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Pulse Propagation in a Two-Mode Waveguide

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The results of an earlier paper, describing pulse propagation in multimode dielectric waveguides with random coupling, are specialized to the two-mode case. Because of their greater simplicity, the results for this special case provide more insight into the mechanism of pulse shortening due to mode coupling. The two-mode theory yields a formula for the width of a pulse carried by coupled guided modes that is found to hold also for four modes, so that it may be true for an arbitrary number of modes. This formula [eq. (23)] contains only the measurable distance required to establish the steady-state power distribution and the length of uncoupled pulses. The pulse length formula is identical with Personick's important result. Our treatment suggests that the characteristic length appearing in this formula may be accessible to measurement.

I. INTRODUCTION

In a series of earlier papers, the theory of multimode propagation in dielectric waveguides was analyzed with the help of stochastic coupled power equations.[1-3] The propagation of Gaussian-shaped pulses in a waveguide with randomly coupled modes was treated quite generally for N modes in Ref. 3. The theory was based on the following form of the stochastic coupled power equations:

$$\frac{\partial P_\nu}{\partial z} + \frac{1}{v_\nu}\frac{\partial P_\nu}{\partial t} = -\alpha_\nu P_\nu + \sum_{\mu=1}^{N} h_{\nu\mu}(P_\mu - P_\nu). \tag{1}$$

P_ν is the average power in mode ν, v_ν the group velocity, α_ν is the attenuation coefficient of mode ν in the absence of coupling to other guided modes, and $h_{\nu\mu}$ is the power coupling coefficient. To second order of perturbation theory, and assuming a Gaussian shape (in time) of the input pulse, the solution of (1) can be expressed as:[3]

$$P_\nu(z, t) = \sum_{i=1}^{N} \frac{2\tau}{\Delta t_i} k_i B_\nu^{(i)} e^{-\alpha_0^{(i)} z} \exp\left\{-\left(\frac{t - z/v}{\Delta t_i/2}\right)^2\right\}. \tag{2}$$

Δt_i, the width of the ith Gaussian function in (2), is given by

$$\Delta t_i = 2(\tau^2 + 4\alpha_2^{(i)} z)^{\frac{1}{2}}. \tag{3}$$

The input pulse with half width τ and amplitude G_ν is assumed to be

$$P_\nu = G_\nu \exp\left(-\frac{t^2}{\tau^2}\right). \tag{4}$$

The coefficient k_i appearing in (2) is determined by the input pulse

$$k_i = \sum_{\nu=1}^{N} G_\nu B_\nu^{(i)}. \tag{5}$$

The vectors with components $B_\nu^{(i)}$ and the parameters $\alpha_o^{(i)}$ are the ith eigenvectors and eigenvalues of an algebraic eigenvalue problem defined in Refs. 2 and 3. The parameter $\alpha_2^{(i)}$ is the second-order perturbation of the eigenvalue $\alpha_o^{(i)}$ and is defined as follows:

$$\alpha_2^{(i)} = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{\left\{\sum_{\nu=1}^{N}\left(\frac{1}{v_\nu} - \frac{1}{v}\right) B_\nu^{(i)} B_\nu^{(j)}\right\}^2}{\alpha_o^j - \alpha_o^i}. \tag{6}$$

v is the average group velocity.

This approximate theory of pulse propagation in multimode waveguides holds for random coupling between the guided modes under the assumption that the correlation length of the coupling function is short compared to the distance over which the mode power P_ν changes appreciably.

II. APPLICATION TO THE TWO-MODE CASE

In its full generality, the theory of multimode pulse operation is hard to evaluate. Computer solutions are being provided in Refs. 2 and 3. Here we want to derive expressions for the special case of two modes, that allows us to gain more insight into the meaning of the theory. The two-mode case has been treated previously by several authors.[4,5] Our results are thus not all new.

For two modes, we write $h_{12} = h_{21} = h$. We now have to solve the eigenvalue problem[2,3]

$$\left.\begin{aligned} (\alpha_o - \alpha_1 - h)B_1 + hB_2 &= 0 \\ hB_1 + (\alpha_o - \alpha_2 - h)B_2 &= 0 \end{aligned}\right\}. \tag{7}$$

