THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 51, No. 7, September, 1972
Printed in U.S.A.

Applications of Multidimensional Polynomial Algebra to Bubble Circuits

By S. V. AHAMED

(Manuscript received November 17, 1971)

The principles of Multidimensional Polynomial Algebra developed in a companion paper[1] *are applied to two T-bar circuits with bubble-no-bubble coding and one double rail circuit with lateral displacement coding. The object of this paper is to indicate the flexibility of the algebra in its use with real circuits and to emphasize the potential of the algebra as a design tool for bubble circuits.*

I. INTRODUCTION

The operation of bubble circuits depends on the accurate functioning of individual elements such as channeling gates, logic gates, generators, etc., at the critical instants of time. When the circuit becomes complicated, it is not easy to comprehend a multiplicity of functions and determine accurately the instant and duration of operations of these critical elements. Further, a bubble circuit cannot be easily altered like a prototype experimental electrical circuit; and it becomes necessary to ascertain the proper functioning of the designed bubble circuit prior to its actual construction.

In this paper the technique developed in Ref. 1 is applied to three circuits, and the operation of each circuit is predicted. In the second example, the design parameters such as the operating time of the gates, their duration, and the overall timing for the circuit, are derived step by step as the algebra progresses.

II. APPLICATION OF ALGEBRA TO A T-BAR STATION SCAN MEMORY

2.1 *The Principle of Operation*

Consider a hundred stations with 25 lines each. The status of each line is stored in a loop* with 2500 periods as shown in Fig. 1. A controlled

* This configuration of the station scan memory was supplied by A. J. Perneski and R. M. Smith.

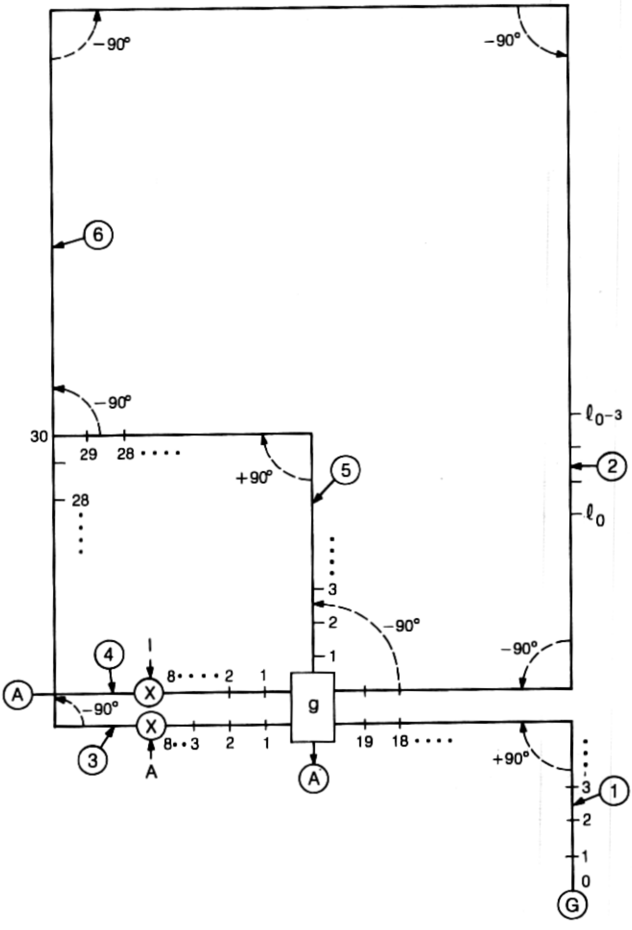


Fig. 1—Block diagram of a station scan memory.

generator G codes the status of each line (active or inactive) by generating a bubble or no bubble during the clock cycles through which the line is being scanned. Thus each line is scanned every 2500 clock cycles. The operation of the gate g is based on the repulsion between bubbles (if any) arriving through elements 1 and 2. One bubble in 1 or 2 passes through the gate to elements 3 or 4 respectively. Two bubbles in 1 and 2 are diverted into the annihilator A and into the element 5. The gate g thus stores the most recent status of each line in the loop through either element 5 or element 3. Two sensors A and I in paths

3 and 4 indicate whether the line has just become active or just become inactive.

The six elements in the circuit are each marked serially so that the bubble streams can be observed in these elements. The gate g does not have an independent status as an element since the streams can only pass through the gate and they cannot be observed within it.

2.2 *The Algebra of the Circuit*

The gate g permits the interaction of the two bubble streams in elements 1 and 2, yielding streams in elements 3, 4 and 5. The truth table for the operation is:

Inputs	Outputs
1 2	3 4 5
0 0	0 0 0
0 1	0 1 0
1 0	1 0 0
1 1	0 0 1

Let the generator G be located 20 periods behind the gate g (including its own location) on element 1. A binary position just generated by G would interact with location (l_0), 20 periods behind the gate on the element 2. Further, let the elements 3 and 5 contain 30 periods each and let the sensors I and A be located 8 periods from g on elements 4 and 3, respectively. These circuit conditions may be algebraically represented as:

g located at Y_1^{20}, $Y_2^{l_0+20}$, Y_3^0, Y_4^0 and Y_5^0

P located at Y_3^{30} and Y_5^{30}

I and A located at Y_4^8 and Y_5^8, respectively.

The circuit operates on a repetitive basis, and it is possible to choose an origin of time at the end of the generation cycle of any one bubble position at G. The origin also corresponds to the end of a coding cycle for a particular line L. The algebra may be carried out with any number of bubble positions in the polynomials. If four positions are chosen to illustrate the use of the algebra, then the four bit string after three cycles from this prechosen origin of time may be written as:

$$u_1 = X^3(a_{01}Y_1^3 + a_{11}Y_1^2 + a_{21}Y_1^1 + a_{31}Y_1^0). \tag{1}$$

At this instant of consideration, the corresponding string of four bubble positions in element 2 may be written as:

$$u_2 = X^3(a_{02}Y_2^{l_0} + a_{12}Y_2^{l_0-1} + a_{22}Y_2^{l_0-2} + a_{32}Y_2^{l_0-3}). \qquad (2)$$

It will take 20 clock cycles for the generated binary position a_{31} to be completely processed by the gate g. Thus with $m_1 = 20$, u_1' and u_2' (see footnote in Sec. 4.3.2 of Ref. 1) can be written as:

$$u_1' = X^{23}(a_{01}Y_1^{23} + a_{11}Y_1^{22} + a_{21}Y_1^{21} + a_{31}Y_1^0), \qquad (3)$$

and

$$u_2' = X^{23}(a_{02}Y_2^{l_0+20} + a_{12}Y_2^{l_0+19} + a_{22}Y_2^{l_0+18} + a_{32}Y_2^{l_0+17}). \qquad (4)$$

