will get locked into the bounded region

$$X = \{x: -\Delta/(1 - \alpha) \leqq x \leqq \Delta/(1 - \alpha)\}. \tag{21}$$

This may be seen by noting from eq. (3) that if x_k is in X, then x_{k+1} must also be in X. Consequently, once x_i is in X, it will never escape. If x_k is not in X, then eq. (3) shows that

$$|x_{k+1}| < |x_k|$$

and if b_k has the correct polarity,

$$|x_{k+1}| < \alpha |x_k| . \tag{22}$$

Hence the values $|x_{k+i}|$ must decrease monotonically as long as x_{k+i} remains outside of X. Since u_k has a positive density, b_i must have positive probability of having either polarity; which means the stronger inequality eq. (22) must hold at some subsequent time instants. Hence, the process must eventually enter the region X.

Iterating eq. (3) backwards yields

$$x_{k+1} = \alpha^k x_1 + \Delta \sum_{i=0}^{k-1} \alpha^i b_{k-i} . \tag{23}$$

This shows that the initial value x_1 is gradually forgotten and the set of allowable values for x_k as k goes to infinity, approaches the set

$$W = \{x: x = \Delta(\pm 1 \pm \alpha \pm \alpha^2 \pm \cdots)\} \tag{24}$$

where all possible sequences of polarities are used to generate values of x. Appendix E proves that, for $\alpha \geqq \frac{1}{2}$, the set W *coincides* with the interval X.*

For the leaky integration case, the stationary distribution G clearly must confine the x component of the state vector to the region X. Furthermore, the following argument shows that G assigns a positive probability to every open subinterval $(y - \epsilon, y + \epsilon)$ with y in X and $\epsilon > 0$. Let $\{c_i\}$ be a suitable binary sequence (generated as in Appendix E) satisfying

$$y = \Delta \sum_{i=0}^{\infty} c_i \alpha^i. \tag{25}$$

Pick the integer N large enough so that

$$\alpha^{N+1} x_1 < \epsilon/3, \quad \text{and} \quad \alpha^N \Delta/(1 - \alpha) < \epsilon/3, \tag{26}$$

and consider the event

$$E = \{b_2 = c_{N-1}, b_3 = c_{N-2}, \cdots, b_{N+1} = c_0\},$$

which has a positive probability for any initial state $\mathbf{s}_1$. Then for $k = N + 1$, eq. (23) can be written in the form

$$x_{N+2} - y = \alpha^{N+1} x_1 + \Delta \alpha^N b_1 - \Delta \sum_{i=N}^{\infty} \alpha^i c_i$$

so that, using eq. (26),

$$| x_{N+2} - y | \leqq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

for x_1 in X. Hence,

$$P\{| x_{N+2} - y | < \epsilon \mid \mathbf{s}_1\} \geqq P\{E \mid s_1\} \tag{27}$$

and averaging eq. (27) over the distribution G for the state $\mathbf{s}_1$ shows that

$$P\{| x_{N+2} - y | < \epsilon\} > 0 \tag{28}$$

where x_{N+2} has the marginal distribution of the first component of G.

VII. CONVERGENCE TO THE LIMITING DISTRIBUTIONS

The following specialized and paraphrased version of a theorem due to J. L. Doob[7] is suited to the delta modulation process:

* For $\alpha < \frac{1}{2}$, it can be shown that W does not coincide with X, in fact W is not even dense in X.

Doob's Theorem: Suppose the Markov process has a stationary distribution G and satisfies the conditions:

(*i*) *For any initial state $\mathbf{s}_1$, if A is a region with $G(A) = 0$, then $F_k(A) \to 0$.*

(*ii*) *If a region A satisfies $p(\mathbf{s}, A) = 1$ for $\mathbf{s}$ in A, then $G(A) = 1$.*

Then either

$$F_k \to G \quad \text{for any initial state}$$

in which case the process is aperiodic, or there exist m disjoint sets $A_0, A_1, \cdots, A_{m-1}$ with $G(\bigcup_0^m A_i) = 1$ and $p(\mathbf{s}, A_{i+1}) = 1$ if $\mathbf{s}$ is in A_i for $i = 0, 1, \cdots, m-2$ and if $\mathbf{s}$ is in A_{m-1}, then $p(\mathbf{s}, A_0) = 1$. In this case, the process is said to be periodic with period m. In particular for $m = 2$, there exist two distributions G_0 and G_1 with $G = \frac{1}{2}(G_0 + G_1)$, $G_1(A_1) = 1$, $G_0(A_0) = 1$, and for any initial state in A_1, $F_{2k+1} \to G_1$ and $F_{2k} \to G_0$, while for any initial state in A_0, $F_{2k+1} \to G_0$ and $F_{2k} \to G_1$.

