THE BELL SYSTEM TECHNICAL JOURNAL

DEVOTED TO THE SCIENTIFIC AND ENGINEERING
ASPECTS OF ELECTRICAL COMMUNICATION

Volume 52 | May–June 1973 | Number 5

Distortion Produced by Band Limitation of an FM Wave

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(Manuscript received November 21, 1972)

The bandwidth required to transmit an FM wave is related to how much distortion is allowed in the signal. Here expressions are developed for the distortion (interchannel interference) produced when an FDM-FM wave passes through an ideal filter. The signal is represented by a flat (PM) band of Gaussian noise. The formulas obtained hold only for small rms frequency deviation, but fortunately this is an important case in microwave communication systems. The theoretical expressions agree well with Monte Carlo results published recently by Anuff and Liou.

I. INTRODUCTION

When a frequency-modulated wave passes through a filter, distortion is produced in the signal by nonlinearity in the filter phase shift (usually the chief offender) and by the filter attenuation. Much effort has been spent in devising methods for computing this distortion.

A related problem is "What radio frequency bandwidth is required to transmit a given FM wave?" An approximate answer, known as "Carson's rule," states that the required bandwidth $2f_h$ is given by[1]

$$2f_h = 2B + 2D_{\max}, \tag{1}$$

where B is the bandwidth of the baseband signal and $D_{\max}$ is the maximum amount the instantaneous frequency deviates from the carrier frequency. Note that (1) implies a conventional FM system. This is the only type we shall consider in this paper. We shall not be concerned with single-sideband FM or other schemes for reducing the rf bandwidth.

Carson's rule has been revised recently by Anuff and Liou.[2] They make use of Monte Carlo calculations of the interchannel interference produced when an FM wave carrying a multichannel signal passes through an ideal filter. The ideal filter has zero attenuation and phase shift within the passband, and infinite attenuation outside the band. Monte Carlo calculations of interchannel interference in microwave systems have also been made by Grierson and McGee.[3]

Here we make a beginning on the analysis (in contrast to Monte Carlo) required to calculate the interchannel interference produced by an ideal filter.

The FM wave is $\cos [\omega_o t + \varphi(t)]$ where $\omega_o = 2\pi f_o$ and $\varphi(t)$ is a stationary, zero-mean Gaussian process with the two-sided power spectrum

$$W_\varphi(f) = \begin{cases} W_o, & |f| \leqq B \\ 0, & |f| > B. \end{cases} \tag{2}$$

In (2), W_o is a constant and B is the top baseband frequency. In order to represent an idle channel at frequency f_c, we take $W_\varphi(f) = 0$ in the narrow slots $f_c \leqq |f| \leqq f_c + \Delta f_c$, Δf_c being so small that $W_\varphi(f)$ can be replaced, without appreciable error, by W_o in the integrals appearing in the analysis.

The mean-square value of $\varphi(t)$ and the rms frequency deviation D are given by the ensemble averages

$$\begin{aligned} \langle \varphi^2(t) \rangle &= 2W_o B \text{ (rad)}^2 \\ D^2 &= \langle (\varphi'(t)/2\pi)^2 \rangle = 2W_o B^3/3 \text{ (Hz)}^2 \end{aligned} \tag{3}$$

where $\varphi'(t) = d\varphi(t)/dt$. This $\varphi(t)$ gives a convenient approximation to the preemphasized wave assumed by Anuff and Liou. A representative value of $D_{\max}$ in (1) is $4D$.

The ideal filter passband extends from $f_o - f_h$ to $f_o + f_h$. It is assumed that $2f_h/f_o \ll 1$ and that $nB < f_h < (n+1)B$ where n is a positive integer.

Our aim is to apply results from the theory of Volterra series to obtain an expression for the dominant portion of the interchannel interference when the normalized rms frequency deviation D/B becomes small.

For the moment, consider one-sided power spectra. Now the power spectrum of $\varphi(t)$ extends from 0 to B and has the value $2W_o$. The average signal power (FM) appearing in the channel $(f, f + \Delta f)$ when it is busy is

$$S = (2\pi f)^2 2W_o \Delta f \text{ (rad/s)}^2. \tag{4}$$

Let N be the average interchannel interference power which appears in the same channel. The value of N depends upon whether the channel is idle or busy. When the channel is idle, the interference can be heard as crosstalk noise. In our expressions for N/S, we assume that our particular channel is idle, that all the other channels are busy, and that N/S is the limit obtained as Δf tends to zero.

The nature of our results is illustrated by the following expression for N/S in the top baseband channel:

$$\begin{aligned} N/S &= \left[\frac{3}{2}\frac{D^2}{B^2}\left(n + 1 - \frac{f_h}{B}\right)\right]^{2n} C_{on} + 0[(D/B)^{4n+2}], \\ C_{on} &= \frac{1}{(2n)!}\sum_{k=1}^{n} \frac{(2n-2k)!(2k-1)!}{(k-1)!k!^2(k+1)!}\left(\frac{1}{(n-k)!}\right)^4. \end{aligned} \tag{5}$$

Here the integer n is determined by the filter semibandwidth f_h and the relation $nB < f_h < (n + 1)B$. The first three values of C_{on} are $C_{o1} = 1/4$, $C_{o2} = 5/96$, and $C_{o3} = 19/10368$. For large n, C_{on} tends to $2^{2n+1}/[n!^4\pi n(n + 2)]$.*

Equations (5) are a special case, $f = B$, of (52) which gives N/S in a channel whose frequency f satisfies $f_h - nB \leqq f \leqq B$. When $0 \leqq f < f_h - nB$, N/S is of order $(D/B)^{4n+4}$ and the formulas corresponding to (52) do not appear to be known. However, comparison with Monte Carlo values plotted by Grierson and McGee[3] indicates that replacing n by $n + 1$ in (52) [n still given by $nB < f_h < (n + 1)B$] gives an expression for N/S which is not greatly in error when f is in $0 < f \leqq f_h - nB$. The simplest instance of (52) holds for $n = 1$, $B < f_h < 2B$, and f in the range $f_h - B \leqq f \leqq B$:

$$N/S = \frac{1}{4}\left(\frac{3}{2}\frac{D^2}{B^2}\right)^2\left[\left(2 - \frac{f_h}{B}\right)^2 - \left(1 - \frac{f}{B}\right)^2\right] + 0(D^6/B^6). \tag{6}$$

The explicit part of (6) decreases to zero as f decreases from B to $f_h - B$. For $0 \leqq f < f_h - B$, N/S is $0(D^8/B^8)$.

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* I am indebted to a reviewer for the observation that the presence of the factor $n!^{-4}$ in C_{on} and the behavior of the curves in Fig. 1 strongly suggest that the formulas give useful results subject only to D/f_h (instead of the more restrictive D/B) being small.

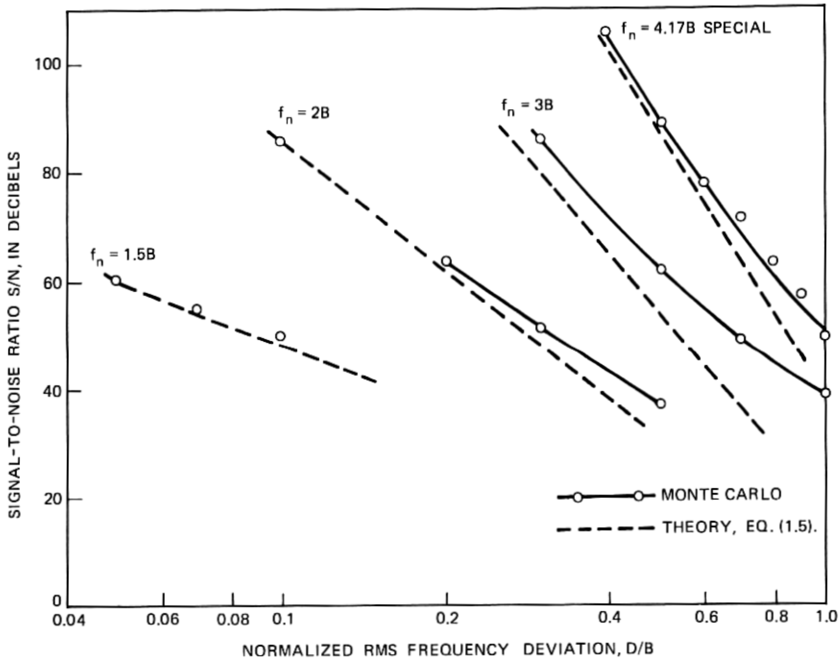

Fig. 1—Signal-to-noise ratio in top channel. The dashed lines show eq. (5) for flat baseband phase modulation. The Monte Carlo curve $4.17B$ is for flat baseband PM, and the curves $1.5B$, $2B$, and $3B$ are for the typical preemphasis used by Anuff and Liou.