The last positions a_{31} and a_{32} would be processed by the gate at the end of the 23rd cycle. Now the conversion of u_1' and u_2' to u_3, u_4 and u_5 is feasible by the truth tables for inputs and outputs, and by the spatial conversions:

$$Y_1^{20+z} \rightarrow Y_3^z,\ Y_4^z \quad \text{or} \quad Y_5^z$$

$$Y_2^{20+z} \rightarrow Y_3^z,\ Y_4^z \quad \text{or} \quad Y_5^z\ .$$

Hence, if a_{01}, a_{11}, a_{21} and a_{31} are 0, 1, 0 and 1; and a_{02}, a_{21}, a_{22} and a_{23} are 1, 1, 1 and 0, then:

$$u_3 = X^{23}(a_{03}Y_3^3 + a_{13}Y_3^2 + a_{23}Y_3^1 + a_{33}Y_3^0) \qquad (5a)$$

$$u_4 = X^{23}(a_{04}Y_4^3 + a_{14}Y_4^2 + a_{24}Y_4^1 + a_{34}Y_4^0) \qquad (5b)$$

$$u_5 = X^{23}(a_{05}Y_5^3 + a_{15}Y_5^2 + a_{25}Y_5^1 + a_{35}Y_5^0) \qquad (5c)$$

where a_{03}, a_{13}, a_{23} and a_{33} are 1, 0, 1 and 0; a_{04}, a_{14}, a_{24} and a_{34} are 0, 0, 0 and 1; and a_{05}, a_{15}, a_{25} and a_{35} are 0, 1, 0 and 0, respectively.

When u_3 and u_4 are multiplied by X^5Y^5, the exponents of Y_3 and Y_5 are both 8. The sensors A and I can read the status during the 28th (i.e., 23 + 5) clock cycle from the prechosen origin of time. In this case, the origin of time corresponds to the start of the coding cycle for the particular line the status of which has just been read, and the 28 cycles indicate the delay between coding the status at G, and reading the change in status at A or I of any particular line L.

Now consider the merger of the streams 3 and 5 at P, i.e., at Y_3^{30} and Y_5^{30}. The merger is complete when the exponent of Y reaches 30 which leads to

$$u_6 = X^{30}Y^{30}\cdot(u_3 + u_5),$$

or

$$u_6 = X^{53}(a_{06}Y_6^3 + a_{16}Y_6^2 + a_{26}Y_6^1 + a_{26}Y_6^0), \qquad (6)$$

with a_{06}, a_{16}, a_{26} and a_{36} being 0, 1, 0 and 1. The position a_{06} reaches the merger point Y_6^0 during the 50th cycle, a_{16} during the 51st cycle and so on.

If the Y_6^3 is 2447 (i.e., 2500–53) periods* from $Y_2^{l_o}$, then

$$u_2 = X^{2447} Y^{2447} \cdot u_6 ,$$

thus leading to the polynomial u_2

$$u_2 = X^{2500}(a_{02} Y_2^{l_o} + a_{12} Y_2^{l_o - 1} + a_{22} Y_2^{l_o - 2} + a_{32} Y_2^{l_o - 3}). \qquad (7)$$

The origin of time was chosen at the end of a coding cycle of a particular line. At the end of 2500 clock cycles, the cyclic process is repeated and the next set of calculations may be started at this instant.

2.1.1 *Effect of Corners in the Circuit*

Corners were ignored in the polynomial calculation in the previous section. Considering their effects, we have

$$u_1' = X^{20} Y^{20+\frac{1}{4}} u_1 \ ; \quad \text{and,} \quad u_2' = X^{20} Y^{20-\frac{1}{4}} u_2 \qquad (8a; 8b)$$

from Sec. 4.3.4 in Ref. 1. However, at the gate g, the two inputs u_1 and u_2 should be in phase. This condition implies that the generator G should generate the position a_{14} half a clock cycle after the instant as assumed in the previous calculation, leading to

$$u_3 = X^{-\frac{1}{2}} Y^{-\frac{1}{2}} \cdot u_1' = X^{22\frac{1}{2}} \sum_{i=0}^{i=3} a_{i3} Y_3^{(2\frac{3}{4} - i)} \qquad (9)$$

$$u_4 = X^{22\frac{1}{2}} \sum_{i=0}^{i=3} a_{i4} Y_4^{(2\frac{3}{4} - i)} . \qquad (10)$$

The element u_5 has a -90 degree corner at the gate, and hence,

$$u_5 = X^{22\frac{1}{2}} \sum_{i=0}^{i=3} a_{i5} Y_5^{(2\frac{1}{2} - i)} . \qquad (11)$$

The constants a_{ij} ($i = 0$ through 3, $j = 3$ through 5) are the same as their previous values. The sensors A and I should be read during the cycle ending at $22\frac{1}{2} + (8 - 2\frac{3}{4}) = 27\frac{3}{4}$ clock cycles, after the generation of the first data position a_{01}.

The polynomial u_3 makes a -90 degree corner, and u_5 makes compensating $+90$ and -90 degree turns, before reaching P, yielding

$$u_3' = X^{30} Y^{29\frac{3}{4}} u_3 \ ; \quad \text{and,} \quad u_5' = X^{30} Y^{30} \cdot u_5 \qquad (9a; 11a)$$

* It can be seen that the top section of the storage loop need not have two independent element numbers 2 and 6; but such a numbering facilitates the representation and its boundary may be considered to lie at any period z located at Y_6^z and $Y_2{}^{l_o 2450+z}$.

leading to

$$u_6 = X^{52\frac{1}{2}} \sum_{i=0}^{i=3} a_{i6} Y_6^{(2\frac{1}{2}-i)}. \tag{12}$$

If incremental space positions are considered, u_6 after half a clock cycle is

$$u_6 = X^{53} \sum_{i=0}^{i=3} a_{i6} Y_6^{3-i}. \tag{13}$$

The polynomial u_6 has moved around two -90 degree corners before it reaches l_0. Hence

$$u_6' = X^{2447} Y^{2446\frac{1}{2}} \cdot u_6 = X^{2500} \sum_{i=0}^{i=3} a_{i6} Y^{2449\frac{1}{2}-i}$$

$$= X^{2500} \sum_{i=0}^{i=3} a_{i2} Y_2^{l_0-\frac{1}{2}-i}. \tag{14}$$

An additional half cycle is now necessary to obtain incremental space positions, which results in

$$u_2 = X^{\frac{1}{2}} Y^{\frac{1}{2}} u_6' = X^{2500\frac{1}{2}} \sum_{i=0}^{i=3} a_{i2} Y_1^{l_0-i} \tag{15}$$

indicating that the bubble position a_{01} generated by G is at l_0, $2500\frac{1}{2}$ clock cycles from the prechosen origin of time. In this case it is at l_0 $2500\frac{1}{2}$ cycles after its generation (having lost $\frac{1}{2}$ cycle at the two -90 degree corners of the loop), and it is going in a direction opposite to the direction at the prechosen origin of time, i.e., at the instant of its generation.