For ideal integration, Section VI shows that there are no transient levels for the x_k process and Appendix A shows that d_k has a positive probability of lying in any region of n space with nonzero volume. Hence, there is no transient set A with $G(A) = 0$ except for trivial sets with $F_k(A) = 0$ for each k. For leaky integration the only nontrivial sets A are regions where the x component lies outside of X and for such regions, Section VI shows that $F_k(A) \to 0$. Therefore, condition (*i*) is satisfied for both types of integration.

Furthermore, the ergodicity requirement (*ii*) is also seen to be satisfied for both ideal and leaky integration from the results of Section VI.

For ideal integration the process clearly has period 2, since, if A_0 is the set of all state vectors with x components taking on even integer multiples of Δ, and A_1 is the complementary set, then $A_0 \cup A_1$ has probability one under the limiting distribution and the transition probability function has the requisite property implying the state vector alternates between A_0 and A_1. For leaky integration the process is aperiodic since eq. (27) holds for all N sufficiently large so the process cannot satisfy the requirements for periodicity, hence $F_k \to G$. Since a sequence of distributions cannot at the same time converge to two different distributions, it follows that the stationary distribution G is unique for both ideal and leaky integration.

VIII. BOUNDING STEADY-STATE QUANTIZING NOISE

The fact that a sequence of distributions converges to a limiting distribution does not imply that moment functions such as mean or variance converge to the corresponding moment of the limiting distribution. However, it does imply that the sequence of expectations of a bounded continuous function converge to the expectation of that function under the limiting distribution. It turns out that this property is sufficient to obtain a bound on the least mean-absolute quantizing error under the stationary distributions.

For ideal integration, eq. (12) can be rewritten in the form

$$\limsup_{m\to\infty} \frac{1}{2m} \sum_{i=1}^{m} E(|e_{2k}| + |e_{2k+1}|) \leqq \mathrm{K} \tag{29}$$

where $\mathrm{K} = c + \Delta/2$, which implies the existence of an even subsequence of time instants t_i $(= 2k_i)$ with

$$\tfrac{1}{2}E(|e_{t_i}| + |e_{t_i+1}|) \leqq \mathrm{K}. \tag{30}$$

Now define the truncating function $J_R(e)$ according to

$$J_R(e) = \begin{cases} 1 & |e| \leqq R \\ 1 - (|e| - R)/\delta & R < |e| < R + \delta. \\ 0 & R + \delta \leqq |e| \end{cases}$$

Then,

$$E(|e_{t_i}| + |e_{t_i+1}|) \geqq E\{|e_{t_i}| J_R(e_{t_i}) + |e_{t_i+1}| J_R(e_{t_i+1})\} \tag{31}$$

and, since the right-hand side is the expectation of a bounded continuous function,

$$E\{|e_{t_i}| J_R(e_{t_i}) + |e_{t_i+1}| J_R(e_{t_i+1})\} \to E_0 |e_j| J_R(e_j) + E_1 |e_{j+1}| J_R(e_{j+1}) \tag{32}$$

where j denotes an even time instant in steady-state operation and E_0 and E_1 denote the expectation under the distributions G_0 and G_1 respectively, or reversed, depending on the parity of x_1. Since eq. (32) holds for each positive R and δ, taking the limit as $\delta \to 0$ and $R \to \infty$ shows that

$$\tfrac{1}{2}(E_0 |e_i| + E_1 |e_{i+1}|) \leqq c + \Delta/2. \tag{33}$$

Thus, the mean-absolute quantizing noise averaged over two consecutive time instants has the bound $c + \Delta/2$ under the limiting distributions for ideal integration.

For leaky integration, the process x_k is bounded with probability one, so that in this case all moments of x_k converge to the corresponding moment under the limiting distribution. Furthermore, in Appendix B, it was shown that the first absolute moment of u_k converges to the corresponding value under the limiting distribution. Together, this implies that $E \mid e_i \mid$ converges to $E_G \mid e_i \mid$. Hence, eq. (12) yields

$$E_G \mid e_i \mid \leqq c + \Delta/2\alpha. \tag{34}$$

This result could also have been obtained by the same argument used to derive eq. (33).

