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Transverse Coupling in Fiber Optics Part I: Coupling Between Trapped Modes

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Two perturbation formulas have been proposed to evaluate the coupling between parallel optical waveguides, one involving a line integral and the other a surface integral. They are shown to be identical. The former expression is preferred because of its greater simplicity. The case of two parallel lossy dielectric slabs is discussed as an example.

I. INTRODUCTION

There has been a renewed interest during the last few years in the evaluation of the transverse* coupling between two parallel open waveguides in connection with integrated optics circuitry[2,3] and long-distance optical communication by bundles of glass fibers.

The coupling between two open waveguides can be obtained by replacing the field of one waveguide by an equivalent current and evaluating the perturbation caused by this current on the other waveguide.[4] A more direct and slightly more general (but essentially equivalent) derivation, based on Lorentz's reciprocity theorem, is given in this paper. A related result, applicable only to lossless fibers, has been used to evaluate the coupling between dielectric rods with circular cross section.[5] The perturbation formula derived in this paper involves an integral along a contour located between the two waveguides. A seemingly different perturbation formula has been recently proposed that involves a surface integral over the cross section.[6] The two formulas are shown to be in fact identical. We will not discuss in detail other coupling formulas such as the ones proposed in Refs. 7 or 3. In Ref. 7, the coupling is obtained by applying the Rayleigh-Ritz

* The word "transverse" is used here to distinguish the problem of two dielectric waveguides lying side by side, where the transfer of power takes place in transverse directions, and the axial coupling between two waveguides placed end to end, where the transfer of power takes place along the z axis (the later arrangement is discussed, for instance, in Ref. 1).

optimization technique to a variational expression. The formula obtained by this method involves surface integrals and is rather complicated. In Ref. 3, analytic expressions were obtained for the coupling between two identical rectangular fibers that agree well with numerical calculations based on the exact field equations. The approach, however, is restricted to fibers with a particular geometry.

II. GENERAL EXPRESSION OF THE COUPLING

Let the time dependence of the sources be denoted $\exp(-\kappa t)$. Maxwell's equations are, in a source-free region with scalar permittivity ϵ and permeability μ_o,

$$\nabla \times \mathbf{E} = \kappa\mu_o\mathbf{H}, \tag{1a}$$

$$\nabla \times \mathbf{H} = -\kappa\epsilon\mathbf{E}. \tag{1b}$$

Any two solutions $(\mathbf{E}, \mathbf{H})$ and $(\mathbf{E}_a, \mathbf{H}_a)$ of eq. (1) satisfy the relation

$$\nabla \cdot \mathbf{J} = 0, \tag{2a}$$

where

$$\mathbf{J} = \mathbf{E}_a \times \mathbf{H} - \mathbf{E} \times \mathbf{H}_a. \tag{2b}$$

Integrating over a volume V, Lorentz reciprocity theorem is obtained

$$\int_S (\mathbf{E}_a \times \mathbf{H} - \mathbf{E} \times \mathbf{H}_a) \cdot \mathbf{d}S = 0, \tag{2c}$$

where S denotes the surface enclosing V, and $\mathbf{d}S$ a vector normal to S pointing outward with magnitude dS. Let the medium be uniform along z, that is, ϵ be independent of z. If

$$(\mathbf{E}, \mathbf{H}) \equiv (E_z, \mathbf{E}_t, H_z, \mathbf{H}_t) \exp(\gamma z) \tag{3}$$

denotes a solution of Maxwell's equations, then

$$(\mathbf{E}^+, \mathbf{H}^+) \equiv (-E_z, \mathbf{E}_t, H_z, -\mathbf{H}_t) \exp(-\gamma z) \tag{4}$$

is also a solution of Maxwell's equations. The arguments x, y have been omitted in the above expressions, and the subscripts t stand for "transverse." The field $(\mathbf{E}^+, \mathbf{H}^+)$ is the field adjoint to $(\mathbf{E}, \mathbf{H})$; it describes a wave propagating in an opposite direction in the same medium and at the same frequency. A more general definition of the adjoint field, applicable to nonreciprocal media, can be found in Ref. 8.