III. APPLICATION OF THE ALGEBRA TO (7, 4) HAMMING CODE, MAGNETIC DOMAIN ENCODER

3.1 *Principle of Operation*

A configuration of the encoder[2] is shown in Fig. 2. The four incoming data bits of every code word are generated by G uniformly during 28 t seconds, where t denotes the interval for one rotation of the main magnetic field (see Ref. 2). The data bits are accumulated in adjoining T-bar periods in loop 1 and gated out by g_1 to a duplicator D. One of the two resulting streams is allowed to circulate in loop 3 and the other is divided[3,4] by the generator function

$$g(X) = X^3 + X^2 + 1.$$

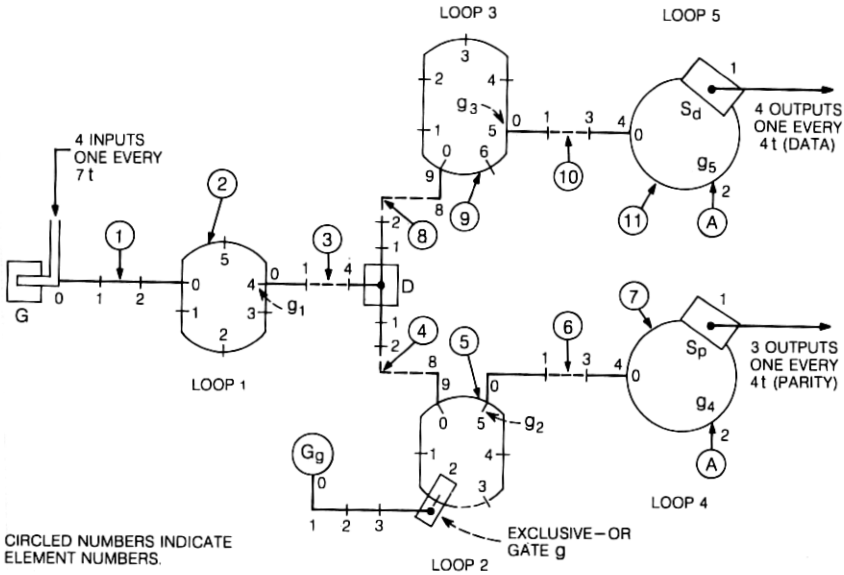


Fig. 2—(7, 4) Hamming encoder with magnetic domains and T-bars.

There are four steps in the division cycle, each step being accomplished as the data stream passes through the gate g. After 4 steps and 3 circulations of the stream in loop 2 (6 periods), the remainder (the three parity bits) are gated by g_2 from loop 2 to loop 4. Meanwhile, the data in the loop 3 (7 periods), also having completed three circulations, is gated by g_3 to loop 5. Loops 4 and 5 have 3 periods each, and the data or parity bit is read by S_d or S_p as the leading bit of the circulating stream in loops 5 or 4 respectively. The gates g_5 and g_4 operate identically by diverting the bit just read by S_d or S_p into the annihilator A. Therefore, the outgoing information (7 bits) is read every $4t$ while the received information (4 bits) is generated every $7t$.

3.2 *The Algebra of the Circuit*

It is necessary to choose an origin of time and proceed in the time domain step by step as the algebra progresses. The exponent of X indicates the number of clock cycles between a prechosen origin of time and the instant of consideration. The binary values of the bit positions are indicated by the values of a, and the location in the circuit is indicated by the subscript of Y (the element number) and the exponent of Y (the particular period in that element).

This circuit operates on a repetitive basis. The information is being continuously received at G, and the coded words are being continuously read at S_d and S_p. The origin of time can thus be chosen at the end of a cycle during which the first binary position of any particular data block is being generated by G.

To facilitate the representation, it is advantageous to divide the algebra of the circuit into a series of operations* corresponding to the subfunctions in the circuit.

If the four positions of the data string generated are represented† and considered 21 clock cycles from the prechosen origin of time,

$$u_1' = X^{21}(a_{01}Y_1^{21} + a_{11}Y_1^{14} + a_{21}Y_1^{7} + a_{31}Y_1^{0})$$
$$= X^{21}\sum_{i=0}^{i=3} a_{i1}Y_1^{7(3-i)}. \tag{16}$$

Operation 1: Transportation of u_1' from generator (element 1) to loop 1 (element 2): With their boundary located at Y_1^3 and Y_2^0, we have (from Sec. 4.3.2 of Ref. 1)

$$u_1'' = X^3Y^3\cdot u_1' = X^{24}\sum_{i=0}^{i=3} a_{i2}Y_1^{7(3-i)+3} \tag{16a}$$

$$u_2' = X_{24}\sum_{i=0}^{i=3} a_{i2}Y_2^{7(3-i)} \tag{17}$$

since $Y_1^{3+\alpha} = Y_2^{\alpha}$.

Operation 2: Looping of u_2' in element 2 with six periods in the loop:

$$u_2 = X^{24}\sum_{i=0}^{i=3} a_{i2}Y_2^{7(3-i)\bmod 6} = X^{24}\sum_{i=0}^{i=3} a_{i2}Y_2^{(3-i)}. \tag{18}$$

Operation 3: Gating the stream u_2 out of loop (element 2) to the path (element 3) between loop and duplicator D: If the gate g_1 is at Y_2^4, then

$$u_3 = X^4Y^4\cdot u_2 = X^{28}\sum_{i=0}^{i=3} a_{i3}Y_3^{(3-i)}, \tag{19}$$

since $Y_2^{4+\alpha} = Y_3^{\alpha}$ when $\alpha \geqq 0$ while the gate g_1 is operational.

DESIGN PARAMETER 1: *Operating the gate* g_1: This gate should be operational for the 25th, 26th, 27th and 28th cycles from the origin of time.

* This type of distinct numbering of operations is suggested when the circuit accomplishes complex functions.

† The prime (s) indicates that the polynomial as such does not represent a bubble stream. But after certain ensuing algebraic operations they will represent observable streams.