Note that the bounds, eqs. (33) and (34), are independent of the spectral density of the input process u_k and are therefore very crude bounds. An important feature is that the mean-absolute quantizing noise is shown to be finite under the stationary distributions. An immediate consequence is that the decoded process x_k has a finite first absolute moment for ideal as well as leaky integration. Since

$$E \mid x_i \mid = E \mid e_i + u_i \mid \leqq E \mid e_i \mid + E \mid u_i \mid,$$

then

$$E_G \mid x_i \mid \leqq 2c + \Delta/2\alpha \tag{35}$$

for both leaky and ideal integration. (Set $\alpha = 1$ for ideal integration.) Possibly of interest also is that this bound may be used to obtain upper bounds on the tail probabilities of the decoded process by using the Chebyshev inequality.

As discussed in Section VI, ideal integration with initial values of x_1 of the form $m\Delta + \theta$ can be handled by replacing u_k by $u_k - \theta$, so that the bound, eq. (35), remains valid if c is replaced by $c + \mid \theta \mid$, and $\mid e_i \mid$ by $\mid e_i \mid - \mid \theta \mid$, which leads to the inequality, eq. (4).

IX. CONCLUSIONS

The results of this paper show that delta modulation indeed possesses the qualitative properties of convergence to a stationary distribution and boundedness of the quantizing noise. Perhaps of greatest interest is the fact that the results also hold for ideal integration, thus justifying the study of this idealized model to obtain an understanding of the usual physical situation of leaky integration.

The use of a Markovian model of the input process has been considered by several authors[8–11] as an approach to determine actual probability distributions for the steady state. The results of the paper show that the Markovian recursion, i.e., the usual Chapman–Kolmogorov equation, will, in fact, converge to the unique stationary distribution.

The technique used here for showing the existence of a stationary distribution (an invariant solution to the Chapman–Kolmogorov equation) extends the method used by this writer[12] for an adaptive filtering algorithm and the earlier results for Feller processes.[13,14]

APPENDIX A

Markovian Imbedding

Suppose initially that the process u_k has zero mean. Then u_k is generated by the recursion

$$u_{k+n} = \alpha_1 u_{k+n-1} + \cdots + \alpha_n u_k + \beta_1 w_{k+n-1} + \cdots + \beta_n w_k \qquad (36)$$

with $\beta \neq 0$. This equation describes the operation of a stable shaping filter with transfer function

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{i=1}^{n} \beta_i z^{n-i}}{z^n - \sum_{i=1}^{n} \alpha_i z^{n-i}}$$

with $A(z)$ having all roots inside the unit circle, $|z| < 1$, and w_k is a white process with zero mean, finite variance and an everywhere-positive probability density function. The requirement that the autocovariance goes to zero for lags approaching infinity is satisfied by the fact that $B(z)$ is of lower degree than $A(z)$.

The state vector $\mathbf{d}_k$ is defined by

$$\begin{aligned} d_{1k} &= u_k - \mu \\ d_{i+1,k} &= d_{i,k+1} - b_i w_k \qquad i = 1, 2, \cdots, n-1 \end{aligned} \qquad (37)$$

which when combined with eq. (36) leads to the state equations

$$\mathbf{d}_{k+1} = A\mathbf{d}_k + \mathbf{b}w_k \qquad (38)$$

characterizing a vector Markov process. The matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 \\ & & \vdots & & & \vdots \\ 0 & 0 & 0 & & \cdots & 1 \\ \alpha_n & \alpha_{n-1} & & & \cdots & \alpha_1 \end{bmatrix}$$

and the constant vector $\mathbf{b} = (b_1, b_2, \cdots, b_n)'$ is determined by solving the equations:

$$b_1 = \beta_1$$

$$b_i = \beta_i + \alpha_{i-1}b_1 + \cdots + \alpha_2 b_{i-2} + \alpha_1 b_{i-1} \quad \text{for} \quad i = 2, 3, \cdots, n.$$

The matrix A is stable since its eigenvalues are the roots of $A(z)$. This state representation is a standard one in the control literature. See for example, Ref. 15, p. 221.

The polynomials $B(z)$ and $A(z)$ may be assumed to have no common roots so that the shaping filter is intrinsically of order n. Then, the state generating eq. (38) is known to be completely controllable.* This means that, for any initial condition at time k, it is always possible to find values for $w_k, w_{k+1}, \cdots, w_{k+n-1}$ to produce any desired value of the state vector $\mathbf{d}_{k+n}$. It follows that since w_k has a positive density, the state vectors $\mathbf{d}_k$ have a positive probability of lying in any region of n-space with nonzero volume.