Let us now consider two open waveguides a and b uniform along the z-axis, and let S be the surface $S_a + S_a' + C_a dz$ shown in Fig. 1. The field $(\mathbf{E}_a, \mathbf{H}_a)$ in eq. (2) is taken as the field of a trapped mode on

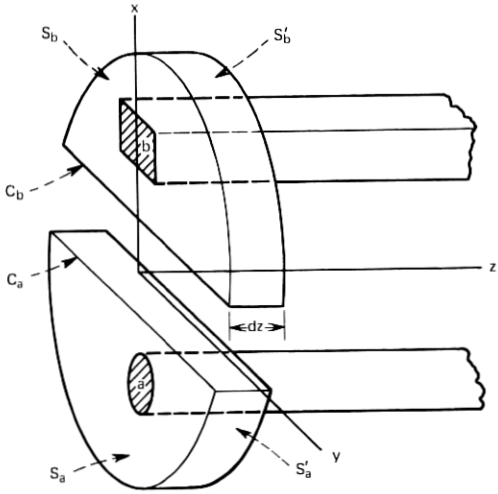

Fig. 1—Coupled dielectric waveguides.

waveguide a in the absence of waveguide b. The dependence of $(\mathbf{E}_a, \mathbf{H}_a)$ on z is denoted $\exp(\gamma_a z)$. The field $(\mathbf{E}^+, \mathbf{H}^+)$ is the adjoint field of a trapped mode of the two coupled waveguides, with an $\exp(-\Gamma z)$ dependence on z. Letting the spacing dz between S_a' and S_a tend to zero, eq. (2) becomes

$$(\gamma_a - \Gamma) \int_{S_a} (\mathbf{E}_a \times \mathbf{H}^+ - \mathbf{E}^+ \times \mathbf{H}_a) \cdot \mathbf{d}S_a$$
$$= -\int_{C_a} (\mathbf{E}_a \times \mathbf{H}^+ - \mathbf{E}^+ \times \mathbf{H}_a) \cdot \mathbf{d}C_a, \quad (5a)$$

where $\mathbf{d}C_a$ is a vector perpendicular to the contour C_a, pointing outward. Proceeding similarly for waveguide b we obtain

$$(\gamma_b - \Gamma) \int_{S_b} (\mathbf{E}_b \times \mathbf{H}^+ - \mathbf{E}^+ \times \mathbf{H}_b) \cdot \mathbf{d}S_b$$
$$= -\int_{C_b} (\mathbf{E}_b \times \mathbf{H}^+ - \mathbf{E}^+ \times \mathbf{H}_b) \cdot \mathbf{d}C_b. \quad (5b)$$

Because the coupling between the two waveguides is small, we can assume that the field $\mathbf{E}$, $\mathbf{H}$ at plane $z = 0$ is the sum of the fields of the two waveguides, that is,

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b, \qquad (6)$$
$$\mathbf{H} = \mathbf{H}_a + \mathbf{H}_b.$$

Substituting these expressions, eqs. (6), in eqs. (5a) and (5b) we observe that the cross terms can be neglected on the left-hand sides (l.h.s.) because ($\mathbf{E}_b$, $\mathbf{H}_b$) is small when ($\mathbf{E}_a$, $\mathbf{H}_a$) is large, and vice versa. On the right-hand sides (r.h.s.), on the contrary, only the cross terms remain, as we can verify by applying Lorentz reciprocity theorem to each waveguide. Multiplying together the l.h.s. and r.h.s. of eq. (5a) and (5b) the desired equation for Γ is obtained.