Operation 4: Transportation of u_3 from gate to duplicator and its duplication: Let the duplicator D be located at Y_3^4. Then

$$u_4 = X^{32} \sum_{i=0}^{i=3} a_{i4} Y_4^{(3-i)}; \qquad u_8 = X^{32} \sum_{i=0}^{i=3} a_{i8} Y_8^{(3-i)}. \tag{20; 21}$$

Operation 5: Transportation of u_4 and u_8 to loops 2 and 3 respectively: If the encoder design has 9 periods in the paths between D and loops 2 and 3 then

$$u_5 = X^9 Y^9 \cdot u_4 = X^{41} \sum_{i=0}^{i=3} a_{i5} Y_5^{(3-i)}, \tag{22}$$

and

$$u_9 = X^9 Y^9 \cdot u_8 = X^{41} \sum_{i=0}^{i=3} a_{i9} Y_9^{(3-i)}. \tag{23}$$

The binary positions a_{ij} ($i = 0$ through 3 and $j = 1, 2, 3, 4, 5, 8$ and 9) have not undergone any transition points in the circuit. However, in the polynomial u_5, the binary values of bubble positions a_{i5} will undergo definite changes as u_5 is divided by the generator function. The effect of these changes may be represented in the algebra by discriminate use of superscripts for a_{i5} which leads to

$$u_5 = X^{41}(a_{05}^0 Y_5^3 + a_{15}^0 Y_5^2 + a_{25}^0 Y_5^1 + a_{35}^0 Y_5^0). \tag{22}$$

Operation 6: Generation of the divisor polynomial u_{g0} (g0 *for the first time*) *by the generator* G_g : It is seen that the instant of start of the generating cycle for the first bit position is not known from the prechosen origin of time. For this reason we may assume that this interval of time is g0 and determine its value as the algebra progresses.

Four binary positions* are generated by G_g . Three cycles after the generation of the first position the bubble stream may be written as:

$$u_{g0} = X^{g0+3}(a_{0g} Y_g^3 + a_{1g} Y_g^2 + a_{2g} Y_g^1 + a_{3g} Y_g^0). \tag{24}$$

Operation 7: Transportation of u_{g0} to gate g: If the designer has allocated 4 periods between the generator G_g and the gate g, then† (Sec. 4.3.2 of Ref. 1)

$$u'_{g0} = X^{g0+7} \sum_{i=0}^{i=3} a_{ig} Y_g^{7-i}. \tag{24a}$$

---

* See Ref. 2 for the details of magnetic domain encoding and decoding.

† The prime indicates that the polynomial u_g in this equation has already undergone the first step of the division cycle, and the binary values are no longer the same as in the previous polynomial.

Operation 8: Transportation of u_5 to gate g: If the designer has permitted 2 periods between the entry of the loop 2 and the gate g then

$$u_5' = X^2Y^2 \cdot u_5 = X^{43} \sum_{i=0}^{i=3} a_{i5}^0 Y_5^{(5-i)}. \tag{25}$$

DESIGN PARAMETER 2: *The instant of generation of u_{g0}* : If the gate g is located at Y_5^2 and Y_g^4 , and if a_{05} in (25) interacts with a_{0g} in (24a), then they should pass through g during the same clock cycle, and the exponents of X associated with a_{05} in (25) and a_{0g} in (24a) when the corresponding exponents of Y_5 and Y_g are 2 and 4, may be equated yielding $43 - 3 = g0 + 7 - 3$ or

$$g0 = 36 \text{ clock cycles.} \tag{26}$$

The implication of this equation is that the generator G_g must generate u_{g0} (if a_{05} is one, Ref. 2) with its bubble position a_{0g} , exactly during the 36th clock cycle from the prechosen origin of time. This is depicted in Fig. 3a.

Operation 9: Exclusive-or operation of u_5 in (22) and u_{g0} in (24): At this stage of the calculation it is necessary to assign binary values for the a_{05}^0 , a_{15}^0 , a_{25}^0 and a_{35}^0 (which are the same as data bits a_{01} , a_{11} , a_{12} and a_{13} respectively) and let these be chosen as 1111 respectively. The values of a_{0g} , a_{1g} , a_{2g} and a_{3g} are 1101 respectively, if the generator function (see Ref. 3 or 4) of the code has a form as defined in Section 3.1.

Now u_5 may be written as*

$$u_{51} = X^{43} \sum_{i=0}^{i=3} a_{i5}^1 Y_5^{(5-i)}, \tag{27}$$

where $a_{i5}^1 = a_{i5}^0 \oplus a_{ig}$ thus yielding $a_{05}^1 = 0$, $a_{15}^1 = 0$, $a_{25}^1 = 1$ and $a_{35}^1 = 0$, since a_{0g} , a_{1g} , a_{2g} and a_{3g} correspond to 1101.

It can be seen that a_{05}^1 is always zero and it can be dropped from the notation, since it is never again activated in the circuit. This leads to

$$u_{51} = X^{43} \sum_{i=1}^{i=3} a_{i5}^1 Y_5^{(5-i)}. \tag{27}$$

Operation 10: Generation of the generator polynomial u_{g1} (g1 *for the second time*) *by the generator* G_g : Four binary positions are generated. Three cycles after the generation of the first position the binary stream

---

* The second subscript 1, of u, indicates that it has gone through the exclusive-OR operation once.

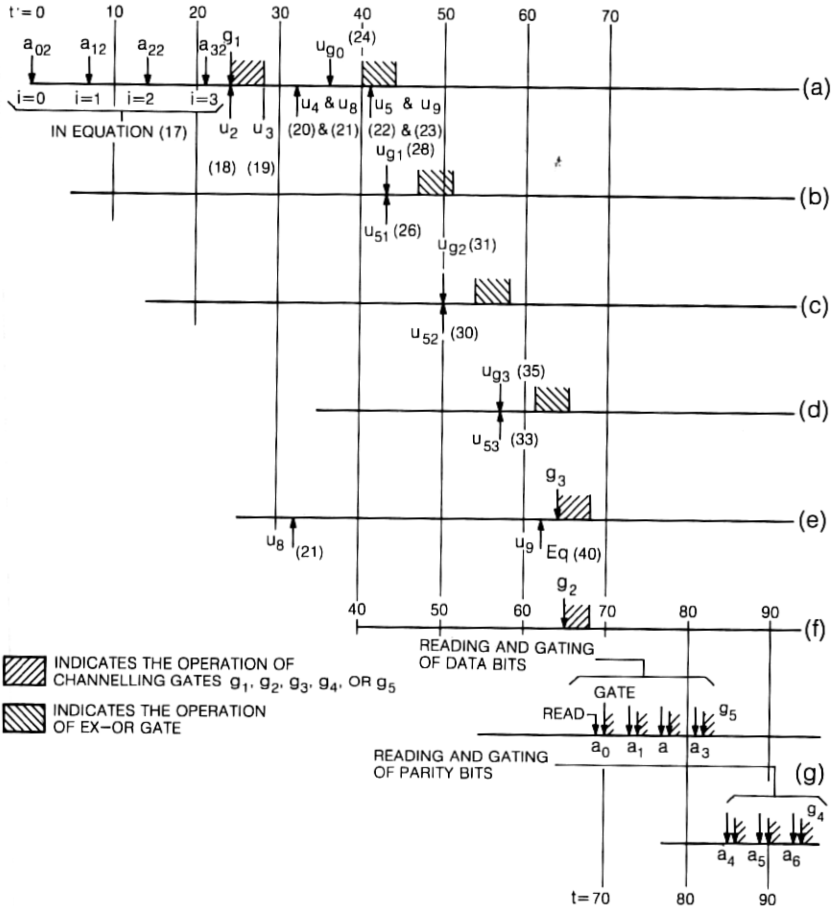


Fig. 3—The timing diagram for the various functions in the (7, 4) Hamming encoder. (a) The generation of data bits a_{01}, a_{11}, a_{21}, a_{31} and subsequent looping, gating, duplicating functions. u_{g0} indicates the generation of the general polynomial by G_g. (b) The generation of u_{g1} by G_g. (c) The generation of u_{g2} by G_g. (d) The generation of u_{g3} by G_g. (e) The polynomial u_9 and its gating to element 10. (f) The parity bits [polynomial u_{54}, eq. (37)] and its gating. (g) The encoded data a_0 through a_6 and its reading by S_d and S_p.

may be represented as

$$u_{g1} = X^{g1+3} \sum_{i=0}^{i=3} a_{ig} Y_g^{(3-i)}, \tag{28}$$

where g1 is another design parameter to be determined as the algebra progresses.