It turns out that the explicit portions of (5) and (52) are obtained by considering modulation terms of order $2n + 1$ and of type $\cos 2\pi[(n + 1)B - nB]t$.

The curves labeled $f_h = 1.5B$, $2B$, and $3B$ in Fig. 1 have been plotted to compare our eq. (5), based on the flat power spectrum (2) for $W_\varphi(f)$, with the Monte Carlo results given by Anuff and Liou for a typical preemphasis curve. The solid lines and dots show Monte Carlo values of S/N for the top baseband channel. The dashed lines are computed from our (5). It is seen that the slopes agree well for small D/B, but for $f_h = 3B$ a separation of about 6 dB appears. For $f_h = 4B$ (not shown) the separation increases to about 12 dB. Most of the separation appears to be due to the difference between (2) and the $W_\varphi(f)$ used by Anuff and Liou. This is indicated by later Monte Carlo computations made by Anuff for the $W_\varphi(f)$ of (2), and labeled $f_h = 4.17B$ in Fig. 1. There is still a separation of 2 or 3 dB. This may be due to the granularity of the Monte Carlo approximation to $W_\varphi(f)$ and also to the fact that the Monte Carlo filter is not quite ideal.

Section II contains a statement of results from the Volterra series theory needed in our analysis. In Section III, the simplest case, $B < f_h < 2B$, involving third-order modulation terms is discussed in some detail. Section IV and Appendices C and D deal with the general $nB < f_h < (n + 1)B$ case. In Section V, formulas are given for the calculation of N/S. Appendices A and B contain material which provides some insight to the general work of Section IV. Appendix A discusses the case $\varphi(t) = A \cos \omega_a t$, and Appendix B treats a simple analog of the FM problem.

All of our work deals with the flat power spectrum $W_\varphi(f)$ defined by (2). The chief obstacle in going to a more general $W_\varphi(f)$ is the evaluation of the multiple integrals which occur in the analysis. Possibly $W_\varphi(f) = Af^\nu$ for $|f| < B$ and $\nu > -1$ could be handled by the procedure used here, but this extension has not been studied seriously.

II. RESULTS NEEDED FROM VOLTERRA SERIES THEORY

Because the carrier frequency f_o is at the center of the ideal filter passband, the even-order modulation products vanish. In the notation of Ref. 4, the Volterra series with the even terms equal to zero is

$$y(t) = \frac{1}{1!}\int_{-\infty}^{\infty} du_1 g_1(u_1)x(t - u_1) + \frac{1}{3!}\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 \int_{-\infty}^{\infty} du_3 g_3(u_1, u_2, u_3) \prod_{k=1}^{3} x(t - u_k) + \cdots. \quad (7)$$

When $x(t)$ is a stationary, zero-mean Gaussian process with two-sided power spectrum $W_x(f)$, the Mircea-Sinnreich[5] series for the two-sided power spectrum $W_y(f)$ of $y(t)$ becomes [eqs. (14) and (160) of Ref. 4]:

$$\begin{aligned} W_y(f) = W_x(f)\Big| G_1(f) &+ \frac{1}{1!2}\int_{-\infty}^{\infty} df_1' W_x(f_1')G_3(f, f_1', -f_1') \\ &+ \frac{1}{2!2^2}\int_{-\infty}^{\infty} df_1' \int_{-\infty}^{\infty} df_2' W_x(f_1')W_x(f_2')G_5(f, f_1', -f_1', f_2', -f_2') + \cdots \Big|^2 \\ &+ \frac{1}{3!}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 W_x(f_1)W_x(f_2)W_x(f - f_1 - f_2) \\ &\times \Big| G_3(f_1, f_2, f - f_1 - f_2) \\ &+ \frac{1}{1!2}\int_{-\infty}^{\infty} df_1' W_x(f_1')G_5(f_1, f_2, f - f_1 - f_2, f_1', -f_1') + \cdots \Big|^2 \\ &+ \frac{1}{5!}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 \int_{-\infty}^{\infty} df_3 \int_{-\infty}^{\infty} df_4 W_x(f_1) \cdots \\ &\qquad W_x(f_4)W_x(f - f_1 - \cdots - f_4)|G_5(\cdots) + \cdots|^2 + \cdots. \end{aligned} \quad (8)$$

Here, $G_m(f_1, f_2, \cdots, f_m)$ is the m-fold Fourier transform of $g_m(t_1, \cdots, t_m)$, i.e., the mth-order transfer function.

We shall need another result which can be derived from the analysis of Section VII of Ref. 4. Let $x(t)$ and $y(t)$ be as in (7) and (8), and let

$$y_L(t) = \int_{-\infty}^{\infty} du_1 \, g_1(u_1) x(t - u_1) \tag{9}$$

be the linear part of $y(t)$. Then the power spectrum of $y(t) - y_L(t)$ is given by

$$W_{y-y_L}(f) = [\text{Series for } W_y(f) \text{ with } G_1(f) \text{ replaced by } 0]. \tag{10}$$

This result can be established by using the series (152) of Ref. 4 for $\langle y(t + \tau) z(t) \rangle$ to evaluate the four ensemble averages appearing in the autocorrelation function of $y(t) - y_L(t)$.

In problems in which $\cos [2\pi f_o t + \varphi(t)]$ enters a filter with transfer function $K(f)$, the normalized transfer function

$$\Gamma(f) = K(f_0 + f)/K(f_0) \tag{11}$$

appears. For the ideal filter of our problem, $\Gamma(f) = 1$ when $-f_h < f < f_h$ and $\Gamma(f) = 0$ when $|f| > f_h$. Furthermore, the power spectrum $W_\theta(f)$ of the output phase angle $\theta(f)$ is given by the expression obtained from (8) by replacing $W_x(f)$ by $W_\varphi(f)$ and $G_1(f_1)$, $G_3(f_1, f_2, f_3), \cdots$ by [Mircea[6] and (52), (71), and (72) of Ref. 4]:

$$\begin{aligned}
G_{\theta 1}(f_1) &= \Gamma(f_1), \\
G_{\theta 3}(f_1, f_2, f_3) &= j^2[\Gamma(f_1 + f_2 + f_3) - \Gamma(f_1)\Gamma(f_2 + f_3) \\
&\quad - \Gamma(f_2)\Gamma(f_1 + f_3) - \Gamma(f_3)\Gamma(f_1 + f_2) \\
&\quad + 2\Gamma(f_1)\Gamma(f_2)\Gamma(f_3)], \\
G_{\theta 5}(f_1, \cdots, f_5) &= j^4[(12345) - 1!\sum_5{}' (1)(2345) - 1!\sum_{10}{}' (12)(345) \\
&\quad + 2!\sum_{10}{}' (1)(2)(345) + 2!\sum_{15}{}' (1)(23)(45) \\
&\quad - 3!\sum_{10}{}' (1)(2)(3)(45) + 4!(1)(2)(3)(4)(5)], \\
&\quad \vdots \\
G_{\theta m}(f_1, \cdots, f_m) &= j^{m-1} \sum_{\ell=1}^{m} (-1)^{\ell-1}(\ell - 1)! \sum_{(\nu;\, \ell,\, m)} \sum_N{}' \\
&\quad \times \Gamma(f_1 + \cdots + f_{\nu_1})\Gamma(f_{\nu_1+1} + \cdots + f_{\nu_1+\nu_2}) \cdots \\
&\qquad \times \Gamma(f_{m-\nu_\ell+1} + \cdots + f_m).
\end{aligned} \tag{12}$$