The equation system (7) has the solution

$$\alpha_o^{(1)} = h + \frac{\alpha_1 + \alpha_2}{2} - \left[\frac{(\alpha_2 - \alpha_1)^2}{4} + h^2\right]^{\frac{1}{2}} \tag{8}$$

$$\alpha_o^{(2)} = h + \frac{\alpha_1 + \alpha_2}{2} + \left[\frac{(\alpha_2 - \alpha_1)^2}{4} + h^2\right]^{\frac{1}{2}} \tag{9}$$

for the first and second eigenvalue. The two components of the first eigenvector are

$$B_1^{(1)} = \frac{h}{\left[\frac{(\alpha_2 - \alpha_1)^2}{4} + h^2\right]^{\frac{1}{4}}\left\{\alpha_1 - \alpha_2 + 2\left[\frac{(\alpha_2 - \alpha_1)^2}{4} + h^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}} \tag{10}$$

$$B_2^{(1)} = \frac{\left\{\alpha_1 - \alpha_2 + 2\left[\frac{(\alpha_2 - \alpha_1)^2}{4} + h^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}}{2\left[\frac{(\alpha_2 - \alpha_1)^2}{4} + h^2\right]^{\frac{1}{4}}}. \tag{11}$$

The components of the second eigenvector can be expressed in terms of the components of the first eigenvector.

$$B_1^{(2)} = B_2^{(1)} \qquad B_2^{(2)} = -B_1^{(1)}. \tag{12}$$

III. DISCUSSION OF THE TWO-MODE CASE

In the special case $\alpha_1 = \alpha_2 = 0$ we have the eigenvalues

$$\alpha_o^{(1)} = 0 \tag{13}$$

and

$$\alpha_o^{(2)} = 2h, \tag{14}$$

while the eigenvectors are

$$B_1^{(1)} = \frac{1}{\sqrt{2}} \qquad B_2^{(1)} = \frac{1}{\sqrt{2}} \tag{15}$$

and

$$B_1^{(2)} = \frac{1}{\sqrt{2}} \qquad B_2^{(2)} = -\frac{1}{\sqrt{2}}. \tag{16}$$

There are several interesting features apparent in this special solution. In the absence of loss, both modes carry equal power. We see immediately from (15) and (16) that the sum of the squares of the components of each eigenvector adds up to unity, while the inner product of the two vectors vanishes. This is a general property that is also shared by the solutions (10) through (12).

The "loss coefficient" $\alpha_o^{(1)}$ of the term with $i = 1$ in (2) vanishes, so that this term does not decrease in amplitude as the pulse moves along the z-axis. The "loss coefficient" $\alpha_o^{(2)}$ of the second term of (2) (with $i = 2$) is equal to twice the coupling coefficient h, so that this term becomes vanishingly small for large values of z. This too is a general feature of (2). The lowest order eigenvalue $\alpha_o^{(1)}$ is smaller than all other eigenvalues, so that only the first term of the series (2) remains for large values of z while all the other terms have become vanishingly small. Even though $\alpha_o^{(1)}$ is not zero in the general (lossy) case, the multimode waveguide always reaches a steady state which is described by the first term of the series expansion in (2), provided that the modes are coupled. Only in the lossless case do we find $\alpha_o^{(1)} = 0$. It is noteworthy that the "loss terms" $\exp(-\alpha_o^{(i)} z)$ $(i > 1)$ all become vanishingly small even in the absence of losses. A steady-state distribution of mode power versus mode number is thus established that is independent of the initial excitation of the waveguide. The decay of the higher-order terms in (2) does not necessarily indicate power loss. We see indeed, from the solution (16), that the sum of the components of the second eigenvector adds up to zero indicating that no power is carried by the second term ($i = 2$) of (2) in the absence of loss. The individual terms of (2) must not be confused with waveguide modes. They have no independent physical meaning except for the first term with $i = 1$, which is the steady-state power distribution. It is apparent that it is sufficient to study the behavior of the first term in (2) alone, since all other terms (the second term is the only other term in the two-mode case) become negligible for large values of z.

We can easily define a characteristic distance that is required for the steady state to establish itself. Once the exponential factor $\exp(\alpha_o^{(1)} - \alpha_o^{(2)})z$ has become small, the steady state is reached. We thus define the characteristic length as follows

$$L_s = \frac{\kappa}{\alpha_o^{(2)} - \alpha_o^{(1)}}. \tag{17}$$

The parameter κ is a number of order unity. For $\kappa = 1$ we have $\exp(\alpha_o^{(1)} - \alpha_o^{(2)})L_s = 1/e$. Thus $\kappa = 1$ is too small to consider the steady state as reached. However, we can still define L_s by (17) with $\kappa = 1$. If we use $\kappa = 4.6$, we have $\exp(\alpha_o^{(1)} - \alpha_o^{(2)})L_s = 0.01$. This number is small enough to consider the second term in (2) as negligibly small. For the two-mode case we obtain from (14) and (17) for the case of low losses

$$L_s = \frac{\kappa}{2h}. \tag{18}$$

Finally, we study the steady-state pulse width which follows from (3), with $i = 1$. We also neglect τ in this equation, assuming that the pulse has spread to a size much longer than the input pulse. From (6), (8), (9), (10), and (11) we find

$$\Delta t_1 = 4\Delta T \frac{h}{[(\alpha_2 - \alpha_1)^2 + 4h^2]^{\frac{3}{4}}} \frac{1}{\sqrt{L}}. \tag{19}$$

