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Exact Solutions to Some Deterministic and Random Transmission Line Problems

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A special class of transmission lines is considered, in which the modes decompose into two noninteracting sets. Both a single transmission line with constant characteristic impedance and variable propagation factor, and two transmission lines with equal propagation factors and variable coupling, in which the forward modes do not interact with the backward modes, are investigated. Exact expressions are obtained for the reflection and transmission coefficients when a section of such a transmission system connects two semi-infinite transmission systems consisting of constant impedance and admittance lines. These results hold for arbitrarily varying propagation factors and coupling; and while they are of independent interest in the case of deterministic variations, we make an application of them here in the case of stochastic variations.

Exact results are obtained for the ensemble averages of the transmission coefficient and transmitted power, and their variances, for the inserted section of single line, when the variable propagation factor is a random function involving either a Gaussian process or the random telegraph process. Asymptotic results are also obtained in the general case of weak fluctuations and long inserted sections. Analogous results may be obtained for the inserted section of two lines when they are randomly coupled, and the results are given in the case of matched lines, for which no reflections occur. Finally, some of the time domain statistics for lossless lines are considered, and expressions are derived for the ensemble averages of the transmitted pulse, due to pulses incident on the inserted section.

I. INTRODUCTION

This paper deals with a special class of the generalized equations of telegraphy. The starting point is the following simple observation. Consider the telegraphist equations in the frequency domain for a single transmission line

$$\frac{d\mathcal{V}}{dx} = -\mathfrak{z}(x)\mathcal{I}(x),$$

$$\frac{d\mathcal{I}}{dx} = -\mathcal{Y}(x)\mathcal{V}(x), \tag{1}$$

where $\mathcal{V}(x)$ and $\mathcal{I}(x)$ are the time Fourier transforms of the voltage and current in the line, and $\mathfrak{z}(x)$ and $\mathcal{Y}(x)$ are the impedance and admittance, respectively. Then, if the characteristic impedance

$$K = \sqrt{\mathfrak{z}(x)/\mathcal{Y}(x)} \tag{2}$$

is a constant independent of x, it is simple to show that (1) has the two fundamental solutions

$$\mathcal{V}_+(x) = \exp\left\{-\int_0^x \Gamma(\xi)\,d\xi\right\}, \qquad \mathcal{I}_+(x) = \frac{1}{K}\mathcal{V}_+(x),$$

$$\mathcal{V}_-(x) = \exp\left\{\int_0^x \Gamma(\xi)\,d\xi\right\}, \qquad \mathcal{I}_-(x) = -\frac{1}{K}\mathcal{V}_-(x), \tag{3}$$

where

$$\Gamma(x) = \sqrt{\mathfrak{z}(x)\mathcal{Y}(x)}. \tag{4}$$

This simple result has several interesting consequences. Note that $\mathcal{V}_+(x)$ and $\mathcal{I}_+(x)$ describe a wave moving to the right and $\mathcal{V}_-(x)$ and $\mathcal{I}_-(x)$ describe a wave moving to the left (the time factor is assumed to be $e^{i\omega t}$). Furthermore, the wave moving to the right does not induce a reflected wave moving to the left, and vice versa, except possibly at the beginning or termination of the line.

This decomposition into noninteracting right and left moving waves suggests that a similar decomposition may exist in the case of n coupled transmission lines. This is the subject of Section II. It is shown there, that when a condition analogous to (2) is satisfied by the impedance and admittance matrices of the system, the system of $2n$ equations can be decomposed into two noninteracting sets of n equations. Under some circumstances, the n fundamental solutions of one system correspond to waves moving to the right; and the n fundamental solutions of the other system correspond to the left moving waves. In particular, we show that the model of two interacting waves recently studied by Rowe and Young[1,2] corresponds to just such a decomposition.

In addition, the form of the solution (3) for the single transmission line makes these models particularly convenient to study when $\mathfrak{z}(x)$

and $\mathcal{Y}(x)$ are stochastic processes. In Section III, we consider a finite section of length l of lossless transmission line with variable impedance and admittance satisfying condition (2). This section of line connects two semi-infinite lossless transmission lines having constant inductances and capacitances. We derive various expressions for the reflection and transmission coefficients of the inserted section. Using these expressions, in Section IV, we calculate exact expressions for the ensemble average of $T(\omega, l)$, the transmission coefficient, and of $| T(\omega, l) |^2$ and $| T(\omega, l) |^4$, when $\mathfrak{z}(x)$ and $\mathcal{Y}(x)$ are particular stochastic processes. Asymptotic results are obtained in the general case of weak fluctuations in $\mathfrak{z}(x)$ and $\mathcal{Y}(x)$, and long inserted sections. These results are based on a limit theorem of Khas'minskii,[3] and the details are given in the Appendix.

In Section V, we study a more complicated model consisting of two coupled transmission lines. Here the model involves a finite length, l, of two transmission lines having equal propagation constants, with variable inductive and capacitive coupling, connecting two semi-infinite, constant impedance and admittance lines. The self-impedance and admittance of the inserted section are also constant, and the semi-infinite lines are uncoupled. We derive expressions for the reflection and transmission matrices of the inserted section. Exact results may be obtained for the ensemble averages of the elements of the reflection and transmission matrices in the case of random coupling, for particular stochastic processes, using the results of Section IV. Since the results are quite lengthy in the general case, we give them only in the case of matched lines, so that no reflections occur.

Finally, in Section VI, we consider some of the time domain statistics of our models. Exact expressions are derived for the ensemble average of the transmitted pulse, due to a pulse incident on the inserted section of single transmission line with random inductance and capacitance. It is of interest to note that if the fluctuations in the propagation factor are described by a Gaussian process, the transmitted wave violates causality. This is not the case when the fluctuations are described by the random telegraph process. Analogous results are obtained for the transmitted pulses, due to pulses incident on the inserted section of two randomly coupled transmission lines, in the case of matched, lossless lines.

II. CLASS OF TRANSMISSION LINES

The generalized equations of telegraphy are the starting point of this paper:

$$\frac{\partial}{\partial x}\mathbf{V}(x, t) = -\mathbf{R}(x)\mathbf{I}(x, t) - \mathbf{L}(x)\frac{\partial}{\partial t}\mathbf{I}(x, t), \tag{5a}$$

$$\frac{\partial}{\partial x}\mathbf{I}(x, t) = -\mathbf{G}(x)\mathbf{V}(x, t) - \mathbf{C}(x)\frac{\partial}{\partial t}\mathbf{V}(x, t). \tag{5b}$$

These equations are typically used to describe the time and space variations of the current and voltage along n coupled transmission lines.[4] In this case, $\mathbf{V}$ and $\mathbf{I}$ are column vectors whose elements $V_p(x, t)$ and $I_p(x, t)$, $p = 1, 2, \cdots, n$, are, respectively, the voltage of the pth line relative to some fixed voltage, and the current in the pth line. $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$, and $\mathbf{C}$ are $n \times n$ matrices, the resistance, inductance, conductance, and capacitance respectively, which typically are functions of the distance x along the lines.