The Γ's and f's have been omitted and the subscripts written within parentheses in $G_{\theta 5}$. In $G_{\theta m}$ the summation over ℓ and $(\nu; \ell, m)$ is essentially a summation over the partitions of m, ℓ being the number of parts and $\nu_1, \nu_2, \cdots, \nu_\ell$ the parts:

$$\begin{aligned} \nu_1 + \nu_2 + \cdots + \nu_\ell &= m, \\ 1 \leqq \nu_1 \leqq \cdots &\leqq \nu_\ell. \end{aligned} \tag{13}$$

The summation $\sum_N'$ extends over the N (not to be confused with the N denoting noise power) nonidentical products that can be obtained by permuting the subscripts on the f's. The number of terms in the summation $\sum_N'$ is

$$N = m!/\nu_1!\nu_2! \cdots \nu_\ell! r_1! r_2! \cdots r_k! \tag{14}$$

where r_1 is the number of equal ν's in the first run of equalities in the arrangement $\nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_\ell$, r_2 the number in the second run, etc. When the ν's are unequal, the r's do not appear. A more complete explanation of the notation is given in (24) to (29) of Ref. 4.

In our work, $G_{\theta(2n+1)}$ will be either 0 or -1 when $n \geqq 1$.

When $\varphi(t)$ is bandlimited to $|f| \leqq B$ and f_h exceeds B, the linear portion of $\theta(t)$ is equal to $\varphi(t)$. This can be seen formally by assuming $\varphi(t)$ to have a Fourier transform $F(f)$ which vanishes for $|f| > B$. Then, from (9) and $G_{\theta 1}(f) = \Gamma(f) = 1$ for $|f| < f_h$, it follows that

$$\begin{aligned} \theta_L(t) &= \int_{-\infty}^{\infty} du\, g_{\theta 1}(u)\, \varphi(t-u) \\ &= \int_{-\infty}^{\infty} df\, G_{\theta 1}(f) F(f) e^{i2\pi f t} \\ &= \int_{-B}^{B} df F(f) e^{i2\pi f t} = \varphi(t). \end{aligned} \tag{15}$$

Most of our analysis will consist of using the combination of (8) and (10) to obtain expressions for $W_{\theta-\varphi}(f)$, the power spectrum of the difference $\theta(t) - \varphi(t)$ between the output and input phase angles.

III. $W_{\theta-\varphi}(f)$ WHEN $B < f_h < 2B$

In this section we take $B < f < 2B$, $f_h - B \leqq f \leqq B$, and assume D/B (and consequently $W_o B$) to be small. The power spectrum of the output phase angle θ is, from (8) with θ in place of y,

$$\begin{aligned} W_\theta(f) = W_\varphi(f) \bigg| \Gamma(f) &+ \frac{1}{1!2} \int_{-B}^{B} df_1' W_\varphi(f_1') G_{\theta 3}(f, f_1', -f_1') + 0(W_o^2 B^2) \bigg|^2 \\ &+ \frac{1}{3!} \int_{-B}^{B} df_1 \int_{-B}^{B} df_2\, W_\varphi(f_1) W_\varphi(f_2) W_\varphi(f - f_1 - f_2) \\ &\times |G_{\theta 3}(f_1, f_2, f - f_1 - f_2) + 0(W_o B)|^2 + 0(W_o^5 B^4). \end{aligned} \tag{16}$$

From (2), $W_\varphi(f_1')$ and $W_\varphi(f_i)$, $i = 1, 2$, can be replaced by W_o in the integrals. However, $W_\varphi(f - f_1 - f_2)$ will be retained for the present because it serves to make the integral vanish when $|f - f_1 - f_2| > B$. For completeness, we shall carry the first line in (16) along in the analysis even though it will vanish when we calculate the crosstalk noise in an idle channel represented by a slot in W_φ at $(f, f + \Delta f)$.

Since the linear portion of $\theta(t)$ is equal to $\varphi(t)$, the power spectrum $W_{\theta-\varphi}(f)$ of $\theta(t) - \varphi(t)$ is given by (16) with $\Gamma(f)$ in the first line replaced by zero:

$$W_{\theta-\varphi}(f) = W_\varphi(f)\left|\frac{1}{1!}\int_o^B df_1' W_o G_{\theta 3}(f, f_1', -f_1')\right|^2 + \frac{1}{3!}\int_{-B}^B df_1 \int_{-B}^B df_2 W_o^2 W_\varphi(f - f_1 - f_2)|G_{\theta 3}(f_1, f_2, f - f_1 - f_2)|^2 + 0(W_o^4 B^3). \quad (17)$$

In obtaining (17), we have used the fact that the integrand in the first line is an even function of f_1'.

Examination of (17) shows that the dominant terms in $W_{\theta-\varphi}(f)$ are $0(W_o^3 B^2)$ and hence correspond to third-order modulation. When f does not lie in an idle channel (i.e., $W_\varphi(f) \neq 0$), some of the third-order terms in $W_\theta(f)$ arise from the cross term $\Gamma(f)0(W_o^2 B^2)$ which requires a knowledge of $G_{\theta 5}$ for its evaluation. For this reason, we prefer to deal with $W_{\theta-\varphi}(f)$ [instead of $W_\theta(f)$] which requires only $G_{\theta 3}$ for the calculation of all its third-order terms.

When $0 \leqq f \leqq B$ and $0 \leqq f_1' \leqq B$, as in (17), all of the Γ's in

$$G_{\theta 3}(f, f_1', -f_1') = -\Gamma(f) + \Gamma(f)\Gamma(0) + \Gamma(f_1')\Gamma(f - f_1') + \Gamma(-f_1')\Gamma(f + f_1') - 2\Gamma(f)\Gamma(f_1')\Gamma(-f_1') \quad (18)$$

are unity except possibly $\Gamma(f + f_1')$ which is unity if $f + f_1' < f_h$ and zero if $f + f_1' > f_h$. Hence, $G_{\theta 3}(f, f_1', -f_1')$ is zero if $f_1' < f_h - f$ and is -1 if $f_h - f < f_1'$. It follows that

$$\frac{1}{1!}\int_o^B df_1' W_o G_{\theta 3}(f, f_1', -f_1') = \begin{cases} -(B - f_h + f)W_o, & f \geqq f_h - B \\ 0, & f \leqq f_h - B \end{cases}. \quad (19)$$

The function $W_\varphi(f - f_1 - f_2)$ vanishes for $|f - f_1 - f_2| > B$, and the function

$$G_{\theta 3}(f_1, f_2, f - f_1 - f_2) = -\Gamma(f) + \Gamma(f_1)\Gamma(f - f_1) + \Gamma(f_2)\Gamma(f - f_2) + \Gamma(f - f_1 - f_2)\Gamma(f_1 + f_2) - 2\Gamma(f_1)\Gamma(f_2)\Gamma(f - f_1 - f_2) \quad (20)$$

vanishes in part of the square $f_1 = \pm B$, $f_2 = \pm B$. The result is that, as will be shown, the region of integration for the double integral in (17) reduces to the shaded areas shown in Fig. 2. In Fig. 2, it is assumed that $f_h - B \leqq f \leqq B$. When $0 \leqq f \leqq f_h - B$, the double integral in (17) is zero because $G_{\theta 3}$ is zero.