$$(\Gamma - \gamma_a)(\Gamma - \gamma_b) = c_a c_b / P_a P_b, \tag{6a}$$

where

$$c_{a,b} = \int_{C_{a,b}} (\mathbf{E}_{a,b} \times \mathbf{H}_{b,a}^{+} - \mathbf{E}_{b,a}^{+} \times \mathbf{H}_{a,b}) \cdot \mathbf{d}C_{a,b}, \tag{6b}$$

$$P_{a,b} = \int_{S_{a,b}} (\mathbf{E}_{a,b} \times \mathbf{H}_{a,b}^{+} - \mathbf{E}_{a,b}^{+} \times \mathbf{H}_{a,b}) \cdot \mathbf{d}S_{a,b}. \tag{6c}$$

Because the coupling takes place only if $\gamma_a \sim \gamma_b$, the coupling c_a (resp. c_b) is independent of the choice of the contour C_a (resp. C_b) as long as it surrounds only one waveguide. By choosing the two contours as coincident in the region where the fields of the two trapped modes have a significant intensity and using eq. (4), we find that c_a is equal to c_b. It is shown in the appendix that our result, eq. (6), can be expressed in the form given in Ref. 6. The expression, eq. (6), however, is simpler to evaluate.

Let us now assume that the contours C_a, C_b coincide with the y axis and are closed at infinity where the fields vanish. The general expression, eq. (6), becomes

$$(\Gamma - \gamma_a)(\Gamma - \gamma_b) = c^2 / P_a P_b, \tag{7}$$

where

$$c = \frac{1}{2} \int_{-\infty}^{+\infty} (E_{ay} H_{bz} + E_{az} H_{by} - E_{by} H_{az} - E_{bz} H_{ay}) dy,$$

and

$$P_a = \int \int_{-\infty}^{+\infty} (\mathbf{E}_a \times \mathbf{H}_a) \cdot \mathbf{z} dx dy,$$

$$P_b = \int \int_{-\infty}^{+\infty} (\mathbf{E}_b \times \mathbf{H}_b) \cdot \mathbf{z} dx dy,$$

where $\mathbf{z}$ denotes the unit vector directed along the z axis.

Let us specialize eq. (7) to symmetrical stratified dielectric waveguides such as the slabs shown in Fig. 2. The fields are assumed to be independent of y. For TE waves the electric field has only one component $E_y \equiv E$. We have, from Maxwell's equations, eq. (1)

$$E_x = E_z = H_y = 0, \tag{8a}$$

$$H_z = (\kappa\mu_o)^{-1} dE/dx, \tag{8b}$$

$$H_x = -(\gamma/\kappa\mu_o)E. \tag{8c}$$

Equation (7) thus reduces to the simpler form

$$(\Gamma - \gamma_a)(\Gamma - \gamma_b) = (1 - \kappa^2/\gamma^2)E_a^2 E_b^2 \Big/ \left(\int_{-\infty}^{+\infty} E_a^2 dx \int_{-\infty}^{+\infty} E_b^2 dx \right), \tag{9}$$

the fields E_a and E_b of the uncoupled waveguides being evaluated at the same point between the two waveguides.

III. COUPLING BETWEEN LOSSY DIELECTRIC SLABS

If the waveguides are homogeneous dielectric slabs of thickness $2d$ and complex permittivity ϵ we have

$$E = \exp(\mp\gamma_{xo}x) \tag{10a}$$

above or below the slabs and, for even modes,

$$E = \cosh(\gamma_x x)/\cosh(\gamma_x d) \tag{10b}$$

within the slabs (obvious changes in the origin of the x axis were made). In eqs. (10a) and (10b) we have defined

$$\gamma_{xo}^2 \equiv k_o^2 - \gamma^2, \qquad \text{Real }(\gamma_{xo}) > 0, \tag{11a}$$

$$\gamma_x^2 \equiv k_o^2 n^2 - \gamma^2, \tag{11b}$$

$$k_o^2 \equiv \kappa^2 \epsilon_o \mu_o, \tag{11c}$$

$$n^2 \equiv \epsilon/\epsilon_o.$$

The propagation constant γ is known to satisfy

$$\gamma_x \tanh(\gamma_x d) + \gamma_{xo} = 0. \tag{12}$$

(See, for instance, Ref. 6.) Substituting E from eqs. (10a) and (10b) in eq. (9) we obtain, using eq. (12),