We will, for the most part, find it convenient to work in the frequency plane, and so we introduce the Fourier transforms

$$\mathsf{V}(x, \omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{V}(x, t)e^{-j\omega t}\,d\omega, \tag{6a}$$

$$\mathsf{I}(x, \omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{I}(x, t)e^{-j\omega t}\,d\omega. \tag{6b}$$

Then V and I satisfy the equations

$$\frac{d\mathsf{V}}{dx} = -\mathsf{Z}(x)\mathsf{I}, \qquad \frac{d\mathsf{I}}{dx} = -\mathsf{Y}(x)\mathsf{V}, \tag{7}$$

where

$$\mathsf{Z}(x) = \mathbf{R}(x) + j\omega\mathbf{L}(x), \qquad \mathsf{Y}(x) = \mathbf{G}(x) + j\omega\mathbf{C}(x) \tag{8}$$

are the impedance and admittance matrices, respectively. It should be noted that the frequency domain equations appear in other contexts,[5] but there, the frequency dependance of $\mathsf{Z}(x)$ and $\mathsf{Y}(x)$ is generally more complicated. The remainder of this section is devoted to some general properties of the frequency domain eqs. (7).

It follows from (7) that

$$\frac{d}{dx}(\mathsf{V}^t\mathsf{I}^* + \mathsf{V}^{*t}\mathsf{I}) = -\mathsf{I}^t[\mathsf{Z}^t(x) + \mathsf{Z}^*(x)]\mathsf{I}^* - \mathsf{V}^t[\mathsf{Y}^t(x) + \mathsf{Y}^*(x)]\mathsf{V}^*, \tag{9}$$

where t denotes transpose and $*$ denotes complex conjugate. Hence, for lossless lines

$$\mathsf{Z}^t(x) + \mathsf{Z}^*(x) = 0, \qquad \mathsf{Y}^t(x) + \mathsf{Y}^*(x) = 0. \tag{10}$$

We wish to consider the class of coupled transmission lines for which

$$\mathsf{Z}(x) = \mathbf{K}\mathsf{Y}(x)\mathbf{K}, \tag{11}$$

where $\mathbf{K}$ is a nonsingular constant matrix. Note that, in the lossless case, the first condition in (10) follows from the second if $\mathbf{K}^t = \mathbf{K}^*$. When (11) is satisfied, the solutions of (7) may be split into two groups, namely

$$\mathsf{V} = \mathbf{K}\mathsf{I}, \qquad \frac{d\mathsf{V}}{dx} = -\mathbf{K}\mathsf{Y}(x)\mathsf{V}, \tag{12}$$

and

$$\mathsf{V} = -\mathbf{K}\mathsf{I}, \qquad \frac{d\mathsf{V}}{dx} = \mathbf{K}\mathsf{Y}(x)\mathsf{V}. \tag{13}$$

If the lines are appropriately matched at both ends, then either one set of solutions or the other occurs, and reflections are avoided. Since $\mathbf{K}$ is constant, this matching is independent of the length of the lines.

As a particular example, let $n = 2$ and

$$\mathbf{K} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \qquad \mathsf{Y} = \begin{bmatrix} \dfrac{\Gamma_1}{K_1} & \dfrac{-jc(x)}{(K_1K_2)^{\frac{1}{2}}} \\ \dfrac{-jc(x)}{(K_1K_2)^{\frac{1}{2}}} & \dfrac{\Gamma_2}{K_2} \end{bmatrix}. \tag{14}$$

Thus, from (11),

$$\mathsf{Z} = \begin{bmatrix} \Gamma_1 K_1 & -j(K_1K_2)^{\frac{1}{2}}c(x) \\ -j(K_1K_2)^{\frac{1}{2}}c(x) & \Gamma_2 K_2 \end{bmatrix}. \tag{15}$$

Corresponding to (12), we have

$$\begin{aligned} \frac{d\mathcal{V}_1}{dx} + \Gamma_1\mathcal{V}_1 &= jc(x)\left(\frac{K_1}{K_2}\right)^{\frac{1}{2}}\mathcal{V}_2 , \\ \frac{d\mathcal{V}_2}{dx} + \Gamma_2\mathcal{V}_2 &= jc(x)\left(\frac{K_2}{K_1}\right)^{\frac{1}{2}}\mathcal{V}_1 . \end{aligned} \tag{16}$$

The substitutions $\mathcal{V}_1 = K_1^{\frac{1}{2}}I_1$, $\mathcal{V}_2 = K_2^{\frac{1}{2}}I_2$ lead to the equations for two coupled modes traveling in the same direction, which have been considered previously.[1,2] We remark that if we choose $\mathbf{K} = \left[\begin{smallmatrix} K_1 & 0 \\ 0 & -K_2 \end{smallmatrix}\right]$ instead, then we are led to equations for two modes traveling in opposite directions.

Next, we consider a particular class of transmission lines, satisfying (11), for which

$$\mathbf{K}\mathsf{Y}(x) = \Gamma(x)\mathbf{I} - jc(x)\mathbf{A}, \tag{17}$$

where $\mathbf{I}$ is the unit matrix of order n, and $\mathbf{A}$ is a constant matrix. Then there are solutions of (12) and (13) of the form

$$\mathbf{V} = \mathbf{b} \exp \left\{ \mp \int [\Gamma(x) - j\lambda c(x)]\, dx \right\} = \pm K \mathbf{I}, \tag{18}$$

where $\mathbf{b}$ is a constant vector satisfying

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{b} = 0. \tag{19}$$

The eigenvalues λ are given by $|\lambda \mathbf{I} - \mathbf{A}| = 0$.

We will assume that $c(x)\mathbf{A}$ describes only the coupling between lines, so that $\mathbf{A}$ has diagonal elements equal to zero. The case $n = 1$ (for which $\mathbf{A} = 0$, and $\lambda = 0$ is the only eigenvalue) corresponds to the well-known case of a single line with constant characteristic impedance and variable propagation factor. This case is considered further in subsequent sections. The case $n \geqq 2$ corresponds to n transmission lines with identical propagation factors and variable coupling. Such a situation might arise in the consideration of n twisted pairs in a cable, although the relationship (11) is not too realistic. In Section V, we consider the case $n = 2$ corresponding to $\Gamma_1 = \Gamma_2$ in (14), so that (17) holds.

III. SINGLE TRANSMISSION LINE

In this and the following section, we study in some detail the following example. Consider an infinitely long, lossless, single transmission line ($n = 1$ in the classification of Section II) which for $x < 0$ has the constant impedance and admittance $\mathfrak{Z}_0 = j\omega L_0$, $\mathfrak{Y}_0 = j\omega C_0$, for $x > l$ has the constant impedance and admittance $\mathfrak{Z}_l = j\omega L_l$ and $\mathfrak{Y}_l = j\omega C_l$, while the central section $0 < x < l$ has the variable impedance and admittance $\mathfrak{Z}(x) = j\omega L(x)$, $\mathfrak{Y}(x) = j\omega C(x)$. A wave traveling to the right in the region $x < 0$ will be partially reflected and partially transmitted on striking the central region in $0 < x < l$. We study the transmitted wave under the assumption that $\mathfrak{Z}(x)$ and $\mathfrak{Y}(x)$ satisfy condition (11), i.e.,

$$K = \sqrt{\mathfrak{Z}(x)/\mathfrak{Y}(x)} = \sqrt{L(x)/C(x)} \tag{20}$$

is a positive constant independent of x.

Although this case is probably hard to realize physically, it is nevertheless of considerable interest, since it is mathematically simple enough so that many interesting questions about it can be answered.