In the present case, $B < f_h < 2B$, it is convenient to set

$$f_3 = f - f_1 - f_2 \tag{21}$$

so that the lines $f_3 = \pm B$, or $f_1 + f_2 = f \pm B$, mark boundaries outside of which $W_\varphi(f - f_1 - f_2)$ is zero. Equation (21) also enables us to write the boundaries $f_1 = f - f_h$ and $f_2 = f - f_h$ as $f_2 + f_3 = f_h$ and $f_1 + f_3 = f_h$, respectively, as shown in Fig. 2.

The expression (20) for $G_{\theta 3}(f_1, f_2, f - f_1 - f_2)$ is equal to -1 in the shaded areas of Fig. 2. This follows from the fact that all of the Γ's in (20) are unity except possibly $\Gamma(f - f_1)$, $\Gamma(f - f_2)$, and $\Gamma(f_1 + f_2)$, which are 0 when their arguments exceed f_h. The possibilities $f - f_1 < -f_h$ and $f - f_2 < -f_h$ are ruled out because $f > 0$, and $f_1 + f_2 < -f_h$ is discarded because it makes $f_3 > B$. Performing the integration over the shaded areas in Fig. 2 is equivalent to adding

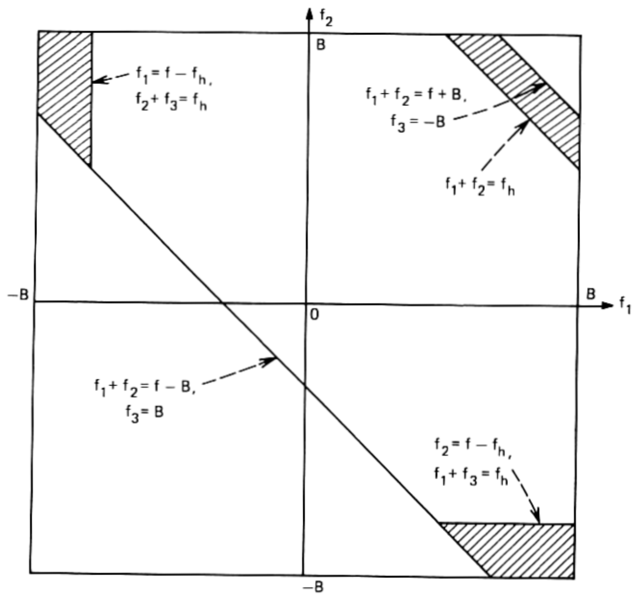

Fig. 2—The three areas of integration for the double integral in eq. (17) for $W_{\theta-\varphi}(f)$.

the areas and gives

$$\frac{1}{3!}\int_{-B}^{B} df_1 \int_{-B}^{B} df_2 W_o^2 W_\varphi(f - f_1 - f_2)\,|G_{\theta 3}(f_1, f_2, f - f_1 - f_2)|^2 = \tfrac{1}{4}[(2B - f_h)^2 - (B - f)^2]W_o^3 \quad (22)$$

when $f_h - B \leqq f \leqq B$. As mentioned earlier, the double integral vanishes when $f \leqq f_h - B$.

The main result of this section is obtained by substituting the values (19) and (22) of the integrals in the expression (17) for $W_{\theta-\varphi}(f)$:

$$W_{\theta-\varphi}(f) = W_\varphi(f)(B + f - f_h)^2 W_o^2 + \tfrac{1}{4}W_o^3[(2B - f_h)^2 - (B - f)^2] + 0(W_o^4 B^3) \quad (23)$$

where f_h and f satisfy $B < f_h < 2B$ and $f_h - B \leqq f \leqq B$, respectively. When $0 \leqq f \leqq f_h - B$, the $G_{\theta 3}$'s are zero in the corresponding ranges of integration and it follows from (16) (with $\Gamma(f)$ replaced by zero) that

$$W_{\theta-\varphi}(f) = 0(W_o^5 B^4). \quad (24)$$

It also appears that the third-order part of $W_{\theta-\varphi}(f)$ is constant when $B < f < f_h$.

Although it may not be obvious in Fig. 2, the areas of the three shaded regions are equal, and each contributes the same amount to $W_{\theta-\varphi}(f)$. There is an underlying symmetry which becomes evident when the boundaries of the three regions are written as follows:

$$\begin{array}{ccc} f_1 = B & f_2 = B & f_3 = B \\ f_2 = B & f_3 = B & f_1 = B \\ f_3 = -B & f_1 = -B & f_2 = -B \\ f_1 + f_2 = f_h & f_2 + f_3 = f_h & f_3 + f_1 = f_h. \end{array} \quad (25)$$

Furthermore, the double integral in (17) can be written as

$$\frac{1}{3!}\int df_1 \int df_2 \int df_3\, W_\varphi(f_1)W_\varphi(f_2)W_\varphi(f_3)\delta(f - f_1 - f_2 - f_3) \times |G_{\theta 3}(f_1, f_2, f_3)|^2 \quad (26)$$

where, replacing $\delta(x)$ by the limit as $\epsilon \to 0$ of $h(x) = 1/\epsilon$ for $|x| < \epsilon/2$ and $h(x) = 0$ for $|x| > \epsilon/2$, the integration extends over three portions of a three-dimensional slab bounded by the planes $f_1 + f_2 + f_3 = f \pm \epsilon/2$. The three portions are cut out of the slab by the planes defined by eqs. (25). When the integration is accomplished by integrating with respect to f_3 first (the thickness of the slab is $\epsilon/3^{\frac{1}{2}}$ and f_3 is integrated over a length ϵ), the areas of integration for f_1 and f_2 are those shown in Fig. 2.

Thus the twofold integral is equal to the sum of three equal contributions where each contribution can be regarded as arising from a region near one of the corners of a three-dimensional cube. It turns out that the corresponding $2n$-fold integral encountered later is equal to the sum of $(2n+1)!/n!(n+1)!$ contributions arising from regions near $(2n+1)!/n!(n+1)!$ of the 2^{2n+1} corners of a $(2n+1)$-dimensional cube. The corners are those whose $(2n+1)$ coordinates consist of $(n+1)$ plus B's and n minus B's.

IV. $W_{\theta-\varphi}(f)$ WHEN $nB < f_h < (n+1)B$

For f_h and f such that $nB < f_h < (n+1)B$, $n = 1, 2, \cdots$, and $f_h - nB \leqq f \leqq B$, the dominant terms in $W_{\theta-\varphi}(f)$ are given by

$$\begin{aligned} W_{\theta-\varphi}(f) &= W_\varphi(f)\left|\frac{W_o^n}{n!}\int_o^B df_1'\cdots\int_o^B df_n' \right. \\ &\quad \left. \times\, G_{\theta(2n+1)}(f, f_1', -f_1', \cdots, f_n', -f_n')\right|^2 \\ &\quad + \sum_{k=1}^{n}\frac{W_o^{2k}}{(2k+1)!}\int_{-B}^{B} df_1\cdots\int_{-B}^{B} df_{2k}\, W_\varphi(f - f_1 - \cdots - f_{2k}) \\ &\quad \times \left|\frac{W_o^{n-k}}{(n-k)!}\int_o^B df_1'\cdots\int_o^B df_{n-k}'\, G_{\theta(2n+1)}(f_1, \cdots, f_{2k}, f - f_1 \right. \\ &\quad \left. - \cdots - f_{2k}, f_1', -f_1', \cdots, f_{n-k}', -f_{n-k}')\right|^2 + 0(W_o^{2n+2}B^{2n+1}) \end{aligned} \tag{27}$$

where for $k = n$ it is understood that the quantity within the absolute value signs becomes $G_{\theta(2n+1)}(f_1, \cdots, f_{2n}, f - f_1 - \cdots f_{2n})$. No $G_{\theta m}$ for $m < 2n + 1$ appears in (27) because, from Appendix D, all such terms vanish over the region of integration.