Operation 11: Transportation of u_{g1} *to gate* g:

$$u'_{g1} = X^{g1+7} \sum_{i=0}^{i=4} a_{ig} Y_g^{(7-i)}. \tag{28a}$$

Operation 12: Looping of u_{51} *(26) once in loop 2:* Loop 2 has six periods and after 7 clock cycles we have from Sec. 4.3.3 of Ref. 1

$$u'_{51} = X^7 Y^{7 \bmod 6} \cdot u_5 = X^7 Y \cdot u_{51} = X^{50} \sum_{i=1}^{i=3} a_{15}^1 Y_5^{(6-i)}. \tag{29}$$

DESIGN PARAMETER 3: *The instant of generation of* u_{g1} : If the encoder is to function properly (see Ref. 2) then a_{0g} in (28a) should interact with a_{15}^1 . A calculation similar to that in operation 8 yields g1 = 43. It is to be noted that G_g generates its first binary position a_{0g} in (28a), during the 43rd clock cycle from the prechosen origin of time, which corresponds to the generation of a_{01} by G. (See Fig. 3b.)

Operation 13: Exclusive-or function between u'_{51} *and* u_g : For the correct functioning of the encoder, the generator G_g generates the sequence of four bubble positions 1101, only if the leading bubble position in polynomial u'_{51} is one. In this case it can be seen that a_{15}^1 is zero, so the generator G_g generates a sequence of 4 bubble positions whose binary values are zero thus leading to

$$u_{52} = X^{50} \sum_{i=1}^{i=3} a_{i5}^2 Y^{(6-i)} + a_{45} X^{50} Y_5^2 , \tag{30}$$

where $a_{i5}^2 = a_{i5}^1 \oplus a_{(i-1)g}$ thereby yielding $a_{15}^2 = 0$, $a_{25}^2 = 1$, $a_{35}^2 = 0$, $a_{45}^2 = 0$ since a_{0g} , a_{1g} , a_{2g} and a_{3g} are 1101. Once again, a_{15}^1 being always zero, can be dropped from the equation leading to

$$u_{52} = X^{50} \sum_{i=2}^{i=4} a_{i5}^2 Y_5^{(6-i)}. \tag{30a}$$

The last term in (30) and (30a) is the binary position a_{3g} of (28a) with the exponent of X being 50 since g1 was calculated as 43 clock cycles.

Operation 14: Generation of generator polynomial u_{g2} (g2 *for the third time*) *by the generator* G_g : Four binary positions are generated. Three *cycles:* after the generation of the first binary position, the binary position is written as

$$u_{g2} = X^{g2+3} \sum_{i=0}^{i=3} a_{ig} Y_g^{(3-i)}, \tag{31}$$

where g2 will be evaluated as a design parameter.

Operation 15: Transportation of u_{g2} to gate:

$$u'_{g2} = X^{g2+7} \sum_{i=0}^{i=3} a_{ig} Y_g^{(7-i)}. \tag{31a}$$

Operation 16: Looping u_{52} in (30) in loop 2 once for 7 clock cycles:

$$u'_{52} = X^7 Y \cdot u_{52} = X^{57} \sum_{i=2}^{i=4} a_{i5}^2 Y_5^{(7-i)}. \tag{32}$$

DESIGN PARAMETER 4: *The instant of generation of u_{g2}* : Equating the exponents of X associated with the interacting terms a_{0g} in (31a) and a_{25} in (32) when they pass through the gate g we have g2 = 50 clock cycles.

Operation 17: Exclusive-or function:

$$u_{53} = X^{57} \sum_{i=3}^{i=5} a_{i5}^3 Y_5^{7-i}, \tag{33}$$

where a_{35}^3 , a_{45}^3 , a_{55}^3 may be evaluated as 101 respectively, since a_{1g} , a_{2g} and a_{3g} are 101.

Operation 18: Generation of u_{g3} : Three cycles after the generation of a_{0g} we have

$$u_{g3} = X^{g3+3} \sum_{i=0}^{i=3} a_{ig} Y_g^{3-i}. \tag{34}$$

Operation 19: Transportation of u_{g3} :

$$u'_{g3} = X^{g3+7} \sum_{i=0}^{i=3} a_{ig} Y_g^{7-i}. \tag{35}$$

Operation 20: Looping of u_{53} in loop 2

$$u'_{53} = X^{64} \sum_{i=3}^{i=5} a_{i5}^3 Y_5^{(8-i)}. \tag{36}$$

DESIGN PARAMETER 5: *The instant of generation of u_{g3}* : The value of g3 can be calculated as 57 clock cycles (by equating the exponents of X in (35) and (36); see also Fig. 3d).

Operation 21: Exclusive-or function between u'_{53} and u'_{g3} :

$$u_{54} = X^{64} \sum_{i=4}^{i=6} a_{i5}^4 Y_5^{(8-i)}, \tag{37}$$

where a_{45}^4 , a_{55}^4 , a_{65}^4 correspond to 111 respectively (since a_{1g} , a_{2g} and a_{3g} are 101, thus leading to the parity bits for the (7, 4) Hamming code with the four data bits as 111 and 1.

Operation 22: Gating of parity bits from loop 2: Now the parity bits at Y_5^4, Y_5^3 and Y_5^2 can be channeled out of the loop 2 into the element 6 by the action of the gate g_2 located at Y_5^5. This function may be represented as

$$u_5' = X^3Y^3 \cdot u_{54} = X^{67} \sum_{i=4}^{i=6} a_{i5} Y_5^{11-i}$$
$$= X^{67}(a_{45}Y_5^7 + a_{55}Y_5^6 + a_{65}Y_5^5), \tag{38}$$

since $Y_5^{5+\alpha} = Y_6^{\alpha}$ when $\alpha > 0$ thereby leading to

$$u_6 = X^{67}(a_{46}Y_6^2 + a_{56}Y_6^1 + a_{66}Y_6^0). \tag{39}$$

DESIGN PARAMETER 6: *The operation of the gate* g_2 : The gate is located at Y_5^5 and Y_6^0 and it can be seen that a_{46}, a_{56} and a_{66} reach Y_6^0 when the exponent of X is 65, 66 and 67, indicating that the gate must operate during these three cycles to channel the bits for loop 2.