Let

$$\Gamma_\alpha = \sqrt{\mathfrak{Z}_\alpha \mathfrak{Y}_\alpha} = j\omega \sqrt{L_\alpha C_\alpha} = j\omega\gamma_\alpha ,$$
$$K_\alpha = \sqrt{\mathfrak{Z}_\alpha/\mathfrak{Y}_\alpha} = \sqrt{L_\alpha/C_\alpha}, \qquad \alpha = 0, l. \tag{21}$$

Then in $x \leqq 0$ and $x \geqq l$, we can solve eqs. (7) simply. In $x \leqq 0$, we have a solution

$$\mathcal{V}(x) = e^{-\Gamma_0 x} + R(\omega, l)e^{\Gamma_0 x},$$

$$\mathcal{I}(x) = \frac{1}{K_0}(e^{-\Gamma_0 x} - R(\omega, l)e^{\Gamma_0 x}). \tag{22}$$

This represents a plane wave $e^{-\Gamma_0 x}$ moving to the right and a reflected wave $R(\omega, l)e^{\Gamma_0 x}$ moving to the left, where $R(\omega, l)$, the reflection coefficient, is a function of ω and l. Similarly, in $x \geqq l$, the solution can be written

$$\mathcal{V}(x) = T(\omega, l)e^{-\Gamma_l(x-l)} = K_l \mathcal{I}(x), \tag{23}$$

representing a transmitted plane wave moving to the right, where $T(\omega, l)$ is the transmission coefficient.

In $0 \leqq x \leqq l$ we define the propagation factor

$$\Gamma(x) = \sqrt{\mathfrak{z}(x)\mathcal{Y}(x)} = j\omega K C(x) = j\omega \frac{L(x)}{K}. \tag{24}$$

Then, from (18), we can write the general solution of (7) in $0 \leqq x \leqq l$ as

$$\mathcal{V}(x) = \mathrm{A}\mathcal{V}_+(x) + \mathrm{B}\mathcal{V}_-(x),$$

$$\mathcal{I}(x) = \frac{1}{K}(\mathrm{A}\mathcal{V}_+(x) - \mathrm{B}\mathcal{V}_-(x)), \tag{25}$$

when A and B are constants and

$$\mathcal{V}_\pm(x) = \exp\left\{\mp\int_0^x \Gamma(\xi)\, d\xi\right\}. \tag{26}$$

We now have a solution depending on four unknown constants A, B, R and T which can be determined from the condition that $\mathcal{V}(x)$ and $\mathcal{I}(x)$ must be continuous at $x = 0$ and $x = l$. The resulting four linear equations are easily solved and yield for the reflection and transmission coefficients

$$R(\omega, l) = \frac{(K - K_0)(K + K_l)\mathcal{V}_-(l) - (K + K_0)(K - K_l)\mathcal{V}_+(l)}{(K + K_0)(K + K_l)\mathcal{V}_-(l) - (K - K_0)(K - K_l)\mathcal{V}_+(l)}, \tag{27}$$

$$T(\omega, l) = \frac{4KK_l}{(K + K_0)(K + K_l)\mathcal{V}_-(l) - (K - K_0)(K - K_l)\mathcal{V}_+(l)}. \tag{28}$$

Notice that if $K = K_0 = K_l$, then $R(\omega, l) = 0$.

We confine our study to the transmitted wave, although the reflected wave can be studied equally well by the techniques we employ. In particular, there is the easily proved energy conservation relationship:

$$1 = |R(\omega, l)|^2 + \frac{K_0}{K_l} |T(\omega, l)|^2. \tag{29}$$

Before proceeding, let us assume instead that a section of line $0 \leqq x \leqq l$ is driven by a voltage source in series with an impedance Z_0 and that the line is terminated in an impedance Z_l, as shown in Fig. 1. Then, $\mathcal{V}(x)$ and $\mathcal{I}(x)$ are still given by (25), but the boundary conditions, from which A and B are determined, are now

$$\mathcal{V}(0) + Z_0 \mathcal{I}(0) = E_0 ,$$
$$\mathcal{V}(l) = Z_l \mathcal{I}(l). \tag{30}$$

Then, it is easily shown that

$$\mathcal{V}(l) = \frac{2KZ_lE_0}{(K + Z_0)(K + Z_l)\mathcal{V}_-(l) - (K - Z_0)(K - Z_l)\mathcal{V}_+(l)}, \tag{31}$$

so the transfer impedance in this formulation is essentially identical with the transmission coefficient in the first formulation. We shall continue to use the first formulation.

We now further specialize the model, and let

$$L(x) = L(1 + \epsilon N(x)), \tag{32}$$

where we assume that $0 \leqq \epsilon \leqq 1$ is a dimensionless constant, $L > 0$ is a constant with the dimension of inductance, and $N(x)$ is a (dimensionless) stochastic process with zero mean. It follows that L is the stochastic mean of $L(x)$,

$$L = \langle L(x) \rangle. \tag{33}$$

The symbol $\langle\ \rangle$ will be used throughout to denote the stochastic mean.

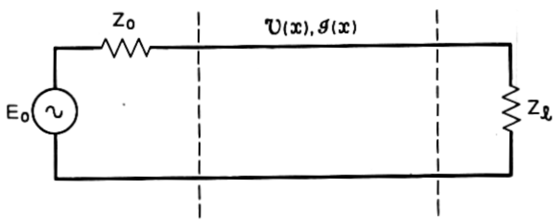

Fig. 1—Diagram of transmission line circuit driven by source E_0 at $x = 0$ with internal impedance Z_0 and terminated at $x = l$ by impedance Z_l.

It follows from (20) that

$$C(x) = C(1 + \epsilon N(x)), \qquad K = \sqrt{\frac{L}{C}}. \tag{34}$$

Further define

$$\gamma = \sqrt{CL}, \tag{35}$$

so that

$$\Gamma(x) = j\omega\gamma(1 + \epsilon N(x)). \tag{36}$$

In addition, we let

$$\theta(l) = \int_0^l N(\xi)\, d\xi, \tag{37}$$

and

$$\rho = \frac{(K - K_0)(K - K_l)}{(K + K_0)(K + K_l)}, \qquad \lambda = \frac{4KK_l}{(K + K_0)(K + K_l)}. \tag{38}$$

Then we can write expression (28) for $T(\omega, l)$ as

$$T(\omega, l) = \lambda e^{-j\omega\gamma(l+\epsilon\theta(l))}[1 - \rho e^{-2j\omega\gamma(l+\epsilon\theta(l))}]^{-1}. \tag{39}$$

We can now derive some series expansions of $T(\omega, l)$ and some of its powers which will prove useful in the next section. Since K, K_0 and K_l are positive,

$$|\rho| < 1, \tag{40}$$

hence we have the geometric series expansion

$$T(\omega, l) = \lambda \sum_{r=0}^{\infty} \rho^r e^{-(2r+1)j\omega\gamma(l+\epsilon\theta(l))}. \tag{41}$$

Next let

$$f(\varphi) = \lambda^2/[(1 - \rho e^{-i\varphi})(1 - \rho e^{i\varphi})]. \tag{42}$$

Then $f(\varphi)$ has the partial fraction expansion

$$f(\varphi) = \frac{\lambda^2}{1 - \rho^2}\left[\frac{1}{1 - \rho e^{i\varphi}} + \frac{\rho e^{-i\varphi}}{1 - \rho e^{-i\varphi}}\right], \tag{43}$$

and hence $f(\varphi)$ has the series expansion

$$f(\varphi) = \frac{\lambda^2}{1 - \rho^2} \sum_{n=-\infty}^{\infty} \rho^{|n|} e^{in\varphi}. \tag{44}$$

Then, it follows that

$$|T(\omega,\, l)|^2 = \frac{\lambda^2}{1-\rho^2} \sum_{r=-\infty}^{\infty} \rho^{|r|} e^{2rj\omega\gamma(l+\epsilon\theta(l))}. \tag{45}$$

We finally need to examine double sums of the form

$$\mathcal{S} = \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \rho^{|r|+|s|} h(r+s). \tag{46}$$