To aid in the evaluation of the integrals which arise in dealing with (27) we shall use[7]

$$\int dx_1 \cdots \int dx_m\, H(\sigma_m) = \frac{1}{(m-1)!}\int_K^L H(z)z^{m-1}dz \qquad K \leqq \sigma_m \leqq L \tag{28}$$

where $K \geqq 0$, $\sigma_m = x_1 + x_2 + \cdots + x_m$, and the integration on the left extends over the region specified by $x_i \geqq 0$, $i = 1, 2, \cdots, m$ and $K \leqq \sigma_m \leqq L$. The integrations with respect to the f_i's in our problem extend over regions where f_i is near $+B$ or $-B$; and we shall use (28) by making the change of variable $f_i = B - x_i$ or $f_i = -B + x_i$.

The $G_{\theta(2n+1)}$ in the second line of (27) is different from 0 (and, from Appendix D, equal to -1) only if

$$f + f_1' + \cdots + f_n' > f_h. \tag{29}$$

Setting $f_i' = B - x_i$, $i = 1, 2, \cdots, n$ carries this inequality into

$$x_1 + x_2 + \cdots + x_n < f + nB - f_h = P - Q \tag{30}$$

where we have introduced the parameters

$$P = (n + 1)B - f_h$$
$$Q = B - f \tag{31}$$

and have assumed $P > Q$. When $P < Q$, the inequality (29) and its analogues for the other terms in (27) cannot be satisfied. Consequently, all the $G_{\theta(2n+1)}$'s are zero and all the modulation terms of order $(2n + 1)$ vanish from (27) when $P < Q$.

From (28) with $m = n$, $K = 0$, $L = P - Q$, and $H(z) = -1$, we get

$$W_\varphi(f)\left|\frac{W_o^n}{n!(n-1)!}\int_o^{P-Q}(-1)z^{n-1}dz\right|^2 = W_\varphi(f)W_o^{2n}(P - Q)^{2n}/n!^4 \tag{32}$$

for the first term on the right in (27).

Now consider the kth term in the sum in (27). For $G_{\theta(2n+1)}$ to be -1 instead of 0, the sum of $(n + 1)$ of its arguments must exceed f_h (Appendix D). It can be shown that $(n - k)$ of the arguments must be $f_1', \cdots, f_{n-k}'$ and that the remaining $(k + 1)$ arguments come from the set of $2k + 1$ elements

$$f_1, f_2, \cdots, f_{2k}, f - f_1 - \cdots - f_{2k}. \tag{33}$$

There are $(2k+ 1)!/(k + 1)!k!$ different choices of $(k + 1)$ items from the set (33). Let $f_1, f_2, \cdots, f_{k+1}$ represent the typical choice and

$$f_1 + f_2 + \cdots + f_{k+1} + f_1' + f_2' + \cdots + f_{n-k}' \tag{34}$$

be the typical sum of $(n + 1)$ elements of $G_{\theta(2n+1)}$ which exceeds f_h. Each sum is associated with a region of integration, one boundary of which is obtained by setting (34) equal to f_h. For $k = 1$, there are three regions and, after the integrations with respect to the f_i''s have been performed, the regions become the ones shown in Fig. 2 with f_h replaced by $f_h - (n - 1)B$. For k arbitrary, the regions correspond to the corners of a $(2k + 1)$-dimensional cube, the corner coordinates consisting of $(k + 1)$ plus B's and k minus B's. By virtue of the type of symmetry shown by (25) and (26) for the case $B < f_h < 2B$, each of the $(2k + 1)!/(k + 1)!$k! regions contributes the same amount to the kth term in (27).

The first step in evaluating the kth term $(k < n)$ is to perform the integrations with respect to $f_1', \cdots, f_{n-k}'$. Suppose that the values of the typical choice $f_1, \cdots, f_{k+1}$ are given. Then for $G_{\theta(2n+1)}$ to be equal to -1, it is necessary that

$$f_1' + \cdots + f_{n-k}' > f_h - f_1 - \cdots - f_{k+1}. \tag{35}$$

Setting $f_i' = B - x_i$, $i = 1, 2, \cdots, n - k$ carries (35) into

$$x_1 + x_2 + \cdots + x_{n-k} < (n - k)B - f_h + f_1 + f_2 + \cdots + f_{k+1}. \quad (36)$$

Using (28) with $m = n - k$, $K = 0$, $H(o) = -1$, and L equal to the right side of (36) shows that the quantity inside the absolute value signs in the kth term is equal to

$$\frac{W_o^{n-k}}{(n-k)!(n-k-1)!}\int_0^L (-1)z^{n-k-1}dz = -W_o^{n-k}L^{n-k}/(n-k)!^2 \quad (37)$$

where $L \geqq 0$.

Next, we integrate with respect to $f_1, f_2, \cdots, f_{k+1}$. The restriction that the right side of (36) be positive gives

$$f_1 + f_2 + \cdots + f_{k+1} > f_h - (n - k)B \quad (38)$$

and the fact that the argument of $W_\varphi(f - f_1 - \cdots - f_{2k})$ must exceed $-B$ gives

$$f_1 + f_2 + \cdots + f_{k+1} < B + f - f_{k+2} - \cdots - f_{2k}. \quad (39)$$

Setting $f_i = B - x_i$ for $i = 1, 2, \cdots, k + 1$ and $f_i = -B + x_i$ for $i = k + 2, \cdots, 2k$ carries (38), (39), and the L in (37) into

$$x_1 + x_2 + \cdots + x_{k+1} < (n + 1)B - f_h = P$$
$$x_i + x_2 + \cdots + x_{k+1} > B - f + x_{k+2} + \cdots + x_{2k} = Q + x_{k+2} + \cdots + x_{2k} \quad (40)$$
$$L - P - x_1 - x_2 - \cdots x_{k+1}.$$

At this stage, the kth term in (27) is, for $k > 1$,

$$\frac{W_o^{2k}}{(2k+1)!}\frac{(2k+1)!}{(k+1)!k!}\int dx_{k+2}\cdots\int dx_{2k}\int dx_1\cdots\int dx_{k+1}W_o \times W_o^{2n-2k}(P - x_1 - \cdots - x_{k+1})^{2n-2k}/(n-k)!^4 \quad (41)$$

where $(2k + 1)!/(k + 1)!k!$ is the number of equally contributing regions of integration. For fixed $x_{k+2}, \cdots, x_{2k}$, the integration with respect to $x_1, \cdots, x_{k+1}$ can be performed by using (28) with $m = k + 1$, $K = Q + x_{k+2} + \cdots + x_{2k}$, $L = P$, and $H(z) = (P - z)^{2n-2k}$. Expression (41) becomes

$$\frac{W_o^{2n+1}}{(k+1)!k!(n-k)!^4}\int dx_{k+2}\cdots\int dx_{2k} \times \frac{1}{k!}\int_{Q+x_{k+2}+\cdots+x_{2k}}^{P}(P - z)^{2n-2k}z^k dz. \quad (42)$$

The integration in (42) extends over the region defined by $x_i \geqq 0$, $i = k+2, \cdots, 2k$, and the inequality obtained by combining the two inequalities in (40):

$$x_{k+2} + \cdots + x_{2k} < P - Q. \tag{43}$$

Using (28) with $m = k - 1$, $K = 0$, $L = P - Q$, and

$$H(z) = \int_{Q+z}^{P} (P - y)^{2n-2k} y^k dy \tag{44}$$

leads to a double integral which can be reduced to a single integral by reversing the order of integration:

$$\frac{1}{(k-2)!}\int_{o}^{P-Q} H(z) z^{k-2} dz = \frac{1}{(k-1)!}\int_{Q}^{P} y^k (P-y)^{2n-2k}(y-Q)^{k-1} dy$$
$$= k \sum_{\ell=0}^{k} \frac{(2n-2k)!(2k-\ell-1)!}{\ell!(k-\ell)!(2n-\ell)!} Q^{\ell}(P-Q)^{2n-\ell}. \tag{45}$$