*Operation 23: Looping of u_9 three times:** The data stream u_9 has been circulating in loop 3 for (3 × 7) clock cycles leading to

$$u_9 = X^{41+21} \sum_{i=0}^{i=3} a_{i9} Y_9^{(24-i) \bmod 7} = X^{62} \sum_{i=0}^{i=3} a_{i9} Y_9^{3-i}. \tag{40}$$

Operation 24: Transportation of u_9 to gate g_3 : This gate is located at Y_9^5 to match the location of the gate g_2 at Y_5^5 (see operation 22). If the gate operates at the appropriate time for four clock cycles, then the polynomial representing the stream in the path between loop 3 and loop 5 is

$$u_{10}' = X^5Y^5 \cdot u_9 = X^{67} \sum_{i=0}^{i=3} a_i Y_9^{(8-i)}. \tag{41}$$

But $Y_9^{5+\alpha} = Y_{10}^{\alpha}$ during gating of g_3 and therefore we have

$$u_{10} = X^{67} \sum_{i=0}^{i=3} a_i Y_{10}^{(3-i)}.$$

DESIGN PARAMETER 7: *The operation of gate* g_3 : The polynomial u_{10} in (41) indicates that the gate g_3 should operate when the bubble positions a_0, a_1, a_2, and a_3 are at Y_{10}^0. Further, when the exponents of X are exactly 64, 65, 66 and 67, the four binary positions a_0, a_1, a_2 and a_3 are in the gate g_3. Hence, the operation of this gate must coincide with the 64th, 65th, 66th and 67th clock cycles from the prechosen origin of time.

* This operation takes place during the 26 clock cycles allocated for operations 6 through 22 for the polynomials u_5, u_{51}, u_{52}, u_{53} and u_6.

The single clock cycle between the operation of gates g_3 and g_2 (Fig. 3e and f) is due to the incomplete fourth rotation of the parity bits u_6. These are gated out by g_3 after 3 rotations and after 4 steps of the division. (See Ref. 2.)

Operation 25: Transportation of data and parity bits to loops 5 and 4: If there are 4 periods in elements 10 and 6, then

$$u'_{11} = X^4 Y^4 \cdot u_{10} = X^{71} \sum_{i=0}^{i=3} a_i Y_{11}^{3-i}, \tag{42}$$

and

$$u_7^* = X^4 Y^4 \cdot u_6 = X^{71} \sum_{i=4}^{i=6} a_i Y_7^{(3-i)}. \tag{43}$$

DESIGN PARAMETER 8: *Reading of data bit a_0 and its gating by g_5 into the annihilator* A: If the sensor S_d is located at Y_{11}^1, then a_0 will be at Y_{11}^1 at X^{69}, indicating that the sensor should be read during the 69th clock cycle from the prechosen origin of time. If the gate g_5 is located at Y_{11}^2, then a_0 is at Y_{11}^2 at X^{70} and it should operate during the 70th cycle. After the gating of a_0 by g_5, we have

$$u_{11} = X^{71} \sum_{i=1}^{i=3} a_i Y_{11}^{(3-i)}. \tag{44}$$

Operation 26: Looping for 2 clock cycles: After 2 clock cycles, a_1 will be at Y_{11}^1, and it can be read again, since

$$u_{11} = X^{73} \sum_{i=1}^{i=3} a_i Y_{11}^{(5-i) \bmod 3}. \tag{45}$$

DESIGN PARAMETERS 9, 10, 11: *Reading and gating of a_1, a_2 and a_3 by S_d and g_5 respectively:* The sensor S_d is read during the 73rd clock cycle and gate g_5 operates during the 74th clock cycle from the origin of time. More operations of the type 27 indicate that S_d should read a_2 during the 77th clock cycle, and g_5 should gate a_2 during the 78th clock cycle, S_d should read a_3 during the 81st clock cycle, and g_4 should gate a_3 during the 82nd clock cycle. These details are plotted in Fig. 3g.

Operation 27: Looping† of parity bits in loop 4: After 10 clock cycles the parity bits in loop 4 may be represented as

---

* It is no longer necessary to carry a second subscript for a, since none of the binary values in any of the polynomials change in the remainder of the circuit.

† This operation takes place during the 10 (i.e., −2 for a_0, and 4 for a_1, a_2 and a_3 each) clock cycles allocated for reading, gating and looping of the four data bits a_0, a_1, a_2 and a_3.

$$u_7 = X^{71+10}(a_4Y_7^{12 \bmod 3} + a_5Y_7^{11 \bmod 3} + a_6Y_7^{10 \bmod 3})$$
$$= X^{81}(a_4Y_7^0 + a_5Y_7^2 + a_6Y_7^1). \quad (46)$$

Operation 28: Looping of u_7 : After 4 clock cycles*

$$u_7 = X^{85}(a_4Y_7^1 + a_5Y_7^0 + a_6Y_7^2). \quad (47)$$

DESIGN PARAMETERS 12, 13 AND 14: *Reading and gating of* a_4 , a_5 *and* a_6 *:* It is seen that a_4 is at Y_7^1 (location of S_p) during the 85th clock cycle. The gate g_4 at Y_7^2 should divert a_4 during the 86th clock cycle. More operations of type 28 indicate that S_p should read a_5 during the 89th clock cycle, and g_4 should gate a_5 during the 90th clock cycle, S_p should read a_6 during 93rd clock cycle, and g_4 should gate a_6 during the 94th clock cycle. These details are also plotted in Fig. 3g.

A block of data is thus completely processed by the circuit after the 94th clock cycle, and the position of every bit of information may be accurately predicted during any prechosen clock cycle. The design parameters are also accurately determined by the analysis.