If we make the change of summation variables $u = r + s$, $t = r$, we can write $\mathcal{S}$ as

$$\mathcal{S} = \sum_{u=-\infty}^{\infty} h(u)\xi(u), \tag{47}$$

where

$$\xi(u) = \sum_{t=-\infty}^{\infty} \rho^{|t|+|t-u|}. \tag{48}$$

The series $\xi(u)$ can be evaluated simply, and so we obtain

$$\mathcal{S} = \sum_{u=-\infty}^{\infty} \left\{|u| + \frac{1+\rho^2}{1-\rho^2}\right\} \rho^{|u|} h(u). \tag{49}$$

If we now square expression (45) for $|\, T(\omega,\, l)\,|^2$, we obtain a series of the form (46), and so it follows that

$$|T(\omega,\, l)|^4 = \frac{\lambda^4}{(1-\rho^2)^2} \sum_{r=-\infty}^{\infty} \left\{|r| + \frac{1+\rho^2}{1-\rho^2}\right\} \rho^{|r|} e^{2rj\omega\gamma(l+\epsilon\theta(l))}. \tag{50}$$

IV. FREQUENCY DOMAIN STATISTICS

In this section we study some of the frequency domain statistics of the model described in Section III. In particular, we obtain expressions for the stochastic average of $T(\omega,\, l)$ and $|\, T(\omega,\, l)\,|^2$ and examine the standard deviation of these quantities.

It is clear from (41), and (45) and (50) that the problem of calculating $\langle T(\omega, l)\rangle$, $\langle\, |\, T(\omega, l)\,|^2\rangle$ and $\langle\, |\, T(\omega, l)\,|^4\rangle$ has been reduced to the problem of calculating $\langle e^{j\alpha\theta(l)}\rangle$.

Consider first the case where $N(x)$ is a zero-mean, Gaussian random process. Then $\theta(l)$ is a zero-mean, Gaussian random variable with variance[6]

$$\sigma^2(l) = \int_0^l \int_0^l \langle N(x)N(y)\rangle\, dx\, dy, \tag{51}$$

and

$$\langle e^{j\alpha\theta(l)}\rangle = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{j\alpha\theta-\theta^2/2\sigma^2}\,d\theta = e^{-\frac{1}{2}\alpha^2\sigma^2}. \tag{52}$$

Consequently,

$$M_1(\omega, l) = \langle T(\omega, l)\rangle = \lambda \sum_{r=0}^{\infty} \rho^{|r|} e(2r+1, \omega, l), \tag{53}$$

$$M_2(\omega, l) = \langle |T(\omega, l)|^2\rangle = \frac{\lambda^2}{1-\rho^2} \sum_{r=-\infty}^{\infty} \rho^{|r|} e(2r, \omega, l), \tag{54}$$

and

$$\begin{aligned} M_4(\omega, l) &= \langle |T(\omega, l)|^4\rangle \\ &= \frac{\lambda^4}{(1-\rho^2)^2} \sum_{r=-\infty}^{\infty} \left\{|r| + \frac{1+\rho^2}{1-\rho^2}\right\} \rho^{|r|} e(2r, \omega, l), \end{aligned} \tag{55}$$

where

$$e(r, \omega, l) = \exp\{-jr\omega\gamma l - \tfrac{1}{2}r^2\omega^2\gamma^2\epsilon^2\sigma^2(l)\}. \tag{56}$$

If $N(x)$ is white noise, then $\langle N(x)N(y)\rangle = \mathrm{D}_0\,\delta(x-y)$ and

$$\sigma^2(l) = \mathrm{D}_0 l\,, \tag{57}$$

where D_0 is a constant having dimensions of length. If $N(x)$ is a wide-sense stationary Gaussian process,

$$\langle N(x)N(y)\rangle = g(|\,x - y\,|), \tag{58}$$

with continuous $g(\xi)$, then

$$\sigma^2(l) = 2\int_0^l (l-\xi)g(\xi)\,d\xi. \tag{59}$$

In particular, if $g(\xi) = e^{-2b\xi}$, then

$$\sigma^2(l) = \frac{1}{b}\left[l - \frac{1-e^{-2bl}}{2b}\right]. \tag{60}$$

Since $\sigma(0) = 0$ in all these cases, $e(r, \omega, 0) = 1$, and so

$$M_n(\omega, 0) = \left(\frac{\lambda}{1-\rho}\right)^n, \qquad n = 1, 2, 4. \tag{61}$$

In many cases, such as (57) and (60), $\sigma^2(l)$ tends monotonely to ∞ as $l \to \infty$. In these cases we obtain the asymptotics of the moments as $l \to \infty$ simply. For $\omega\gamma\epsilon\sigma(l) > 1$, we have

$$M_1(\omega, l) \approx \lambda e^{-j\omega\gamma l - \frac{1}{2}\omega^2\gamma^2\epsilon^2\sigma^2(l)}, \tag{62}$$

$$M_2(\omega, l) \approx \frac{\lambda^2}{1 - \rho^2} \{1 + 2\rho \cos(2\omega\gamma l) e^{-2\omega^2\gamma^2\epsilon^2\sigma^2(l)}\}, \tag{63}$$

$$M_4(\omega, l) \approx \frac{\lambda^4}{(1 - \rho^2)^3} \{1 + \rho^2 + 4\rho \cos(2\omega\gamma l) e^{-2\omega^2\gamma^2\epsilon^2\sigma^2(l)}\}. \tag{64}$$

The mean-amplitude transmission coefficient $M_1(\omega, l)$ decays exponentially to zero as $l \to \infty$, while the mean-power transmission coefficient tends to the limit

$$\frac{\lambda^2}{1 - \rho^2} = \frac{4KK_l^2}{(K_0 + K_l)(K^2 + K_0K_l)}. \tag{65}$$

This can be explained qualitatively by noting that, when the transmitted amplitudes are averaged over the ensemble, cancellation can take place, while the transmitted powers all have the same sign and so no cancellation can take place on averaging.

It is easily seen from (62) through (64) that the ratio of the standard deviation to the mean of $T(\omega, l)$ is

$$\Sigma_1(\omega, l) \approx \frac{1}{\sqrt{1 - \rho^2}} e^{\frac{1}{2}\omega^2\gamma^2\epsilon^2\sigma^2(l)}, \tag{66}$$

while the ratio of the standard deviation to the mean of $|T(\omega, l)|^2$ is

$$\Sigma_2(\omega, l) \approx \sqrt{\frac{2\rho^2}{1 - \rho^2}} [1 - \rho \cos(2\omega\gamma l) e^{-2\omega^2\gamma^2\epsilon^2\sigma^2(l)}]. \tag{67}$$

For the examples (57) and (60), $\Sigma_1(\omega, l) \to \infty$ exponentially as $l \to \infty$, while $\Sigma_2(\omega, l)$ tends to the limit

$$\sqrt{\frac{2\rho^2}{1 - \rho^2}} = \frac{|K - K_0|\,|K - K_l|}{\sqrt{2K(K_0 + K_l)(K^2 + K_0K_l)}}. \tag{68}$$

To get some feel for the numbers, we note that for

$$\frac{1}{2} \leqq \frac{K_0}{K} = \frac{K_l}{K} \leqq 2, \quad 0 \leqq \sqrt{\frac{2\rho^2}{1 - \rho^2}} \leqq 0.158.$$

However, if K_0 and K_l differ too much from K, this ratio becomes much larger than 1.

