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A Note on Optimal Approximating Manifolds of a Function Class

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Concept of n-width and extremal subspaces, first introduced by Kolmogorov, plays an important part in mathematical problems of approximation of classes of functions and in engineering problems of signal representation and reconstruction. In this short paper, explicit expressions for n-width and extremal subspaces are obtained for a class which is of some engineering importance.

I. INTRODUCTION

Kolmogorov[1] first introduced the concept of n-width as a measure of the degree of approximation of a set Ω of a normed linear space X by linear manifolds of finite dimension. If $\mathfrak{M}$ is an n-dimensional subspace of X, then its deviation $E(\Omega, \mathfrak{M})$ from Ω is defined as

$$E(\Omega, \mathfrak{M}) = \sup_{x \in \Omega} \{ \inf_{y \in \mathfrak{M}} \|x - y\| \} \tag{1}$$

and the n-width

$$d_n(\Omega) = \inf \{ E(\Omega, \mathfrak{M}) : \mathfrak{M} \subset X, \dim \mathfrak{M} = n \}. \tag{2}$$

If the lower bound in (2) is attained by $\mathfrak{M}_*$, then $\mathfrak{M}_*$ is called the n-dimensional extremal subspace.

Since Kolmogorov, several authors, including Lorentz,[2] Tihomirov,[3] Mitjagin,[4] Golomb,[5] and Jerome,[6] have obtained results on n-widths and extremal subspaces of several important function classes. Golomb has obtained expressions for n-widths and the narrowest subspaces for ellipsoids determined by a linear, nonnegative, self-adjoint operator in a Hilbert space. In this paper, these results are extended to the case of a class formed by the intersection of an ellipsoid and a unit sphere. These classes are important in signal representation and reconstruction problems whenever the magnitude of signals has to be constrained because of certain physical reasons.

II. THE MAIN RESULT

Following Golomb, if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are subspaces of a complex Hilbert space, H, with the inner product $\langle \cdot, \cdot \rangle$, let us call $\mathfrak{M}_1$ narrower than $\mathfrak{M}_2$ (written: $\mathfrak{M}_1 \ll \mathfrak{M}_2$) whenever $\mathfrak{M}_1^{\perp} \cap \mathfrak{M}_2 \neq \phi$, ϕ being the null space and $\mathfrak{M}_1^{\perp}$ the orthogonal complement of $\mathfrak{M}_1$ in H. The class Ω under consideration is defined by

$$\Omega_1 = \{f: f \in \mathfrak{D}(A), \langle Af, f \rangle \leqq K\}, \tag{3}$$

$$\Omega_2 = \{f: \langle f, f \rangle = 1\}, \tag{4}$$

and

$$\Omega = \Omega_1 \cap \Omega_2, \tag{5}$$

where A is a linear, nonnegative [i.e., $\langle Af, f \rangle \geqq 0$ for all $f \in \mathfrak{D}(A)$], not necessarily bounded, self-adjoint operator with domain $\mathfrak{D}(A)$ which is dense in H. Let $\lambda \rightarrow E_\lambda$ be the spectral family of A which is continuous from the left and $\mathcal{E}_\lambda$ be the range of E_λ.

The case $\Omega_1 \subset \Omega_2$ is not possible, and the case $\Omega_2 \subset \Omega_1$ leads to the following trivial result:

$$\begin{aligned} E(\Omega, \mathfrak{M}) &= 1, \quad \text{if} \quad \mathfrak{M} \neq H \\ &= 0, \quad \text{if} \quad \mathfrak{M} = H. \end{aligned} \tag{6}$$

Therefore, these two cases are not considered below. It is also assumed that[†] $\sigma_{\min} < \delta^{-2}$, $\sigma_{\min} < K < \delta^{-2}$, where $\sigma_{\min}$ is the minimum value of the spectrum of A.

Theorem 1:

$$(i) \quad E(\Omega, \mathfrak{M}) \geqq \left\{\frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}\right\}^{\frac{1}{2}}, \qquad \text{if} \quad \mathfrak{M} \ll \mathcal{E}_{\delta^{-2}}, \tag{7}$$

$$(ii) \quad E(\Omega, \mathfrak{M}) = \left\{\frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}\right\}^{\frac{1}{2}}, \qquad \text{if} \quad \mathfrak{M} = \mathcal{E}_{\delta^{-2}}. \tag{8}$$

Proof: By definition,

$$\mathfrak{M} \ll \mathcal{E}_{\delta^{-2}} \Rightarrow \mathfrak{M}^{\perp} \cap \mathcal{E}_{\delta^{-2}} \neq \phi.$$

Then let $\mathfrak{M}^{\perp} \cap \mathcal{E}_{\delta^{-2}} = G$ and consider the following two cases:

(*A*) $\mathcal{E}_{\sigma_{\min}}$ is not empty.