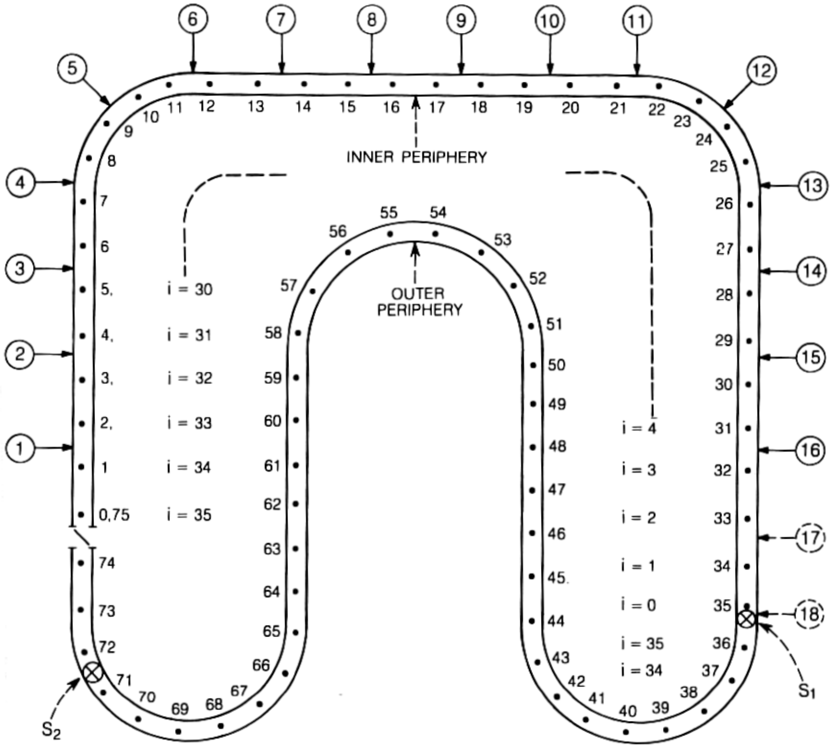
VI. APPLICATION OF THE ALGEBRA TO 16 LINE LDC MAGNETIC DOMAIN LINE SCANNER

4.1 *Principle of Operation*

The circuit for the line scanner is shown in Fig. 4. Sixteen inputs, 1 through 16, carry telephone line currents in a loop for each line. When the line current is not sensed (i.e., on-hook status) during a scan interval, a bubble in the position (1–2), (3–4), $\cdots$ (31–32) of a 75-position loop is moved from the outer periphery to the inner periphery of the loop, thus effecting a *L*ateral *D*isplacement *C*oding.

The laterally displaced bubble positions are moved under two sensors S_1 and S_2 . The spacing between these is arranged to sense the status of a particular line at two instants of time. When the sensor S_2 is sensing the contents of a certain cell coded by the circuit in line j, (j = 1 through 16) at an instant t, then the sensor S_1 is sensing the contents of the cell Y coded by the current in the same line j at $t + \delta$. (The value of δ is the duration required to move the bubbles from sensor S_1 to S_2 and is also the scanning interval of the lines.) When there is no discrepancy between the readings of S_1 and S_2 , then there is no change in status of the line (on hook or off hook) during the scanning interval and vice versa. The status of the jth line is coded during the movement of the

* It is interesting to note that the extra clock cycle between the gating of g_3 and g_2 (design parameter 7) is really necessary for correct functioning at this stage.



i DENOTES THE BUBBLE POSITION IN STREAM.
0-74 DENOTES THE LOCATION NUMBERS.

S_1 AND S_2 ARE SENSORS TO DETECT ON-HOOK STATUS OF CODED BUBBLE POSITIONS.

Fig. 4—The schematic diagram of 16 Telephone-Line-Scanner.

bubble across the cell from a position $(2j - 1)$ to $2j$. A conductor carrying a single phase current around the loop propagates the bubble positions around its periphery at a uniform speed. Further, when the bubble positions pass through the cell 74–0, they are all positioned towards the outer periphery of the loop, and are reset ready to be coded again.

4.2 *Algebra of the Circuit*

The origin of time may be chosen at the start of a coding cycle which repeats every 36 clock cycles. The lateral displacement coding occurs during the single clock cycle immediately after all the cells 0 through 35 contain bubbles at the outer periphery of the loop. The start of this cycle would then constitute the origin of time, and the polynomial representing the stream in this section of the loop is*

$$u_{00} = X^0 \sum_{i=0}^{i=35} a_i Y^{35-i}, \tag{48}$$

where $i = 0$ denotes the leading bubble position in the stream; and a_i is the uncoded status of the bubble position. Next consider the bubble stream in the section 36 to 71 of the loop. This stream carries the status of each line from the last cycle, and the polynomial representing this stream at the prechosen origin of time is

$$u_{10} = X^0 \sum_{i=0}^{i=35} a_i Y^{71-i}, \tag{49}$$

where i indicates the bubble position in the stream. Similarly, the polynomial u_2 between 72 and 74 may be written as:

$$u_{20} = X^0 \sum_{i=33}^{i=35} a_i Y^{107-i}. \tag{50}$$

The value of i ranges from 33 to 35 since there are only three locations 72, 73, and 74 to accommodate the last three bubble positions of the stream coded during their own coding cycle. The contents of the entire loop at the origin of time may be written as:

$$\begin{aligned} u_0 &= u_{00} + u_{10} + u_{20} \\ &= X^0\left(\sum_{i=0}^{i=35} a_i Y^{35-i} + \sum_{i=0}^{i=35} a_i Y^{71-i} + \sum_{i=33}^{i=35} a_i Y^{107-i}\right). \end{aligned} \tag{51}$$

Now consider the movement of the bubble stream represented by (51); for m ($m \leqq 36$) clock cycles, the resulting polynomial according to Sec. 4.3.2 of Ref. 1 is:

$$\begin{aligned} u_1 = X^m\Bigg(&\sum_{i=0}^{i=m-1} a_i Y^{m-1-i} + \sum_{i=0}^{i=35} a_i Y^{35+m-i} \\ &+ \sum_{\substack{i=0 \\ \text{or} \\ i=m-3}}^{i=35} a_i Y^{71+m-i} + \sum_{i=33+m}^{i=35} a_i Y^{107-i}\Bigg) \\ = {}& u_{01} + u_{11} + u_{21} + u_{31}, \end{aligned} \tag{52}$$

where u_{01} represents the first m binary positions of u_{00} being generated in the section 0 through 35 of the loop; u_{11} results from the translatory

* There is no need to subscript Y since there is only one element (the loop) in the circuit.

movement (Sec. 4.3.2 of Ref. 1) of u_{00} in (51), and u_{21} results from the movement of u_{10} in (51). The lower limit of i in u_{21} should be chosen according to the value of m. When m is less than 3, the value of i equals 0 is appropriate, since the leading bubble position a_0 is within the maximum number of binary bubble positions (i.e., 74) in the loop. When m exceeds three, the leading bubble a_0 of u_{10} in (51) is transformed as the fourth term of u_{01} in (52). The polynomial u_{31} is the transformation of u_{20} when m is less than 3. When m exceeds two, u_{31} drops out of (52), since the lower limit of i exceeds the upper limit of 35.

Examine the polynomial u_1 in (52), when $m = 0$, u_{01} drops out of the equation, and u_{11} , u_{21} and u_{31} will assume the roles of the polynomials u_{00} , u_{10} and u_{20} in (51). Next observe the polynomial u_1 in (52); when m reaches a value of 36, u_{01} , u_{11} and u_{21} will assume the roles of u_{00} , u_{10} and u_{20} in (51), and u_{31} drops out of the equation. In essence, we have two cyclic processes taking place simultaneously, the first one being in the time dimension and repeating every 36 clock cycles, the second one being in the spatial dimension and repeating every 75 periods. The effect of the first cyclic process may be eliminated by always considering m as (m mod 36). The effect of the second one may be eliminated by always considering the exponent (e) for Y as (e mod 75).