As a second example, consider the case where $N(x)$ is the random telegraph process.[7] It is an ensemble of square wave functions $\{N(x)\}$, such that each sample function $N(x)$ can assume only the values ± 1. For fixed x, a sample function chosen at random will equal $+1$ or -1

with equal probability. The probability $p(n, x)$ of a given sample function changing sign n times in an interval of length x is given by the Poisson process

$$p(n, x) = \frac{(bx)^n}{n!} e^{-bx}, \qquad (n = 0, 1, 2, \cdots). \tag{69}$$

This process has zero mean and correlation function

$$\langle N(x)N(y) \rangle = e^{-2b|x-y|}. \tag{70}$$

The probability density function for the integral of the random telegraph process has been derived by McFadden,[8] and in our notation

$$P(\theta, l) = \tfrac{1}{2}e^{-bl}\Big[\delta(l - \theta) + \delta(l + \theta)$$

$$+ b\Big\{I_0[b(l^2 - \theta^2)^{\frac{1}{2}}] + \frac{lI_1[b(l^2 - \theta^2)^{\frac{1}{2}}]}{(l^2 - \theta^2)^{\frac{1}{2}}}\Big\}H(l - \theta)H(l + \theta)\Big], \tag{71}$$

where $H(u)$ is the Heaviside function

$$H(u) = 1, \qquad u > 0,$$

$$H(u) = 0, \qquad u < 0. \tag{72}$$

It follows that[9]

$$\langle e^{j\alpha\theta(l)} \rangle = e^{-bl}\Big[\cosh\{(b^2 - \alpha^2)^{\frac{1}{2}}l\}$$

$$+ \frac{b}{(b^2 - \alpha^2)^{\frac{1}{2}}} \sinh\{(b^2 - \alpha^2)^{\frac{1}{2}}l\}\Big]. \tag{73}$$

If we define

$$f(r, \omega, l) = \exp\{-jr\omega\gamma l - bl\}$$

$$\cdot\Big[\cosh\{(b^2 - r^2\omega^2\gamma^2\epsilon^2)^{\frac{1}{2}}l\} + \frac{b\sinh\{(b^2 - r^2\omega^2\gamma^2\epsilon^2)^{\frac{1}{2}}l\}}{(b^2 - r^2\omega^2\gamma^2\epsilon^2)^{\frac{1}{2}}}\Big], \tag{74}$$

then the expressions for $\langle T(\omega, l)\rangle$, $\langle | T(\omega, l)|^2\rangle$ and $\langle | T(\omega, l)|^4\rangle$ for the random telegraph case can be obtained from (53) through (55) on on replacing $e(r, \omega, l)$ by $f(r, \omega, l)$.

If $\epsilon > 0$ is small enough so that for a given positive integer r_0

$$\eta \equiv (\epsilon r_0\omega\gamma/b)^2 \ll 1, \qquad \eta^2 bl \ll 1,$$

then for $0 \leqq r \leqq r_0$,

$$f(r, \omega, l) = \exp\{-jr\omega\gamma l - \tfrac{1}{2}r^2\omega^2\gamma^2\epsilon^2 l/b\}[1 + 0(\eta) + 0(\eta^2 bl)]. \tag{75}$$

If furthermore $bl \gg 1$, then from (60) for the Gaussian with correlation function (70), we have

$$\sigma^2(l) = \frac{l}{b}\left[1 + 0\left(\frac{1}{bl}\right)\right].$$

Hence, for these two cases, for $0 \leqq r \leqq r_0$,

$$\frac{f(r,\omega,l)}{e(r,\omega,l)} = [1 + 0(\eta) + 0(\eta^2 bl)]. \tag{76}$$

Note that both $e(r, \omega, l)$ and $f(r, \omega, l)$ are exponentially small for $r > r_0$ if ηbl is moderately large. Also, from (74), as $l \to \infty$ the first-order moment of $T(\omega, l)$ tends to zero, and the second moment and its standard deviation tend to the same limits as in the Gaussian case.

We now consider a general case of weak fluctuations in the inductance and capacitance, and long sections of line, so that $0 < \epsilon \ll 1$ in (32), and $l = \Lambda/\epsilon^2$. It is assumed that $N(x)$ is a bounded, zero-mean, wide-sense stationary stochastic process, with correlation function given by (58). An application is made in the Appendix of a limit theorem due to Khas'minskii,[3] in order to determine the behavior of

$$\epsilon\theta(\Lambda/\epsilon^2) = \epsilon \int_0^{\Lambda/\epsilon^2} N(x)\, dx, \tag{77}$$

for Λ bounded and $\epsilon \to 0$.

If the stochastic process $N(x)$ satisfies a certain strong mixing condition,[3] then it is found that the process $\epsilon\theta(\Lambda/\epsilon^2)$ converges weakly to a Markov diffusion process $\Theta(\Lambda)$, with probability density function

$$p(\Theta, \Lambda) = \frac{1}{(2\pi\bar{a}\Lambda)^{\frac{1}{2}}} \exp\left[\frac{-\Theta^2}{2\bar{a}\Lambda}\right], \tag{78}$$

where

$$\bar{a} = 2 \lim_{X\to\infty}\left[\frac{1}{X}\int_0^X (X - z)g(z)\, dz\right], \tag{79}$$

and g is given by (58). If, as we assume,

$$\lim_{X\to\infty}\left[\frac{1}{X}\int_0^X zg(z)\, dz\right] = 0, \tag{80}$$

then

$$\bar{a} = 2\int_0^\infty g(z)\, dz. \tag{81}$$

It follows from (78) that

$$\langle \exp [jr\omega\gamma\epsilon\theta(\Lambda/\epsilon^2)]\rangle \approx \exp(-\tfrac{1}{2}\bar{a}r^2\omega^2\gamma^2\Lambda). \tag{82}$$

Hence, for $0 < \epsilon \ll 1$ and $l = \Lambda/\epsilon^2$, the asymptotic expressions for $\langle T(\omega, l)\rangle$, $\langle |T(\omega, l)|^2\rangle$ and $\langle |T(\omega, l)|^4\rangle$ can be obtained from (53) through (55) on replacing $e(r, \omega, l)$ by

$$h(r, \omega, l) = \exp\{-jr\omega\gamma l - \tfrac{1}{2}r^2\omega^2\gamma^2\epsilon^2\bar{a}l\}, \tag{83}$$

where $\bar{a}$ is given by (81). For the random telegraph case $g(z) = \exp(-2bz)$, so that $\bar{a} = 1/b$, and the consistency of (75) with the above result is noted.

V. TWO COUPLED TRANSMISSION LINES

We consider here the case of two coupled transmission lines described in eqs. (14) through (16), but with identical propagation constants so that

$$\Gamma_1 = \Gamma = \Gamma_2\,. \tag{84}$$

Then Γ is independent of x in (17), and

$$\mathbf{A} = \begin{bmatrix} 0 & (K_1/K_2)^{\frac{1}{2}} \\ (K_2/K_1)^{\frac{1}{2}} & 0 \end{bmatrix}. \tag{85}$$

Thus, the eigenvalues of $\mathbf{A}$ are $\lambda = \pm 1$. Let

$$\xi(x) = \int_0^x c(y)\,dy, \tag{86}$$

it being assumed that $c(x)$ is real, and that $K_1K_2 > 0$. Then, from (14), (18), and (19), the general solution of (7), subject to (11), for this case may be written in the form

$$\mathcal{V}_1 = K_1^{\frac{1}{2}}[e^{i\xi(x)}(Be^{-\Gamma x} + Ce^{\Gamma x}) + e^{-i\xi(x)}(Ae^{-\Gamma x} + De^{\Gamma x})], \tag{87}$$

$$\mathcal{V}_2 = K_2^{\frac{1}{2}}[e^{i\xi(x)}(Be^{-\Gamma x} - Ce^{\Gamma x}) - e^{-i\xi(x)}(Ae^{-\Gamma x} - De^{\Gamma x})], \tag{88}$$

$$\mathcal{I}_1 = K_1^{-\frac{1}{2}}[e^{i\xi(x)}(Be^{-\Gamma x} - Ce^{\Gamma x}) + e^{-i\xi(x)}(Ae^{-\Gamma x} - De^{\Gamma x})], \tag{89}$$

$$\mathcal{I}_2 = K_2^{-\frac{1}{2}}[e^{i\xi(x)}(Be^{-\Gamma x} + Ce^{\Gamma x}) - e^{-i\xi(x)}(Ae^{-\Gamma x} + De^{\Gamma x})]. \tag{90}$$