In this case, take $f_0 \in \mathcal{E}_{\delta^{-2}}$ such that

$$f_0 = f_1 + f_2, \qquad f_1 \in G, \; f_2 \in \mathcal{E}_{\sigma_{\min}}, \tag{9}$$

$$\|f_1\|^2 = \frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}, \tag{10}$$

† If $K < \sigma_{\min}$, Ω will be empty.

and

$$f_2 = \ell \cdot \frac{P_{\mathcal{E}_{\sigma_{\min}}} f_1}{\|P_{\mathcal{E}_{\sigma_{\min}}} f_1\|} \quad \text{if} \quad P_{\mathcal{E}_{\sigma_{\min}}} f_1 \neq 0$$
$$= \ell \cdot e \quad \text{otherwise,} \tag{11}$$

where e is any unit vector in $\mathcal{E}_{\sigma_{\min}}$ and

$$\ell = \sqrt{1 - \|P_{\mathcal{E}^{\perp}_{\sigma_{\min}}} f_1\|^2} - \|P_{\mathcal{E}_{\sigma_{\min}}} f_1\|. \tag{12}$$

This construction of f_0 is illustrated geometrically in Fig. 1, where OA is a vector in G and OB is along the subspace spanned by the projection $P_{\mathcal{E}_{\sigma_{\min}}} f_1$. Then f_0 is the vector sum of f_1 and a vector f_2 along OB such that f_0 has unit magnitude.

It is easy to see that

$$\begin{aligned} \|f_0\|^2 &= \|f_1 + f_2\|^2 \\ &= \|P_{\mathcal{E}_{\sigma_{\min}}} f_1 + f_2\|^2 + \|P_{\mathcal{E}^{\perp}_{\sigma_{\min}}} f_1\|^2 \\ &= 1 \end{aligned} \tag{13}$$

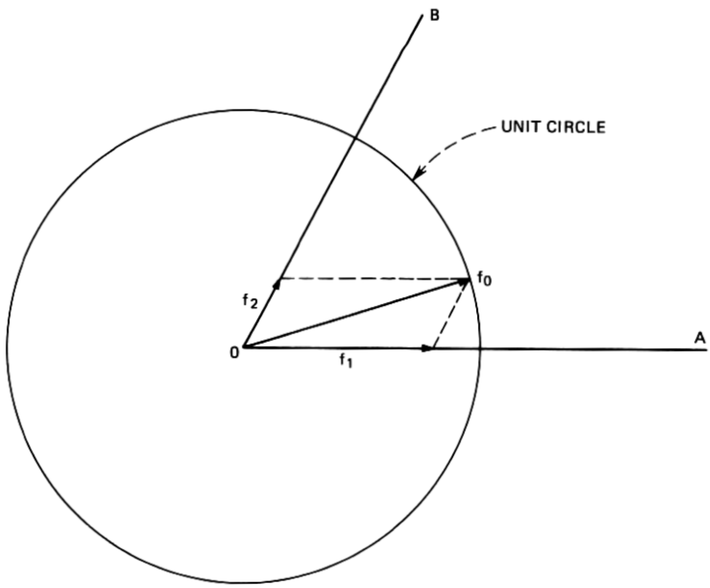

Fig. 1—Geometrical construction of f_0.

and

$$\begin{aligned}\langle Af_0, f_0\rangle &= \int_{\sigma_{\min}}^{\delta^{-2}} \lambda d[\|E_\lambda f_0\|^2] \\ &\leqq \sigma_{\min}[\|P_{\mathcal{E}_{\sigma_{\min}}} f_1 + f_2\|^2] + \delta^{-2}[\|P_{\mathcal{E}^{\perp}_{\sigma_{\min}}} f_1\|^2] \\ &\leqq \sigma_{\min}[1 - \|P_{\mathcal{E}^{\perp}_{\sigma_{\min}}} f_1\|^2] + \delta^{-2}[\|P_{\mathcal{E}^{\perp}_{\sigma_{\min}}} f_1\|^2] \\ &\leqq \|P_{\mathcal{E}^{\perp}_{\sigma_{\min}}} f_1\|^2[\delta^{-2} - \sigma_{\min}] + \sigma_{\min} \\ &\qquad\qquad \leqq K - \sigma_{\min} + \sigma_{\min} = K. \end{aligned} \tag{14}$$