When (45) is used in the expression (42) for the kth term in $W_{\theta-\varphi}(f)$, (42) becomes

$$\frac{W_o^{2n+1}(2n-2k)!}{(k+1)!k!(k-1)!(n-k)!^4} \sum_{\ell=0}^{k} \frac{(2k-\ell-1)! Q^{\ell}(P-Q)^{2n-\ell}}{\ell!(k-\ell)!(2n-\ell)!}. \tag{46}$$

It can be verified that (46) also holds for $k = 1$, even though $k > 1$ was assumed in the derivation. Adding the expression (32) to the sum of (46) from $k = 1$ to n and interchanging the order of summation gives the equation sought in this section:

$$W_{\theta-\varphi}(f) = W_{\varphi}(f) W_o^{2n}(P-Q)^{2n} n!^{-4} + W_o^{2n+1} \sum_{\ell=0}^{n} C_{\ell n} Q^{\ell}(P-Q)^{2n-\ell}$$
$$+ 0(W_o^{2n+2} B^{2n+1}) \tag{47}$$

where n is a positive integer, $nB < f_h < (n+1)B$, $P > Q$, P and Q are given by (31), and

$$C_{\ell n} = \frac{1}{\ell!(2n-\ell)!} \sum_{k=\max(1,\ell)}^{n} \frac{(2k-\ell-1)!(2n-2k)!}{(k-\ell)!(k-1)!k!(k+1)!(n-k)!^4}. \tag{48}$$

When $P < Q$, i.e., $f < f_h - nB$, our analysis tells us only that $W_{\theta-\varphi}(f)$ is $0(W_o^{2n+3} B^{2n+2})$.

V. THE NOISE TO SIGNAL RATIO N/S

According to (4) the average signal power (FM) in the channel $(f, f + \Delta f)$ when it is busy is

$$S = (2\pi f)^2 (2W_o)\Delta f. \tag{49}$$

Likewise, the average interchannel interference noise power is

$$N = (2\pi f)^2[2W_{\theta-\varphi}(f)]\Delta f \tag{50}$$

and hence

$$N/S = W_{\theta-\varphi}(f)/W_o. \tag{51}$$

When the channel is idle, $W_\varphi(f)$ is 0 in $(f, f + \Delta f)$, and if all of the other channels are busy, (51) and (47) give

$$\begin{aligned} N/S &= W_o^{2n} \sum_{\ell=0}^{n} C_{\ell n} Q^\ell (P - Q)^{2n-\ell} + 0[(W_o B)^{2n+1}] \\ &= \left[\frac{3}{2}\frac{D^2}{B^2}\right]^{2n} \sum_{\ell=0}^{n} C_{\ell n}\left(1 - \frac{f}{B}\right)^\ell \left(n + \frac{f}{B} - \frac{f_h}{B}\right)^{2n-\ell} \\ &\qquad + 0[(D/B)^{4n+2}] \end{aligned} \tag{52}$$

provided $nB < f_h < (n+1)B$, $f_h - nB \leqq f \leqq B$, and D/B is small. In going to the second line, we have used $W_o B = 3D^2/(2B^2)$ from (3) and the definitions (31) of P and Q. Equations (5) and (6) given as examples in Section I are obtained by setting $f = B$ and $n = 1$, respectively, in (52).

The first few values of $C_{\ell n} \times 10^n$ are listed below.

	$\ell = 0$	1	2	3	4
$n = 1$	2.5	5.0			
2	5.208	19.44	2.083		
3	1.832	9.375	3.906	0.193	
4	0.1994	1.226	1.182	0.2122	0.0060

VI. ACKNOWLEDGMENTS

I want to express my thanks to A. A. Anuff for his suggestions and helpful discussions regarding the comparison of the theory with the Monte Carlo computations. Furthermore, his Monte Carlo computation of the interchannel interference for the $W_\varphi(f)$ used in this paper gave a very welcome confirmation of the theoretical results. I am also indebted to Miss Judith Seery for computing values of N/S from the rather complicated formulas.

APPENDIX A

Sinusoidal Modulation

Some idea of how the FM distortion depends upon the radio frequency bandwidth when the deviation ratio, say A, is small can be obtained from the case $\varphi(t) = A \cos \omega_a t$, $\omega_a = 2\pi f_a$. The carrier frequency and ideal filter are the same as in Section I, but now n is such that $nf_a < f_h < (n+1)f_a$.

The input to the ideal filter is the real part of

$$\exp\left[j\omega_o t + j\varphi(t)\right] = \sum_{m=-\infty}^{\infty} c_m \exp\left(j\omega_o t + jm\omega_a t\right) \tag{53}$$

where $c_m = j^m J_m(A)$, $J_m(A)$ is the Bessel function of order m, and

$$\exp\left(jA\cos\omega_a t\right) = \sum_{m=-\infty}^{\infty} j^m J_m(A)\exp\left(jm\omega_a t\right). \tag{54}$$

Since $2f_h/f_o \ll 1$, the filter output is nearly equal to the real part of

$$\exp\left[j\omega_o t + j\theta(t)\right] = \sum_{m=-n}^{n} c_m \exp\left(j\omega_o t + jm\omega_a t\right). \tag{55}$$

Subtracting (54) from (55) and dividing by $\exp\left[j\omega_o t + j\varphi(t)\right]$ gives

$$e^{j\theta - j\varphi} = 1 - e^{-j\varphi}\left(\sum_{-\infty}^{-n-1} + \sum_{n+1}^{\infty}\right) c_m \exp\left(jm\omega_a t\right) \tag{56}$$

where the argument t has been omitted in $\theta(t)$ and $\varphi(t)$.

Replacing $\exp(-j\varphi)$ on the right by its series obtained from (53) and taking logarithms give the known first-order approximation

$$\theta - \varphi = -Im\left\{\sum_{\ell=-\infty}^{\infty}\left(\sum_{m=-\infty}^{-n-1} + \sum_{m=n+1}^{\infty}\right) c_\ell^* c_m \exp\left[j(m-\ell)\omega_a t\right]\right\} + \text{terms of order } \left[\sum_\ell\left(\sum_m + \sum_m\right)\right]^2. \tag{57}$$

Since A is small and $c_m = j^m J_m(A)$, we have for $m \geqq 0$

$$c_m = c_{-m} = (jA/2)^m/m! + 0(A^{m+2}). \tag{58}$$

The interchannel interference in a microwave system corresponds to the $\exp(j\omega_a t)$ and $\exp(-j\omega_a t)$ components in the expression (57) for $\theta - \varphi$. The largest contribution to these components comes from the values $m = n+1$, $\ell = -n$, and $m = -n-1$, $\ell = n$, respectively:

$$\begin{aligned}(\theta - \varphi)_{\omega_a} &= -Im\left[j^{-n+n+1}e^{j\omega_a t} + j^{-n+n+1}e^{-j\omega_a t}\right]\frac{(A/2)^{2n+1}}{n!(n+1)!} + 0(A^{2n+2}) \\ &= -\frac{2(A/2)^{2n+1}}{n!(n+1)!}\cos\omega_a t + 0(A^{2n+2}).\end{aligned} \tag{59}$$

It follows that the average power in the $\cos\omega_a t$ component of $\theta - \varphi$ is

$$P(A) = \frac{2(A/2)^{4n+2}}{n!^2(n+1)!^2} + 0(A^{4n+3}) \tag{60}$$

and dividing by the average power $A^2/2$ in $\varphi(t) = A \cos \omega_a t$ gives

$$\frac{P(A)}{\text{ave. power in } \varphi} = \frac{(A/2)^{4n}}{n!^2(n+1)!^2} + 0(A^{4n+1}). \tag{61}$$