We suppose that the coupled lines extend from $x = 0$ to $x = l$, so that (87) through (90) hold for $0 \leqq x \leqq l$. For $x < 0$ and $x > l$ we suppose that the transmission system consists of uncoupled lines with constant propagation constants Γ_{p0} and Γ_{pl}, and constant characteristic impedances K_{p0} and K_{pl}, respectively, $(p = 1, 2)$. Then, for an incoming wave on line 1,

$$\mathcal{V}_1 = (e^{-\Gamma_{10}x} + R_1 e^{\Gamma_{10}x}), \qquad \mathcal{V}_2 = R_2 e^{\Gamma_{20}x}, \tag{91}$$

$$\mathcal{I}_1 = \frac{1}{K_{10}}(e^{-\Gamma_{10}x} - R_1 e^{\Gamma_{10}x}), \qquad \mathcal{I}_2 = \frac{-R_2}{K_{20}} e^{\Gamma_{20}x}, \tag{92}$$

for $x \leqq 0$, and

$$\mathcal{V}_1 = T_1 e^{-\Gamma_{1l}(x-l)}, \qquad \mathcal{V}_2 = T_2 e^{-\Gamma_{2l}(x-l)}, \tag{93}$$

$$\mathcal{I}_1 = \frac{T_1}{K_{1l}} e^{-\Gamma_{1l}(x-l)}, \qquad \mathcal{I}_2 = \frac{T_2}{K_{2l}} e^{-\Gamma_{2l}(x-l)}, \tag{94}$$

for $x \geqq l$.

The boundary conditions are that $\mathcal{V}_p$ and $\mathcal{I}_p$, $(p = 1, 2)$, must be continuous at $x = 0$ and $x = l$. The calculation of reflection and transmission coefficients is tedious, but straightforward, so we omit the details and merely state the result. Let

$$\mu_p = \frac{K_{p0}}{K_p}, \qquad \nu_p = \frac{K_{pl}}{K_p}, \qquad (p = 1, 2), \tag{95}$$

and

$$\kappa = (K_2/K_1)^{\frac{1}{2}}, \qquad \chi = \xi(l). \tag{96}$$

Also, define

$$\begin{aligned}\Delta = [(1 + \mu_1)(1 + \mu_2)(1 + \nu_1)(1 + \nu_2)e^{2\Gamma l} - 2(\mu_1 - \mu_2)(\nu_1 - \nu_2) \\ + (1 - \mu_1)(1 - \mu_2)(1 - \nu_1)(1 - \nu_2)e^{-2\Gamma l} \\ - 2(1 - \mu_1\mu_2)(1 - \nu_1\nu_2) \cos 2\chi].\end{aligned} \tag{97}$$

Then it is found that

$$\Delta T_1 = 4\nu_1[(1 + \mu_2)(1 + \nu_2)e^{\Gamma l} - (1 - \mu_2)(1 - \nu_2)e^{-\Gamma l}] \cos \chi, \tag{98}$$

$$\Delta T_2 = 4j\kappa\nu_2[(1 + \mu_2)(1 + \nu_1)e^{\Gamma l} + (1 - \mu_2)(1 - \nu_1)e^{-\Gamma l}] \sin \chi, \tag{99}$$

$$\begin{aligned}\Delta R_1 = [(1 - \mu_1)(1 + \mu_2)(1 + \nu_1)(1 + \nu_2)e^{2\Gamma l} + 2(\mu_1 + \mu_2)(\nu_1 - \nu_2) \\ + (1 + \mu_1)(1 - \mu_2)(1 - \nu_1)(1 - \nu_2)e^{-2\Gamma l} \\ - 2(1 + \mu_1\mu_2)(1 - \nu_1\nu_2) \cos 2\chi],\end{aligned} \tag{100}$$

and

$$\Delta R_2 = -4j\kappa\mu_2(1 - \nu_1\nu_2) \sin 2\chi. \tag{101}$$

We remark that the reflection and transmission coefficients corresponding to an incoming wave on line 2 may be obtained by appropriate interchange of subscripts.

Since the coupling function $c(x)$ is real, χ is also real, from (86) and (96). If k is an integer, then $T_2 = 0$ and $R_2 = 0$ for $\chi = k\pi$, and $T_1 = 0$ and $R_2 = 0$ for $\chi = (k + \frac{1}{2})\pi$. The oscillatory behavior of T_1 and T_2 has been found earlier by Foschini,[10] in the case of matched lines with equal characteristic impedances, i.e., $\mu_p = 1 = \nu_p$, $(p = 1, 2)$, and $\kappa = 1$. In this case there are no reflections.

In the general case, the expressions for the transmission and reflection coefficients T_1, T_2 and R_1, R_2 may be expanded in Fourier series in χ. Thus, in the case of random coupling between the lines, the problem of calculating the expectations of the transmission and reflection coefficients reduces to that of calculating $\langle \exp jr\chi \rangle$ where, from (86) and (96),

$$\chi = \int_0^l c(x)\, dx. \tag{102}$$

We have seen in Section IV how to carry out this calculation if $c(x)$ is a Gaussian or a random telegraph process. Similar remarks apply also to the calculation of the expected transmitted and reflected powers, and their variances. We do not give the results for the general case, although the calculations are straightforward, since the final expressions are somewhat lengthy.

However, we will consider the case of matched lines, for which no reflections occur. Thus, with $\mu_p = 1$, $\nu_p = 1$, $(p = 1, 2)$, we have, from (97) through (99),

$$T_1 = e^{-\Gamma l} \cos \chi, \qquad T_2 = j\kappa e^{-\Gamma l} \sin \chi. \tag{103}$$

This is for unit input voltage on line 1. Interchanging subscripts, for unit input voltage on line 2 we have

$$T_1 = \frac{j}{\kappa} e^{-\Gamma l} \sin \chi, \qquad T_2 = e^{-\Gamma l} \cos \chi, \tag{104}$$

using (96). Thus, if $\mathcal{V}_1(0) = v_1$ and $\mathcal{V}_2(0) = v_2$, then

$$\mathcal{V}_1(l) = T_1 = e^{-\Gamma l}\left(v_1 \cos \chi + \frac{j}{\kappa} v_2 \sin \chi\right),$$

$$\mathcal{V}_2(l) = T_2 = e^{-\Gamma l}(j\kappa v_1 \sin \chi + v_2 \cos \chi). \tag{105}$$

Note that

$$|\kappa T_1|^2 + |T_2|^2 = e^{-(\Gamma + \Gamma^*)l}(|\kappa v_1|^2 + |v_2|^2). \tag{106}$$