Table I relates the values of the exponents of Y for various values of i and m in the polynomials u_{01} , u_{11} , u_{21} , and u_{31} . It also indicates the locations of the first and last bubble positions of streams represented by these polynomials.

4.4 *Implication and Use of the Representation for the Line Scanner*

4.4.1 *Prediction of Bubble Positions*

Consider the tenth ($i = 10$) bubble position in a data stream coded for twenty ($m_1 = 20$) cycles and propagated for ninety ($m = 90$) cycles.

The initial position under consideration is $a_{10}X^{20}Y^{l}$. Table I indicates that this term exists in u_{11} with a value of $l = 45$ yielding the location of this position. When this location is propagated for 90 cycles, the new position is $Y^{(45+90) \bmod 75}$, i.e., Y^{60}; and the corresponding value of m^* is (20 + 90) mod 36, i.e., 2 cycles. The bubble position is then $a_i X^2 Y^{60}$. The only positive value of i which satisfies the constraints on the exponents of both X and Y is 13; and the individual term denoting this bubble position lies in the polynomial u_{21} in (52). This implies that if the original position in u_{11} is $a_{10}X^{20}Y^{45}$, then after 90

* When the exponent of Y is less than $(m - 1)$, it should be concluded that the term is in u_{01}, (see Table I) and is in the dead interval of the circuit.

TABLE I—LOCATION OF BUBBLE STREAMS

Limits of the exponents of Y in the polynomials u_{01}, u_{11}, u_{21}, and u_{31} in (52)

m or m mod 36	$u_{01} = X^m \sum_0^{m-1} a_i Y^{m-1-i}$	$u_{11} = X^m \sum_0^{35} a_i Y^{35+m-i}$	$u_{21} = X^m \sum_{0 \text{ or } m-3}^{35} a_i Y^{71+m-i}$	$u_{31} = X^m \sum_{33+m}^{35} a_i Y^{107-i}$
0	—	35 to 0	71 to 36	
		$(2 = 0$ to $i = 35)$	$(i = 0$ to $i = 35)$	74, 73, 72
1*	$i = 0$	36 to 1	72 to 37	74, 73
		$(i = 0$ to $i = 35)$	$(i = 0$ to $i = 35)$	
2	1,0	37 to 2	73 to 38	74
3	2 to 0	38 to 3	74 to 39	—
4	3 to 0	39 to 4	74 to 40	—
			$(i = \text{m} - 3)$	
20	19 to 0	55 to 20	74 to 56	—
m	$(m - 1)$ to 0	$35 + m$ to m	74 to $36 + m$	—
			$(i = m - 3)$	
34	33 to 0	69 to 34	74 to 70	—
35	34 to 0	70 to 35	74 to 71	—
36†	—	35 to 0	71 to 36	74, 73, 72

* 0–1 constitutes the coding cycle
† See $m = 0$ values

clock cycles the final position in u_{21} is $a_{13}X^2Y^{60}$ serving as the thirteenth bubble position in the data stream. Further, during the coding cycle i.e., as the exponent of X is changing from 0 to 1, the location of the initial position is at Y^{45-20}, i.e., at Y^{25} serving as an active bubble carrying the status of the 13th line in Fig. 4. During its next coding cycle, the same bubble position is at Y^{60-2}, or at Y^{58} corresponding Y^{58-36}, or at Y^{22} during coding, serving as an inactive bubble position between lines 11 and 12 in Fig. 4.

4.4.2 *Detection of the Nonexistence of Bubbles*

For the correct functioning of the line scanner, all the bubble positions should carry bubbles. Sixteen lines are actively used; and the status of these lines is carried by bubbles in positions 1, 3, $\cdots$ 31. The bubble positions 33 and 35 always carry the status of two fictitious lines (on on-hook and next off-hook) to check the correct operation of the overall magnetic and electronic circuitry. Generally, it is also desirable to check if all bubble positions do carry bubbles by sensors S_1 and S_2 which are capable of detecting only the off-hook status of lines. Such an inspection can be effected when each of the bubble positions is arranged to periodically occupy the position 35, which should always carry the off-hook status. All the bubbles are moved to this status as they traverse the position 74.

Examine the bubble position at Y^{35} just prior to coding. After 36 cycles (i.e., next coding) the bubble position now at $Y^{(35-36) \bmod 75}$, i.e., Y^{74} will occupy Y^{35}. In general, after n coding cycles, the present position $Y^{(35-36n) \bmod 75}$ will occupy Y^{35}. The exponent of Y generates a series 35, 74; 38, 2; $\cdots$, $(35 + 3(n)/2)$, $74 + (3(n - 1)/2) \bmod 75$; $\cdots$, 32, 71; repeating every time n reaches 25. This indicates that the bubble positions now occupying Y^{36}, Y^0 $\cdots$, etc., Y^{37}, Y^1 $\cdots$, etc., never occupy Y^{35} at any finite value of n. To eliminate this condition, the electronic circuitry may be programmed* to delay the coding by one clock cycle every 25 coding cycles. This leads to a new location series: 35, 74, 82, 2, $\cdots$, 32, 71; 36, 0, 39, 3, $\cdots$, 33, 72; 37, 1, 40, 4, $\cdots$, 34, 73; 38, 2, $\cdots$, etc. Alternatively, if the loop is designed with 73, 109, 145, etc., periods, then the need for building additional delay circuits will not be necessary. With 73 periods, every bubble position will be located at Y^{35} every 73 coding cycles, or every 2628 (i.e., 73×36) cycles and so on.

* The general concept of shuffling periodically was suggested by D. Denburg.

V. CONCLUSIONS

The polynomial algebra is a flexible mathematical tool available for the step by step design of conceived circuits, and for the sequential verification of their operation. Various design parameters may be calculated accurately.

When the operations of numerous circuits are to be synchronized, the algebra provides an excellent insight into their combined functioning. The effect of errors or defects of certain sections of the overall circuitry may also be accurately analyzed by the algebraic modeling of the circuit operation.

REFERENCES

1. Ahamed, S. V., "Multidimensional Polynomial Algebra for Bubble Circuits," B.S.T.J., this issue, pp. 1535–1558.
2. Ahamed, S. V., "The Design and Embodiments of Magnetic Domain Encoders and Single Error Correcting Decoders for Cyclic Block Codes," B.S.T.J., *51*, No. 2 (February 1972), pp. 461–485.
3. Peterson, W. W., *Error Correcting Codes*, MIT Press, Cambridge, Mass., 1968.
4. Lucky, R. W., Salz, J., and Weldon, Jr., E. J., *Principles of Data Communication*, McGraw-Hill Book Co., 1967.