Hence $f_0 \in \Omega$. Now

$$\begin{aligned}\|P_G f_0\|^2 & \\ &= \langle f_1 + P_G f_2, f_1 + P_G f_2\rangle \\ &= \langle f_1, f_1\rangle + \langle P_G f_2, P_G f_2\rangle + 2\langle f_1, P_G f_2\rangle \\ &= \langle f_1, f_1\rangle + \langle P_G f_2, P_G f_2\rangle \\ &\qquad + \frac{2\ell}{\|P_{\mathcal{E}_{\sigma_{\min}}} f_1\|}\langle f_1, P_G P_{\mathcal{E}_{\sigma_{\min}}} f_1\rangle, \qquad \text{if} \quad P_{\mathcal{E}_{\sigma_{\min}}} f_1 \neq 0 \end{aligned} \tag{15}$$

$$\geqq \|f_1\|^2 = \frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}. \tag{16}$$

Also, if $P_{\mathcal{E}_{\sigma_{\min}}} f_1 = 0$, then the third term in the RHS of eq. (15) is equal to zero and (16) still holds. Equations (13), (14), and (16) together imply that

$$E^2(\Omega, \mathfrak{M}) \geqq E^2(f_0, \mathfrak{M}) \geqq \frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}. \tag{17}$$

(*B*) $\mathcal{E}_{\sigma_{\min}}$ is empty.

Since $\sigma_{\min}$ is a point in the spectrum, a sequence exists of nonincreasing and nonnegative real numbers $\{\delta_i\}$ such that each δ_i is in the spectrum of A and the following conditions are met:

$$\lim_i \delta_i = \sigma_{\min}, \tag{18a}$$

$$\delta_i < K, \qquad \text{for all } i, \tag{18b}$$

and $\mathcal{E}_{\delta_i}$ is not empty (i.e., it contains more than the zero element) for any δ_i. Then, for each δ_i using the same construction as in (*A*) above (using $\mathcal{E}_{\delta_i}$ instead of $\mathcal{E}_{\sigma_{\min}}$), it can be shown that

$$E^2(\Omega, \mathfrak{M}) \geqq \frac{K - \delta_i}{\delta^{-2} - \delta_i}. \tag{19}$$

However, since $(K - \delta_i)/(\delta^{-2} - \delta_i)$ is a nondecreasing sequence of positive numbers tending to $(K - \sigma_{\min})/(\delta^{-2} - \sigma_{\min})$,

$$E^2(\Omega, \mathfrak{M}) \geqq \frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}. \tag{20}$$

This proves (*i*).

(*ii*) Consider now the case in which $\mathfrak{M} = \mathcal{E}_{\delta^{-2}}$. For any $f \in \Omega$,

$$
\begin{aligned}
E^2(f, \mathcal{E}_{\delta^{-2}}) &= \|f - E_{\delta^{-2}}f\|^2 \\
&= \int_{\delta^{-2}}^{\infty} d[\|E_\lambda f\|^2] \\
&\leqq \frac{1}{\delta^{-2} - \sigma_{\min}} \int_{\delta^{-2}}^{\infty} (\lambda - \sigma_{\min}) d(\|E_\lambda f\|^2) \\
&\leqq \frac{1}{\delta^{-2} - \sigma_{\min}} \int_{\sigma_{\min}}^{\infty} (\lambda - \sigma_{\min}) d(\|E_\lambda f\|^2) \qquad (21) \\
&\leqq \frac{K - \sigma_{\min}}{\delta^{-2} - \delta_{\min}}. \qquad (22)
\end{aligned}
$$

Equation (22) follows from (21) because, for all $f \in \Omega$,

$$\int_{\sigma_{\min}}^{\infty} \lambda d[\|E_\lambda f\|^2] \leqq K \tag{23}$$

and

$$\int_{\sigma_{\min}}^{\infty} d[\|E_\lambda f\|^2] = \|f\|^2 = 1. \tag{24}$$

It follows from (22) that

$$E(\Omega, \mathcal{E}_{\delta^{-2}}) \leqq \left\{\frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}\right\}^{\frac{1}{2}}. \tag{25}$$

Theorem 2:

$$E(\Omega, \mathcal{E}_{\delta^{-2}}) = \left\{\frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}\right\}^{\frac{1}{2}} \tag{26}$$

if and only if δ^{-2} *is in the spectrum of* A *and, in this case,*

$$E(\Omega, \mathfrak{M}_{\delta^{-2}}) > \left\{\frac{K - \sigma_{\min}}{\delta^{-2} - \sigma_{\min}}\right\} \tag{27}$$

if $\mathfrak{M} \ll \mathcal{E}_{\delta^{-2}}$.

Proof of this theorem is similar to the proof of Theorem 2 in Golomb[5] and is therefore omitted.

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