APPENDIX B

Convergence in Distribution of $\mathbf{d}_k$

The Markov process defined by eq. (38) can be iterated to obtain

$$\mathbf{d}_{k+1} = A^k d_1 + \sum_{i=1}^{k} A^{k-i} b w_i \,. \tag{39}$$

Since A has all eigenvalues of less than unit modulus, $A^k \to 0$ as $k \to \infty$, so that the first term on the right side of eq. (39) goes to zero with probability one. The second term has the same distribution as

$$v_k = \sum_{i=0}^{k-1} A^i b w_i$$

since the variates w_k are independent and identically distributed. But v_k is a martingale,[16] since

$$E\{v_{k+j} \mid v_k\} = v_k$$

and

$$E \,||\, v_k \,|| \leqq \sum_{i=0}^{k-1} \lambda^i \,||\, b \,||\, E \mid w_i \mid < \frac{||\, b \,||}{1 - \lambda} E \mid w_k \mid$$

is finite, where λ denotes the Euclidean norm of A, $\lambda < 1$. Then by the

* See Theorem 7–8, p. 389 of Ref. 15.

martingale convergence theorem, v_k converges with probability one to a random variable v_∞. Hence, the probability distribution of the vector $\mathbf{d}_{k+1}$ converges to the distribution of v_∞, where

$$v_\infty = \sum_{i=0}^{\infty} A^i b w_i . \tag{40}$$

Since w_k is uncorrelated and has finite variance, it may be seen from eq. (40) that v_∞ also has finite variance. This, together with the convergence in distribution, implies (Ref. 6, p. 252) that the mean absolute value of each component of $\mathbf{d}_k$ converges to the mean absolute value of the corresponding component of v_∞. Consequently,

$$E \mid u_k \mid \to c$$

for any initial state $\mathbf{d}_1$, where c is the mean absolute value of the first component of v_∞.

APPENDIX C

Existence of a Stationary Distribution

Theorem: A Markov process has a stationary distribution if

(*i*) *for any initial state* $\mathbf{s}_1$ *the averaged distributions* G_k *are stochastically bounded, and*
(*ii*) *for any region* A, *let* D *be the set of points* $\mathbf{s}$ *at which the transition probability function* $p(\mathbf{s}, A)$ *is discontinuous and let* N_δ *be the set of all points whose distance from* D *is less than* δ; *then there is a function* $C(\delta)$ *independent of* $\mathbf{s}$ *with* $C(\delta) \to 0$ as $\delta \to 0$ *and for all* $\mathbf{s}$,

$$p(\mathbf{s}, N_\delta) \leqq C(\delta).$$

Proof: Since the sequence of averaged distributions G_k is stochastically bounded, by the Helly selection theorem, Ref. 6, p. 267, there exists a subsequence G_{k_i} converging to a limiting distribution G. From the definition of G_k and T it follows that

$$TG_n = G_n + \frac{1}{n}(F_{n+1} - F_1)$$

so that

$$\mid TG_{n_i}(A) - G_{n_i}(A) \mid \leqq \frac{1}{n_i} \to 0$$

as $i \to \infty$ for any region A. Since G_{n_i} converges to G, it then follows that

$$TG_{n_i} \to G \qquad i \to \infty . \tag{41}$$

It remains to show that

$$TG_{n_i} \rightarrow TG \tag{42}$$

so that eqs. (41) and (42) will imply that $G = TG$, which is the desired result.

To prove (42), note that $G_{n_i} \rightarrow G$ implies that for any bounded continuous function φ of the state vector (s),

$$E_i\varphi(\mathbf{s}) \rightarrow E_G\varphi(\mathbf{s}) \tag{43}$$

where E_i is the expectation under G_{n_i}. If $p(\mathbf{s}, A)$ were continuous in s, eq. (42) would follow from eq. (43) by setting $\varphi(\mathbf{s}) = p(\mathbf{s}, A)$ and noting from eq. (9) that

$$E_ip(\mathbf{s}, A) = TG_{n_i}(A).$$

However, $p(\mathbf{s}, A)$ is not itself continuous and the following argument is needed to complete the proof.

Let $I_\delta(\mathbf{s})$ denote the function which is equal to 1 if s is not in N_δ, and for s in N_δ let $I_\delta(\mathbf{s})$ denote the distance of s from D. Then $I_\delta(s)p(\mathbf{s}, A)$ is bounded and continuous in s for any region A, and so

$$E_i\{I_\delta(\mathbf{s})p(\mathbf{s}, A)\} \rightarrow E_G\{I_\delta(\mathbf{s})p(\mathbf{s}, A)\}, \qquad i \rightarrow \infty.$$