We used $z = L$, with L designating the length of the waveguide. The factor ΔT, the width of the pulse in the absence of coupling, is defined as

$$\Delta T = \left(\frac{1}{v_2} - \frac{1}{v_1}\right)L. \tag{20}$$

Compared to the width ΔT of the uncoupled modes, the pulse length in case of coupled modes is improving with length. The pulse width formula can again be considered in the two limiting cases. If $\alpha_2 - \alpha_1 \ll h$ we have

$$\Delta t_1 = \sqrt{2} \frac{\Delta T}{(hL)^{\frac{1}{2}}}. \tag{21}$$

This formula shows clearly that the pulse length shortens with increased coupling strength. In the other extreme, $h \ll | \alpha_2 - \alpha_1 |$, we have

$$\Delta t_1 = 4\Delta T \frac{h}{[(\alpha_2 - \alpha_1)^3 L]^{\frac{1}{2}}}. \tag{22}$$

It appears strange at first that the pulse length now increases as the coupling strength is increased. However, in this mode of operation, the pulse length is primarily determined by the differential loss of the two modes. If both modes travel uncoupled, one will die out while the other carries the pulse all by itself. In this case, the pulse width is determined only by the dispersion of the surviving mode, which is not included in our theory. For $h = 0$, we thus obtain a vanishing pulse length. As the coupling is increased, power is flowing from the lower loss mode to the high loss mode so that the pulse width is increased by the different delay time of each mode. It is thus clear that a small amount of coupling causes the pulse to lengthen.

Finally, we combine the formula (21) for the low loss case with the formula (18) defining the characteristic length that indicates where the steady state is reached. We thus obtain the interesting result

$$\Delta t_1 = \frac{2}{\sqrt{\kappa}} \Delta T \left(\frac{L_s}{L}\right)^{\frac{1}{2}}. \tag{23}$$

The factor in front of this equation becomes unity if we use $\kappa = 4$. We have seen that this value is large enough to ensure that steady state is essentially reached. Equation (23) has been derived by Personick.[4] We see from our derivation that the characteristic length l_c in Personick's formula can be interpreted as the length that is required to reach the steady-state power distribution.

Equation (23) was derived for the case of only two modes. It is tempting to use this equation also for the multimode case. In order to test the mode dependence of this formula, I solved the four-mode problem under the assumption that all off-diagonal elements of $h_{\nu\mu}$ vanish with the exception of the elements directly adjacent to the main diagonal. All non-vanishing elements of $h_{\nu\mu}$ were set equal to the same value h. For the lossless case, and assuming that the inverses of the group velocities are evenly spaced, the following formula was obtained for the four-mode case.

$$\Delta t_1 = 0.79 \frac{2}{\sqrt{\kappa}} \Delta T \left(\frac{L_s}{L}\right)^{\frac{1}{2}}. \tag{24}$$

L_s is again defined as the length required to achieve the steady state. ΔT is the length of the uncoupled signal for the four-mode case. Since (23) and (24) are essentially identical [(24) is even slightly more favorable], it might be assumed that (23) may hold independently of mode number.

Equation (23) was derived for the case where the coupling coefficient h is larger than the loss coefficients α_1 or α_2. In the opposite case, where the losses determine the rate at which the power distribution approaches the steady state, no simple relationship exists between L_s and Δt. Formula (23) is quite useful for estimating the length of the Gaussian pulse if the distance L_s, at which steady state is reached, can be observed. If it is known that radiation losses are small compared to the coupling coefficient h, the conditions exist for which (23) was derived. However, it may well be that the region of dominance of radiation losses over coupling strength is different for different modes. Experiments have shown that the coupling mechanism in optical fibers consists of two parts.[6] A Rayleigh-type background with a wide mechanical Fourier spectrum is responsible for most of the radiation losses, while a very sharp peak at zero frequencies of the mechanical power spectrum is responsible for most of the coupling between guided modes. The broad spectrum causes more radiation loss for higher-order modes, provided that the coupling is caused by core-cladding interface irregularities. The narrow peak at zero mechanical frequencies couples lower-order modes much more strongly than high-order modes, because of the closer spacing (in β-space) of the low-order modes.[3] The combined effect of these two spectral regions causes a steady-state distribution that favors the lower-order modes. In this situation, it may still be possible to estimate the pulse performance of the multimode waveguide by ignoring those modes that do not carry power in the steady-state distribution and interpret ΔT as the pulse length that would be obtained by the remaining modes in the absence of coupling. The steady-state distance L_s is best observed by launching only the lowest-order modes and measuring the distance that is required until the power versus mode number distribution ceases to change its shape.

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