We let

$$c(x) = cN(x), \tag{107}$$

where $N(x)$ is a dimensionless stochastic process with zero mean. Then, from (37) and (102),

$$\chi = c\theta(l). \tag{108}$$

Let us consider the case when $N(x)$ is a Gaussian process. Then, from (52),

$$\langle e^{ir\chi} \rangle = e^{-\frac{1}{2}r^2c^2\sigma^2(l)}, \tag{109}$$

where $\sigma^2(l)$ is given by (51). Hence, from (105),

$$\langle T_1 \rangle = v_1 e^{-\Gamma l} e^{-\frac{1}{2}c^2\sigma^2(l)}, \qquad \langle T_2 \rangle = v_2 e^{-\Gamma l} e^{-\frac{1}{2}c^2\sigma^2(l)}. \tag{110}$$

Also, after some algebra, it is found that

$$\langle | T_1 |^2 \rangle = \frac{e^{-(\Gamma+\Gamma^*)l}}{2 | \kappa |^2} [(| \kappa v_1 |^2 + | v_2 |^2) + (| \kappa v_1 |^2 - | v_2 |^2) e^{-2c^2\sigma^2(l)}], \tag{111}$$

$$\langle | T_2 |^2 \rangle = \tfrac{1}{2} e^{-(\Gamma+\Gamma^*)l} [(| \kappa v_1 |^2 + | v_2 |^2) - (| \kappa v_1 |^2 - | v_2 |^2) e^{-2c^2\sigma^2(l)}], \tag{112}$$

and

$$\begin{aligned} \langle | \kappa T_1 |^4 \rangle - \langle | \kappa T_1 |^2 \rangle^2 &= \langle | T_2 |^4 \rangle - \langle | T_2 |^2 \rangle^2 \\ &= \tfrac{1}{8} e^{-2(\Gamma+\Gamma^*)l} [1 - e^{-4c^2\sigma^2(l)}] \{ | v_2^2 - \kappa^2 v_1^2 |^2 [1 + e^{-4c^2\sigma^2(l)}] \\ &\quad - 2(| v_2 |^2 - | \kappa v_1 |^2)^2 e^{-4c^2\sigma^2(l)} \}. \end{aligned} \tag{113}$$

The first equality in (113) is a consequence of (106). From (111) through (113) it follows that the ratio of the standard deviation to the mean of $|T_p|^2$ approaches

$$\sum\nolimits_\infty = \frac{| v_2^2 - \kappa^2 v_1^2 |}{2(| \kappa v_1 |^2 + | v_2 |^2)} \tag{114}$$

as $c^2\sigma^2(l) \to \infty$, both for $p = 1$ and $p = 2$.

Analogous results may be obtained when $N(x)$ is the random telegraph process, by using (73), and also in the case of weak general coupling and long sections of the coupled lines, by using (82). We remark that we have previously[2] calculated the average modal powers in these cases for two coupled lines with unequal propagation constants, corresponding to equation (16), using entirely different methods.

VI. TIME DOMAIN STATISTICS

In this section we conclude the study of our models by calculating some of their time domain statistics. Consider first the single line and an incident wave moving to the right in $x < 0$ of the form

$$V(t - \gamma_0 x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{V}(\omega) e^{j\omega(t-\gamma_0 x)} \, d\omega,$$

$$I(t - \gamma_0 x) = \frac{1}{K_0} V(t - \gamma_0 x). \tag{115}$$

Then the transmitted wave in $x > l$ is

$$V_T(t - \gamma_l x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} T(\omega, l) \mathcal{V}(\omega) e^{j\omega(t-\gamma_l(x-l))} \, d\omega,$$

$$I_T(t - \gamma_l x) = \frac{1}{K_l} V_T(t - \gamma_l x), \tag{116}$$

where $T(\omega, l)$ is the transmission coefficient. If we substitute expression (41) for $T(\omega, l)$ into (116) and formally interchange summation and integration, we obtain

$$\begin{aligned} V_T(t - \gamma_l x) &= \lambda \sum_{r=0}^{\infty} \rho^r \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{V}(\omega) \exp \{ j\omega [t - \gamma_l (x - l) \\ &\qquad - (2r + 1)\gamma(l + \epsilon\theta(l))] \} \, d\omega \\ &= \lambda \sum_{r=0}^{\infty} \rho^r V(t - \gamma_l (x - l) - (2r + 1)\gamma(l + \epsilon\theta(l))). \end{aligned} \tag{117}$$

Therefore,

$$\langle V_T(t - \gamma_l x) \rangle = \lambda \sum_{r=0}^{\infty} \rho^r \langle V(t - \gamma_l (x - l) - (2r + 1)\gamma(l + \epsilon\theta(l))) \rangle. \tag{118}$$

In this formulation, the randomness appears just as we should expect in a random change in the electrical length of the central transmission line. From (117) it follows that

$$\begin{aligned} &\langle | V_T(t - \gamma_l x) |^n \rangle \\ &\quad = \lambda^n \left\langle \left| \sum_{r=0}^{\infty} \rho^r V(t - \gamma_l (x - l) - (2r + 1)\gamma(l + \epsilon\theta(l))) \right|^n \right\rangle. \end{aligned} \tag{119}$$

To better understand some of the implications of these formulas, let $V(u)$ be a rectangular pulse,

$$\begin{aligned} V(u) &= 1, \qquad 0 \leqq u \leqq \tau, \\ &= 0, \qquad u < 0,\ \tau < u. \end{aligned} \tag{120}$$

Then if $N(x)$ is one of the Gaussian processes discussed in Section IV,

$$\begin{aligned} &\langle V(t - \gamma_l(x - l) - (2r + 1)\gamma(l + \epsilon\theta(l)))\rangle \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\theta^2/2\sigma^2)} V(t - \gamma_l(x - l) - (2r + 1)\gamma(l + \epsilon\theta))\, d\theta \\ &= \frac{1}{\sqrt{\pi}} \int_{[t-\gamma_l(x-l)-(2r+1)\gamma l-\tau/\sqrt{2}(2r+1)\epsilon\gamma\sigma(l)]}^{[t-\gamma_l(x-l)-(2r+1)\gamma l/\sqrt{2}(2r+1)\epsilon\gamma\sigma(l)]} e^{-\omega^2}\, d\omega \\ &= \frac{1}{2}\left[\operatorname{erfc}\left\{\frac{t - \gamma_l(x - l) - (2r + 1)\gamma l - \tau}{\sqrt{2}\,(2r + 1)\epsilon\gamma\sigma(l)}\right\}\right. \\ &\qquad \left. - \operatorname{erfc}\left\{\frac{t - \gamma_l(x - l) - (2r + 1)\gamma l}{\sqrt{2}\,(2r + 1)\epsilon\gamma\sigma(l)}\right\}\right]. \end{aligned} \tag{121}$$

In (121) erfc (x) is the complementary error function.[11]

Equation (121) shows that the average field violates causality, since at $x = l$, for example, $\langle V(t - (2r + 1)\gamma(l + \epsilon\theta(l)))\rangle$ is positive for all $-\infty < t < \infty$. This is really a consequence of the fact that for a Gaussian process, at any point x, there is a positive probability that a sample function is less than $-\epsilon^{-1}$. Hence, in our model the inductance and capacitance can both become negative, leading to the violation of causality.