But

$$E_ip(\mathbf{s}, A) - E_i\{I_\delta(\mathbf{s})p(\mathbf{s}, A)\} \leqq G_{k_i}(N_\delta) \tag{44}$$

and also

$$E_Gp(s, A) - E_G\{I_\delta(\mathbf{s})p(\mathbf{s}, A)\} \leqq G(N_\delta). \tag{45}$$

But since

$$p(s, N_\delta) \leqq C(\delta)$$

by hypothesis, it follows by averaging over s that

$$F_k(N_\delta) \leqq C(\delta)$$

and therefore

$$G_{k_i}(N_\delta) \leqq C(\delta) \tag{46}$$

and, since $G_{k_i}(N_\delta) \rightarrow G(N_\delta)$, then

$$G(N_\delta) \leqq C(\delta). \tag{47}$$

Combining these results shows that

$$\limsup_{i\rightarrow\infty} | E_ip(s, A) - E_Gp(s, A) | < 2C(\delta),$$

but since δ can be made arbitrarily small, it follows that

$$E_i p(\mathbf{s}, A) \to E_G p(s, A).$$

Hence, eq. (42) holds and the theorem is proved.

APPENDIX D

Existence of $C(\delta)$

From eq. (38), it follows that

$$u_{k+1} = (Ad_k)_1 + b_1 w_k$$

with $b_1 \neq 0$, so that the conditional distribution of u_{k+1} given $\mathbf{s}_k$ is continuously distributed because w_k is continuously distributed. Let $l = (Ad_k)_1$, and let

$$H(x) = P\{b_1 w_k < x\}.$$

Then

$$P\{u_{k+1} < x \mid \mathbf{s}_k\} = H(x - l) \tag{48}$$

which is a uniformly continuous function of x. Therefore, if

$$C(\delta) = \sup_x [H(x + \delta) - H(x - \delta)]$$

then

$$C(\delta) \to 0 \quad \text{as} \quad \delta \to 0.$$

But

$$P\{\mid u_k - x \mid < \delta \mid \mathbf{s}_k\} = H(x + l + \delta) - H(x + l - \delta) \leqq C(\delta)$$

using eq. (48), which proves the existence of a suitable function $C(\delta)$ independent of $\mathbf{s}_k$ and x.

APPENDIX E

Range of the Mapping $y = \pm 1 \pm \alpha \pm \alpha^2 \pm \cdots$

Theorem: For $\frac{1}{2} \leqq \alpha < 1$, *the range of values taken on by the mapping*

$$y = \sum_{i=0}^{\infty} \alpha^i b_i \tag{49}$$

for all binary sequences, $b_i = \pm 1$ *each* i, *is the closed interval* $\mid y \mid \leqq 1/(1 - \alpha)$.

Proof: For each y in the interval $\mid y \mid \leqq a$, with $a = 1/(1 - \alpha)$, generate a binary sequence $b_0, b_1, b_2, \cdots$ according to the algorithm below.

Let

$$p = \tfrac{1}{2}(y + a).$$

If $p \geqq 1$, let $f_0 = 1$ otherwise $f_0 = 0$. Let

$$s_n = \sum_{i=0}^{n} \alpha^i f_i \qquad n = 0, 1, 2, \cdots .$$

For $n = 1, 2, 3, \cdots$, if $p - s_n \geqq \alpha^{n+1}$ let $f_{n+1} = 1$, otherwise $f_{n+1} = 0$. Then

$$b_i = 2f_i - 1, \qquad i = 0, 1, 2, \cdots . \tag{50}$$

To prove that eq. (49) holds for the binary sequence generated in this manner, note first of all that for all $n \geqq 0$,

$$s_n \leqq s_{n+1}, \qquad \text{and} \qquad s_n \leqq p$$

so that $s_n \to s$ for some number s with $s \leqq p$. Suppose that $s < p$. Then there exists a largest integer m satisfying

$$p < s_{m-1} + \alpha^m. \tag{51}$$

Therefore, $f_m = 0$, and $f_{m+i} = 1$ for each $i > 0$. Consequently,

$$p > s = s_{m-1} + \alpha^{m+1} + \alpha^{m+2} + \cdots$$

so that

$$p > s_{m-1} + \alpha^{m+1}/(1 - \alpha). \tag{52}$$

But eqs. (51) and (52) imply that

$$\frac{\alpha^{m+1}}{1 - \alpha} < \alpha^m$$

so that

$$\alpha < \tfrac{1}{2},$$

which is a contradiction. Therefore,

$$p = \sum_{i=0}^{\infty} \alpha^i f_i$$

and so

$$y = \sum_{i=0}^{\infty} \alpha^i (2f_i - a) = \sum_{i=0}^{\infty} \alpha^i b_i ,$$

which proves the theorem.

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