$$\Gamma - \gamma = \pm\gamma^{-1}\gamma_x^2\gamma_{xo}^2(1 + \gamma_{xo}d)^{-1}(\gamma_x^2 - \gamma_{xo}^2)^{-1} \exp(-\gamma_{xo}D), \tag{13}$$

where D denotes the spacing between the slabs. This expression coincides with the result given in Ref. 6 when appropriate changes of notation are made.

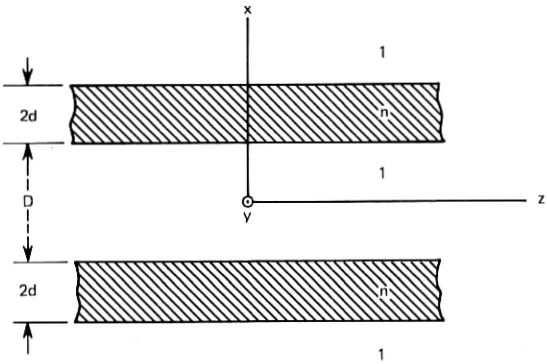

Fig. 2—Coupled dielectric slabs.

Let us now make a general comment. The coupling formula, eq. (6), rests on the existence of a divergenceless quantity, the vector **J** in eq. (2a). Coupling formulas similar to eq. (6) can be derived from other wave equations. For the case of the scalar parabolic wave equation[9] applicable to the propagation of radio waves in atmospheric ducts,* the vector **J** has components

$$J_z = 2k_o E_a^+ E \approx E_a^+ \partial E/\partial z - E \partial E_a^+/\partial z,$$
$$\mathbf{J}_t = E_a^+ \nabla_t E - E \nabla_t E_a^+,$$

where the adjoint field is

$$E_a^+(x, y, z) = E_a(x, y, -z).$$

The above expression for **J** can be obtained by analogy with the equivalent quantum mechanical problem.[9]

In conclusion, we have derived a simple coupling formula which is more general than previous similar expressions[4,5] because it is applicable to lossy fibers. In order to evaluate explicitly the coupling, one needs to know the normalized field of each waveguide, in the absence of the other, along some line located between the two waveguides. For slabs and rods with circular cross section, exact solutions are available. In general, however, we have to resort to numerical techniques or to measurements made at a convenient wavelength on a scaled version of the open waveguide. In a second part of this paper,[10] we will apply eq. (6) to mode-selecting systems.

* A similar equation is applicable to anisotropic fibers that have small transverse variation of permittivity.[8] Note that, in this approximation, a curvature of the fiber axis is equivalent to a constant gradient of refractive index.

APPENDIX

The purpose of this appendix is to show that for lossless fibers the coupling formulas given in Refs. 4 and 6 coincide.

Let (**E**, **H**) denote a field in free space

$$\begin{aligned} \nabla \times \mathbf{E} &= \kappa\mu_o \mathbf{H}, \\ \nabla \times \mathbf{H} &= -\kappa\epsilon_o \mathbf{E}, \end{aligned} \tag{14}$$

and ($\mathbf{E}_b$, $\mathbf{H}_b$) a field in a dielectric with permittivity $\epsilon(\mathbf{r})$

$$\begin{aligned} \nabla \times \mathbf{E}_b &= \kappa\mu_o \mathbf{H}_b, \\ \nabla \times \mathbf{H}_b &= -\kappa\epsilon \mathbf{E}_b. \end{aligned} \tag{15}$$