If $N(x)$ is the random telegraph process, then $\theta(l)$ has the probability density function given in (71). It follows that in this case

$$\begin{aligned} &\langle V(t - \gamma_l(x - l) - (2r + 1)\gamma(l + \epsilon\theta(l)))\rangle \\ &= \tfrac{1}{2}e^{-bl}\Big[V(t - \gamma_l(x - l) - (2r + 1)\gamma l(1 + \epsilon)) \\ &\quad + V(t - \gamma_l(x - l) - (2r + 1)\gamma l(1 - \epsilon)) \\ &\quad + b\int_{-l}^{l}\Big\{I_0[b(l^2 - \theta^2)^{\frac{1}{2}}] \\ &\quad + \frac{lI_1[b(l^2 - \theta^2)^{\frac{1}{2}}]}{(l^2 - \theta^2)^{\frac{1}{2}}}\Big\} V(t - \gamma_l(x - l) - (2r + 1)\gamma(l + \epsilon\theta))\, d\theta\Big]. \end{aligned} \tag{122}$$

If $V(u)$ is the rectangular pulse (120) which arrives at $x = 0$ at $t = 0$, then from (118) and (122), the average transmitted signal is a train of nonrectangular pulses. It is seen easily from (122) that

$$\langle V(t - \gamma_l(x - l) - (2r + 1)\gamma(l + \epsilon\theta(l)))\rangle$$

is a pulse which begins at $t = \gamma_l(x - l) + (2r + 1)\gamma l(1 - \epsilon)$ and ends at $t = \gamma_l(x - l) + (2r + 1)\gamma l(1 + \epsilon) + \tau$. This shows that as long as $\epsilon < 1$, causality is preserved for the average transmitted signal since then $\gamma_l(x - l) + \gamma l(1 - \epsilon) > 0$ for all $l \leqq x$.

The duration of the rth average transmitted pulse is $\Delta t = \tau + 2(2r + 1)\gamma l\epsilon$, while the time between the end of the rth pulse and beginning of the $(r + 1)$th pulse is $2\gamma l - 4(r + 1)\gamma l\epsilon - \tau$. Thus, each pulse in the train is longer than the pulse which preceeded it and all of these are longer than the incident pulse. Furthermore, no matter how short the incident pulse is, the average transmitted pulses eventually begin to overlap.

The transmitted power can be treated in the same way. However, due to the overlapping of the pulses, the analysis is tedious and we do not discuss it here.

Analogous results may be obtained in the case of two coupled transmission lines with identical propagation constants, if we consider lossless lines and suppose that the phase constants and the coupling coefficient are proportional to the frequency, so that

$$\Gamma = j\omega\gamma, \qquad c = \omega\gamma\epsilon, \tag{123}$$

and

$$\Gamma_{p0} = j\omega\gamma_{p0}, \qquad \Gamma_{pl} = j\omega\gamma_{pl}, \qquad (p = 1, 2). \tag{124}$$

It is also assumed that the characteristic impedances K_p, K_{p0}, and K_{pl}, $(p = 1, 2)$ are independent of ω. We will confine our attention to the case of matched lines, so that no reflected waves occur.

Let us consider an incident wave on line 1 moving to the right in $x < 0$, and of the form

$$V_1(t - \gamma_{10}x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{V}_1(\omega) e^{j\omega(t-\gamma_{10}x)}\, d\omega,$$

$$I_1(t - \gamma_{10}x) = \frac{1}{K_{10}} V_1(t - \gamma_{10}x). \tag{125}$$

Then, from (93), (94), (96), (103), (108), and (123) through (125), the transmitted waves in $x > l$ are

$$V_{1T}(t - \gamma_{1l}x) = K_{1l}I_{1T}(t - \gamma_{1l}x)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{V}_1(\omega) e^{-j\omega\gamma l} \cos(\omega\gamma\epsilon\theta(l)) e^{j\omega(t-\gamma_{1l}(x-l))}\, d\omega$$
$$= \tfrac{1}{2}\{V_1(t - \gamma_{1l}(x - l) - \omega\gamma[l - \epsilon\theta(l)])$$
$$+ V_1(t - \gamma_{1l}(x - l) - \omega\gamma[l + \epsilon\theta(l)]\}, \tag{126}$$

and

$$V_{2T}(t - \gamma_{2l}x) = K_{2l}I_{2T}(t - \gamma_{2l}x)$$
$$= \frac{j}{\sqrt{2\pi}} \left(\frac{K_2}{K_1}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \mathcal{V}_1(\omega) e^{-j\omega\gamma l} \sin(\omega\gamma\epsilon\theta(l)) e^{j\omega(t-\gamma_{2l}(x-l))}\, d\omega$$
$$= \frac{1}{2}\left(\frac{K_2}{K_1}\right)^{\frac{1}{2}} \{V_1(t - \gamma_{2l}(x - l) - \omega\gamma[l - \epsilon\theta(l)])$$
$$- V_1(t - \gamma_{2l}(x - l) - \omega\gamma[l + \epsilon\theta(l)])\}. \tag{127}$$

The transmitted waves in $x > l$ corresponding to an incident wave on line 2 moving to the right in $x < 0$, and of the form

$$V_2(t - \gamma_{20}x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{V}_2(\omega) e^{j\omega(t-\gamma_{20}x)}\, d\omega,$$
$$I_2(t - \gamma_{20}x) = \frac{1}{K_{20}} V_2(t - \gamma_{20}x), \tag{128}$$

are obtained by interchanging the subscripts 1 and 2 in (126) and (127). The transmitted waves corresponding to incident waves on both lines are obtained by linear superposition. The ensemble averages of the transmitted waves may be calculated as before.

APPENDIX

We here apply a limit theorem due to R. Z. Khas'minskii,[3] in order to determine the limiting probability density function for the process $\epsilon\theta(\xi/\epsilon^2)$, for bounded ξ, and $\epsilon \to 0$. Now, from (37),

$$\frac{d}{dx}[\epsilon\theta(x)] = \epsilon N(x), \qquad \epsilon\theta(0) = 0. \tag{129}$$

We assume that $N(x)$ is a bounded, zero-mean, wide-sense stationary stochastic process, with correlation function given by (58). Since θ is a scalar rather than a vector, and moreover the derivative of θ depends only on x, and not on θ, we have just about the simplest nontrivial application of the limit theorem.

There is only one quantity to be considered, namely,

$$a(x, y) = \langle N(x)N(y) \rangle = g(| x - y |), \tag{130}$$

from (58). Then, in accordance with Khas'minskii's definition,

$$\begin{aligned} \bar{a} &= \lim_{X\to\infty} \left[\frac{1}{X} \int_{x_0}^{X+x_0} \int_{x_0}^{X+x_0} a(x, y)\, dx\, dy \right] \\ &= 2 \lim_{X\to\infty} \left[\frac{1}{X} \int_0^X (X - z) g(z)\, dz \right], \end{aligned} \tag{131}$$

after some integrations by parts. As required by the hypotheses of the limit theorem, $\bar{a}$ is independent of x_0. It is also required that the stochastic process $N(x)$ satisfy a certain strong mixing condition, and the reader is referred to Khas'minskii's paper for a precise statement of this condition.

For the case under consideration, the limit theorem states that, on the interval $0 \leqq \xi \leqq \xi_0$, where ξ_0 is an arbitrary positive number, the process $\epsilon\theta(\xi/\epsilon^2)$ converges weakly as $\epsilon \to 0$ to a Markov diffusion process $\Theta(\xi)$ with zero drift and diffusion coefficient $\bar{a}$. The drift coefficient $\bar{K}$ is zero since the right-hand side of eq. (129) is independent of θ. The probability density function of the limit process satisfies the equation

$$\frac{\partial p}{\partial \xi} = \frac{1}{2} \bar{a} \frac{\partial^2 p}{\partial \Theta^2}, \qquad p(\Theta, 0) = \delta(\Theta), \tag{132}$$

in view of the initial condition $\Theta(0) = 0$. Thus,

$$p(\Theta, \xi) = \frac{1}{(2\pi\bar{a}\xi)^{\frac{1}{2}}} \exp\left[\frac{-\Theta^2}{2\bar{a}\xi} \right]. \tag{133}$$

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