If, instead of the pure sinusoidal signal $A \cos \omega_a t$, the signal $\varphi(t)$ were a very narrow band of Gaussian noise centered at the frequency f_a, its envelope R would fluctuate slowly according to a Rayleigh probability density

$$p(R) = \sigma^{-2} R \exp(-R^2/2\sigma^2) \tag{62}$$

where σ^2 is the average power in $\varphi(t)$. Replacing A in $P(A)$ by R and averaging with the help of (62) shows that the average of the total power in the components of $\theta - \varphi$ clustered around f_a is

$$\begin{aligned} N &= \int_0^\infty P(R)p(R)dR \\ &= \frac{(2n+1)!\sigma^{4n+2}}{n!^2(n+1)!^2\, 2^{2n}} + 0(\sigma^{4n+3}). \end{aligned} \tag{63}$$

This expression for N can be checked by using $W_x(f) = \sigma^2\delta(|f| - f_a)/2$ in place of $W_x(f) = W_o$, $|f| < B$, in the analysis of Sections II to V. Division by the average power $S = \sigma^2$ in $\varphi(t)$ gives

$$\frac{N}{S} = \frac{(2n+1)!\sigma^{4n}}{n!^2(n+1)!^2\, 2^{2n}} + 0(\sigma^{4n+1}). \tag{64}$$

In $\varphi(t) = A \cos \omega_a t$, A is the deviation ratio and in (64) σ is the rms frequency deviation ratio. The fact that N/S varies as σ^{4n} in (64) agrees with the case in which $\varphi(t)$ has a flat spectrum. However, (64) is larger by roughly the factor $(2n+1)!$

APPENDIX B

Simple Analogue of Relation Between φ and θ

The relation between the FM input φ and output θ is somewhat similar to the relation between x and y given by

$$y = x + \frac{a}{(2n+1)!} x^{2n+1} \tag{65}$$

where a is real and x is a stationary, zero-mean Gaussian process with two-sided spectrum $W_x(f)$ and autocorrelation function $R_\tau \equiv R(\tau) = \langle x(t+\tau)x(t)\rangle$. We are given $W_x(f)$ and want to find $W_y(f)$ and $W_{y-x}(f)$.

Our aim here is to obtain some insight regarding the origin of the various terms in the series (17) and (27) for $W_{\theta-\varphi}(f)$.

From Volterra series theory, taking (65) to be the series, we get $G_1(f_1) = 1$, $G_{2n+1}(f_1, f_2, \cdots, f_{2n+1}) = a$, and $G_m = 0$ for all other values of m[Ref. 4, (22), (23)]. The Mircea-Sinnreich series [Ref. 4, (156), (160)] for $W_y(f)$ becomes

$$W_y(f) = W_x(f)\left|1 + \frac{1}{n!2^n}\int_{-\infty}^{\infty} df_1' \cdots \int_{-\infty}^{\infty} df_n' \, W_x(f_1') \cdots W_x(f_n')a\right|^2$$
$$+ \frac{1}{3!}\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 \, W_x(f_1)W_x(f_2)W_x(f - f_1 - f_2)$$
$$\times \left|\frac{1}{(n-1)!2^{n-1}}\int_{-\infty}^{\infty} df_1' \cdots \int_{-\infty}^{\infty} df_{n-1}' \, W_x(f_1') \cdots W_x(f_{n-1}')a\right|^2$$
$$+ \cdots$$
$$+ \frac{1}{(2n+1)!}\int_{-\infty}^{\infty} df_1 \cdots \int_{-\infty}^{\infty} df_{2n} \, W_x(_1) \cdots W_x(f_{2n})$$
$$\times W_x(f - f_1 - f_2 - \cdots - f_{2n})|a|^2. \quad (66)$$

According to (10), the power spectrum $W_{y-x}(f)$ of $y - x$ is equal to the expression obtained by replacing the 1 [i.e., $G_1(f)$] by zero in the first line of eq. (66) for $W_y(f)$.

In this particular example, $W_y(f)$ can be obtained as the Fourier transform of the autocorrelation function $\langle y(t)y(t+\tau)\rangle$. Let $y(t)$, $y(t+\tau)$, $x(t)$, $x(t+\tau)$ be denoted by y_1, y_2, x_1, x_2, respectively. Then

$$\langle y_1y_2\rangle = \langle x_1x_2\rangle + \frac{a}{(2n+1)!}\left[\langle x_1 x_2^{2n+1}\rangle + \langle x_1^{2n+1}x_2\rangle\right]$$
$$+ \frac{a^2}{(2n+1)!^2}\langle x_1^{2n+1}x_2^{2n+1}\rangle. \quad (67)$$

From

$$\langle \exp\,(jux_1 + jvx_2)\rangle = \exp\left[-2^{-1}(u^2+v^2)R_0 - uvR_\tau\right]$$

we have the known results

$$\langle x_1 x_2^{2n+1}\rangle = \langle x_1^{2n+1}x_2\rangle = (2n+1)!R_\tau R_o^n/(n!2^n)$$
$$\langle x_1^{2n+1}x_2^{2n+1}\rangle = \sum_{k=0}^{n} \frac{(2n+1)!^2 R_\tau^{2k+1}(R_o/2)^{2n-2k}}{(2k+1)!(n-k)!^2} \quad (68)$$

Substituting in (67) and taking the Fourier transform:

$$W_y(f) = \int_{-\infty}^{\infty} e^{-i2\pi f\tau}\langle y_1 y_2\rangle d\tau$$
$$= \int_{-\infty}^{\infty} d\tau e^{-i2\pi f\tau}\left[R_\tau + \frac{2aR_\tau R_o^n}{n!2^n} + \frac{a^2R_\tau(R_o/2)^{2n}}{n!^2}\right.$$
$$\left. + \sum_{k=1}^{n}\frac{a^2R_\tau^{2k+1}(R_o/2)^{2n-2k}}{(2k+1)!(n-k)!^2}\right]. \quad (69)$$

The point being made in this appendix is that the terms in (69) have [after using $1 + 2\alpha + \alpha^2 = (1 + \alpha)^2$] a one-to-one correspondence with the terms in the Mircea-Sinnreich series (66). This can be seen with the help of

$$R_o = \int_{-\infty}^{\infty} W_x(f_i')df_i',$$

$$\int_{-\infty}^{\infty} e^{-i2\pi f\tau}R_\tau^m d\tau = \int_{-\infty}^{\infty} df_1 \cdots \int_{-\infty}^{\infty} df_{m-1} W_x(f_1)\cdots W_x(f_{m-1}) \times W_x(f - f_1 - \cdots - f_{m-1}). \quad (70)$$

APPENDIX C

Inequalities for Sums of Frequencies

Let $f_1, f_2, \cdots, f_{2n+1}$ denote the $(2n + 1)$ arguments of any one of the $G_{\theta(2n+1)}$'s occurring in the expression (27) for $W_{\theta-\varphi}(f)$. They satisfy the relations

$$|f_i| \leqq B, \qquad i = 1, 2, \cdots, 2n + 1$$
$$f_1 + f_2 + \cdots + f_{2n+1} = f \quad (71)$$

where f satisfies $0 < f \leqq B$ and is the frequency at which $W_{\theta-\varphi}(f)$ is being evaluated.

We shall call a set of r of the f_i's an "r-tuple" and the sum of the f_i's the "sum" of the r-tuple.