It is easy to show that these fields satisfy the relation

$$\int_S (\mathbf{E} \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}) \cdot \mathbf{d}S = \kappa \int_V (\epsilon - \epsilon_o) \mathbf{E} \cdot \mathbf{E}_b dV \tag{16}$$

in any source-free volume V bounded by S. Let now the surface S be the surface $S_b + S_b' + C_b dz$ shown in Fig. 1, ($\mathbf{E}_b$, $\mathbf{H}_b$) be the field of a trapped mode of waveguide b with an exp $(\gamma_b z)$ dependence on z, and (**E**, **H**) be the adjoint field ($\mathbf{E}_a^+$, $\mathbf{H}_a^+$) of a trapped mode of waveguide a, with an exp $(-\gamma_a z)$ dependence on z. The field ($\mathbf{E}_a^+$, $\mathbf{H}_a^+$) satisfies eq. (14) inside the surface S that we have just defined. If the two trapped modes are degenerate, that is, if $\gamma_a = \gamma_b$, the contributions of the two surfaces S_b and S_b' on the l.h.s. of eq. (16) cancel out. Therefore, letting dz tend to zero, eq. (16) becomes

$$\int_{C_b} (\mathbf{E}_a^+ \times \mathbf{H}_b - \mathbf{E}_b \times \mathbf{H}_a^+) \cdot \mathbf{d}C_b = \kappa \int_{S_b} (\epsilon - \epsilon_o) \mathbf{E}_a^+ \cdot \mathbf{E}_b dS_b. \tag{17}$$

A similar relation can be obtained for waveguide a. Our coupling equation, eq. (6), can therefore be written in the form given in Ref. 6, except for the fact that in eq. (6) $\mathbf{E}_a^+$ and $\mathbf{E}_b$ represent fields at the same frequency. In Ref. 6 the field $\mathbf{E}_a^+$ is defined at the opposite angular frequency $-\kappa$, that is, $\mathbf{E}_a^+$ is replaced by $\mathbf{E}_a^{+*}$, where the asterisk denotes complex conjugation. For lossless fibers this difference is unimportant because $\mathbf{E}_a^+$ can be assumed real.

REFERENCES

1. J. A. Arnaud, "Mode Coupling in First-Order Optics," J. Opt. Soc. Am., *61*, June 1971, p. 751.
2. S. E. Miller, "Integrated Optics: An Introduction," B.S.T.J., *48*, No. 7 (September 1969), p. 2059.
3. E. A. J. Marcatili, "Dielectric Rectangular Waveguide and Directional Coupler for Integrated Optics," B.S.T.J., *48*, No. 7 (September 1969), p. 2071.

4. J. A. Arnaud, "General Properties of Periodic Structures," Sections B and E, in *Crossed Field Microwave Devices*, Vol. 1, E. Okress ed., New York: Academic Press, 1961.
5. R. Vanclooster and P. Phariseau, "The Coupling of Two Parallel Dielectric Fibers," Physica, *7*, June 1970, pp. 485 and 501.
6. D. Marcuse, "The Coupling of Degenerate Modes in Two Parallel Dielectric Waveguides," B.S.T.J., *50*, No. 6 (July-August 1971), pp. 1791–1816.
7. M. Matsuhara and N. Numagai, "Coupling Theory of Open-Type Transmission Lines and its Application to Optical Circuits," Electron. Commun. Japan, *55*, April 1972, p. 102.
8. J. A. Arnaud, "Biorthogonality Relations for Bianisotropic Media," J. Opt. Soc. Am., *63*, February 1973, p. 238.
9. V. A. Fock, "Theory of Radio-Wave Propagation in an Inhomogeneous Atmosphere for a Raised Source," Bull. Acad. Sciences de l'URSS, sér. phys. *14*, No. 1, p. 70 (1950). In Russian. Translation in V. A. Fock, *Electromagnetic Diffraction and Propagation Problems*, Oxford: Pergamon Press, 1965, Chapter 14.
10. J. A. Arnaud, "Transverse Coupling in Fiber Optics, Part II," to appear in B.S.T.J., *56*, No. 4 (April 1974).