

# B. S. T. J. BRIEF

## Proof of a Convexity Property of the Erlang B Formula

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(Manuscript received January 5, 1972)

### I. INTRODUCTION

Consider Poisson traffic offered to a group of  $n$  trunks. Blocked calls are cleared, and call holding times are independent and identically distributed. In equilibrium, the call congestion is given by the well-known Erlang B formula<sup>1</sup>

$$B(n, a) = \frac{\frac{a^n}{n!}}{\sum_{k=0}^n \frac{a^k}{k!}},$$

where  $a$  is the offered load in erlangs. We prove here that

$$B(n, a) - B(n + 1, a) < B(n - 1, a) - B(n, a) \quad (1)$$
$$n = 1, 2, \dots,$$

i.e., strict convexity with respect to the number of trunks.<sup>†</sup> For a trunk group with sequential assignment of offered calls, these inequalities simply state that the load carried on the last trunk is monotonically decreasing with the number of trunks—a commonly accepted fact which is basic to economic alternate routing in network engineering.<sup>2</sup> Nevertheless, analytical verification of (1) has apparently not been published. The proof given here offers one approach to verifying convexity properties for other loss systems as well.

### II. DEVELOPMENT OF MAIN RESULT

For an  $m$ -trunk group with Poisson offered traffic, consider a class of operating policies of the form:

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<sup>†</sup> Syski<sup>1</sup> mentions some results for the analytic continuation of  $B(\cdot, a)$ , but convexity on the integers does not seem to follow.

If a call arrives when there are  $i$  calls in progress,  $i < m$ , accept the call with probability  $\delta_i$ , where  $\delta_i$  is fixed.

A policy may be represented by a vector  $\delta(m) = (\delta_0, \delta_1, \dots, \delta_{m-1})$ . For any policy  $\delta(m)$ , identifying states with the number of calls in progress, a stationary solution to the birth-and-death equations exists<sup>1</sup> with carried load given by:

$$c(\delta(m)) = \sum_{k=1}^m k \frac{a^k}{k!} \delta_0 \cdots \delta_{k-1} \left( 1 + \sum_{k=1}^m \frac{a^k}{k!} \delta_0 \cdots \delta_{k-1} \right)^{-1}.$$

*Lemma 1:* The carried load  $c(\delta(m))$  is maximized by the unique optimal policy  $1(m)$  defined by  $\delta_i = 1, i = 0, 1, \dots, m-1$ , i.e., if there is an empty trunk, accept the call.

This result is obvious and a proof is straightforward: if  $i$  is the first index such that  $\delta_i < 1$  for  $\delta(m)$ , it is easy to show that

$$\frac{\partial c}{\partial \delta_i}(\delta(m)) > 0.$$

The result is also implicit in the fundamental inequalities developed by Beneš.<sup>3</sup> Several proofs for the lemma were later given in Ref. 4.

For a group of  $n$  trunks, with sequential assignment, let the carried load on the  $i$ th trunk be given by  $a_i = a_i(a)$ ,  $a > 0$ . Since  $a_1 + a_2 + \dots + a_n = a(1 - B(n, a))$ , the inequalities (1) hold if and only if

$$a_1 > a_2 > a_3 \cdots$$

The main result is:

*Theorem 1:* The sequence  $a_1, a_2, \dots$ , where  $a_i = a(B(i-1, a) - B(i, a))$ , satisfies  $a_1 > a_2 > \dots$  for any positive offered load  $a$ .

*Proof:* Suppose not, i.e., that for some  $n$  and  $a > 0$ ,  $a_{n-1} \leq a_n$ . We shall show that this leads to a contradiction of Lemma 1. Thus, consider a group of  $n-1$  trunks with calls placed according to:

- (i) Sequential assignment for the first  $n-2$  trunks.
- (ii) Overflow calls from the first  $n-2$  trunks are offered to the last trunk according to the status of a dummy trunk. If the dummy trunk is free, the call is rejected, and a dummy call with the same holding time distribution is placed on the dummy trunk. If the dummy trunk is busy, the call is offered to the last trunk.

For this system, define

$$\bar{P}_i = \text{Prob (call put up} \mid i \text{ real calls in progress)}.$$

It is easy to see that  $\bar{P}_i = 1, i = 0, 1, \dots, n-3$ , and that  $\bar{P}_{n-2} < 1$ . Moreover, the carried load is equal to that for an  $n-1$  trunk group system corresponding to the policy  $\delta^*(n-1) = (\bar{P}_0, \dots, \bar{P}_{n-2})$ . But the carried load  $c(\delta^*(n-1))$  is given by  $a_1 + a_2 + \dots + a_{n-2} + a_n$ . Since  $a_{n-1} \leq a_n$ ,

$$c(\delta^*(n-1)) \geq a_1 + a_2 + \dots + a_{n-1} = c(1(n-1))$$

which contradicts Lemma 1. This completes the proof.

*Remark:* Subsequent to the appearance of this development in unpublished form, other proofs have been given. Krupp<sup>5</sup> gives an algebraic proof. Descloux<sup>6</sup> gives both an algebraic proof, and a proof of the equivalent result to (1) for renewal input. Buchner and Neal<sup>7</sup> also give a proof for the generalization to renewal input. A reviewer pointed out that the result can be proved by comparing occupancy probabilities on the  $n$ th trunk for sequential assignment, and for sequential assignment modified so that an overflow call from the first  $n-2$  trunks chooses one of the last two trunks equally likely if both are free.

#### REFERENCES

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