First we show that

$$f - nB \leqq \text{sum of any } (n + 1)\text{-tuple} \leqq f + nB. \quad (72)$$

Let $f_1 + f_2 + \cdots + f_{n+1}$ represent the typical $(n + 1)$-tuple sum. Then (72) follows upon using $|f_i| \leqq B$ on the right side of

$$f_1 + \cdots + f_{n+1} = f - f_{n+2} - \cdots - f_{2n+1}. \quad (73)$$

The next inequality is

$$-nB \leqq \text{sum of any } r\text{-tuple} \leqq nB \quad (74)$$

where $r = 1, 2, \cdots, n, n + 2, \cdots, (2n + 1)$. When $r \leqq n$, (74) follows from $|f_i| \leqq B$, and when $r \geqq n + 2$ it can be proved by using equations similar to (73).

The number of different $(n + 1)$-tuples is $(2n + 1)!/(n + 1)!n!$ If, for given set of values of the f_i's, any one of the $(n + 1)$-tuples, call it A, has a sum greater than nB, the sum of any one of the remaining $(n + 1)$-tuples satisfies

$$f - nB \leqq \text{sum of any } (n + 1)\text{-tuple except } A \leqq nB. \quad (75)$$

The left inequality follows from (72). To obtain the right inequality, note that all $(n + 1)$ elements (the f_i's) in A must be positive. Consider any other $(n + 1)$-tuple, say C. Then A contains k elements $1 \leqq k \leqq n$ which are not in C. Let $f_1, \cdots, f_{n+1}$ represent the elements of C so that the left side of (73) gives the sum of C. Then the right side of (73) contains k elements of A. Since the elements of A are positive, the right side of (73) is less than $f + (n - k)B$, and (75) follows from

$$f + (n - k)B \leqq f + (n - 1)B \leqq nB. \qquad (76)$$

APPENDIX D

Values of $G_{\theta(2n+1)}(f_1, \cdots, f_{2n+1})$

The notation used in this appendix is the same as that in Appendix C.

Let $G_{\theta(2n+1)}$ stand for $G_{\theta(2n+1)}(f_1, f_2, \cdots, f_{2n+1})$ where the f_i's satisfy the relations (71). Here we show that

$$G_{\theta(2n+1)} = 0, \qquad f_h > (n + 1)B \qquad (77)$$

where $n \geqq 1$ and f_h is the ideal filter semibandwidth. Furthermore, for a given set of $f_1, f_2, \cdots, f_{2n+1}$, it has been shown in Appendix C that there is at most only one $(n + 1)$-tuple, the sum of which exceeds nB. There may be none. When $nB < f_h < (n + 1)B$ with $n \geqq 1$ we shall show that

$$G_{\theta(2n+1)} = -1 \text{ if one } (n + 1)\text{-tuple sum} > f_h, \qquad (78)$$

$$G_{\theta(2n+1)} = 0 \text{ if no } (n + 1)\text{-tuple sum} > f_h. \qquad (79)$$

The inequalities (72) and (74) show that all of the Γ's in $G_{\theta(2n+1)}$ are unity (*i*) when $f_h > (n + 1)B$ or (*ii*) when no $(n + 1)$-tuple sum exceeds f_h where $nB < f_h < (n + 1)B$. Therefore, to prove (77) and (79), it is sufficient to show that $G_{\theta m}(f_1, \cdots, f_m)$, $m \geqq 2$, is zero when all of the Γ's in its expression (12) are equal to unity.

Consider the sum

$$\sum_{(\nu;\ell,m)} \sum_N{}' \Gamma(f_1 + \cdots + f_{\nu_1}) \cdots \Gamma(f_{m-\nu_\ell+1} + \cdots + f_m). \qquad (80)$$

When all of the Γ's = 1, this sum is equal to the number of different ways m different objects $(f_1, f_2, \cdots, f_m)$ can be put in ℓ identical boxes with no box empty (the ℓ pairs of parentheses enclosing the arguments of the ℓ Γ's). From combinatorial theory, this number is $S(\mathrm{m}, \ell)$ the Stirling number of the second kind given by the generating equation[8], for $n \geqq 1$,

$$t^n = \sum_{k=1}^{n} S(n, k)t(t - 1) \cdots (t - k + 1). \qquad (81)$$

To illustrate (80), let $m = 4$ and $\ell = 2$. Equations (13) show that the sum over $(\nu;\, \ell,\, m)$ in (80) now extends over the partitions of $m = 4$ which have $\ell = 2$ parts. There are two such partitions: $\nu_1 = 1$, $\nu_2 = 3$ and $\nu_1 = 2$, $\nu_2 = 2$. From (14), the corresponding values of N are $4!/1!3! = 4$ and $4!/2!2!2! = 3$, respectively. Hence the sum (80) is equal to $4 + 3 = 7$. This agrees with the known value $S(4, 2) = 7$. To return to the box problem, the 7 different ways of putting 4 different objects into 2 identical boxes with neither box empty is indicated by

$$(1)(234), \quad (2)(134), \quad (3)(124), \quad (4)(123),$$
$$(12)(34), \quad (13)(24), \quad (14)(23).$$

The expression (12) for $G_{\theta m}$ consists of the sum from $\ell = 1$ to $\ell = m$ of $j^{m-1}(-1)^{\ell-1}(\ell - 1)!$ times the sum (80). When all the Γ's are unity this gives

$$G_{\theta m} = j^{m-1} \sum_{\ell=1}^{m} (-1)^{\ell-1}(\ell - 1)!S(m, \ell) = \begin{cases} 1, & m = 1 \\ 0, & m > 1 \end{cases} \tag{82}$$

where the summation is accomplished by dividing (81) by t and then letting $t \to 0$. Setting $m = 2n + 1$ then gives (77) and (79).

Now we turn to (78). Let $f_{n+1} + f_{n+2} + \cdots + f_{2n+1}$ be the single $(n + 1)$-tuple whose sum exceeds f_h. Then $\Gamma(f_{n+1} + \cdots + f_{2n+1}) = 0$ and all the other Γ's in $G_{\theta(2n+1)}$ are unity. The problem is to determine the contribution of all of the terms in $G_{\theta(2n+1)}$ containing $\Gamma(f_{n+1} + \cdots + f_{2n+1})$. Subtracting this contribution from 0 will give the value of $G_{\theta(2n+1)}$.

Setting $m = 2n + 1$ in (12) shows that the terms in $G_{\theta(2n+1)}$ containing $\Gamma(f_{n+1} + \cdots + f_{2n+1})$ as a factor are those for which ℓ and the parts ν_i of the partition of $(2n + 1)$ into ℓ parts are such that

$$\begin{array}{lrl} \ell = 2, & \nu_1 = n, & \nu_2 = n + 1, \\ \ell = 3, & \nu_1 + \nu_2 = n, & \nu_3 = n + 1, \\ & \vdots & \\ \ell = n + 1, & \nu_1 + \cdots + \nu_n = n & \nu_{n+1} = n + 1. \end{array} \tag{83}$$

Therefore, with $k = \ell - 1$, the terms are the product of

$$j^{2n} \sum_{k=1}^{n} (-)^k k! \sum_{(\nu;\, k,\, n)} \sum_N{}' \Gamma(f_1 + \cdots + f_{\nu_1}) \cdots \Gamma(f_{n-\nu_k+1} + \cdots + f_n) \tag{84}$$

and $\Gamma(f_{n+1} + \cdots + f_{2n+1})$ where now

$$\nu_1 + \nu_2 + \cdots + \nu_k = n$$
$$\leqq \nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_k$$
$$N = n!/\nu_1! \cdots \nu_k! r_1! r_2! \cdots.$$

When all of the Γ's in (84) are unity, (84) becomes

$$j^{2n} \sum_{k=1}^{n} (-)^k k! S(n, k) = j^{2n}(-1)^n = 1 \tag{85}$$

where the summation is performed by setting $t = -1$ in the generating equation (81). Subtracting the contribution (85) from 0 gives the value $G_{\theta(2n+1)} = -1$ stated in (78).

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