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On the Behavior of Minimax Relative Error FIR Digital Differentiators

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Optimum (in a minimax relative error sense) linear phase FIR digital differentiators can be designed in an efficient manner using a Remez optimization procedure. This paper presents data on wideband differentiators designed with even and odd values of N, the filter impulse response duration in samples. Based on these data, several interesting observations can be made, including:

(i) Differentiators with even values of N have peak relative errors which are approximately one to two orders of magnitude smaller than identical bandwidth differentiators with odd values of N, and with the same number of multiplications per sample in a direct convolution realization.

(ii) The smaller the bandwidth of the differentiator, the faster the decrease of the peak relative error with increasing N.

(iii) The larger the value of N, the faster the decrease of the peak relative error with decreasing bandwidth.

These observations lead to the conclusions that the bandwidth of a differentiator should be made as small as possible, and that even values of N should be used whenever possible. Complete tables of values of the impulse response coefficients are included for several wideband differentiators for even and odd values of N.

I. INTRODUCTION

In the past few years a great deal of work has been done in devising filter design techniques capable of obtaining optimum approximations (in the Chebyshev or minimax sense) to a prescribed frequency response characteristic. A general-purpose algorithm now exists[1] for the design of such optimum linear phase finite-duration impulse response (FIR) approximations to any desired multiband filter, differentiator, or Hilbert transformer. Since differentiators are an integral part of many practical systems,[2–4] it is the purpose of this paper to present new data on the characteristics of optimum FIR differentiators, as an aid in making informed decisions concerning their use.

II. DISCRETE-TIME DIFFERENTIATORS

A differentiator is a system whose output is the derivative of its input. The frequency response of a differentiator is purely imaginary and is proportional to frequency. A sequence of samples of the derivative of a band-limited signal can be obtained by filtering a sequence of samples of the signal with a digital filter that approximates the ideal frequency response of a differentiator over the bandwidth of the signal. Therefore, digital filters having this type of frequency response are also called differentiators.

The frequency response of the ideal digital differentiator with a delay of τ samples is

$$\begin{aligned} H_d(e^{j\omega}) &= j\omega e^{-j\omega\tau} & 0 \leqq \omega \leqq \pi \\ &= j(\omega - 2\pi)e^{-j(\omega-2\pi)\tau} & \pi < \omega \leqq 2\pi. \end{aligned} \tag{1}$$

The impulse response corresponding to eq. (1) is obtained as the inverse Fourier transform of eq. (1) and is given by

$$h_d(n) = \frac{1}{2\pi}\left[\int_0^{\pi} j\omega e^{-j\omega\tau}e^{j\omega n}d\omega + \int_{\pi}^{2\pi} j(\omega - 2\pi)e^{-j(\omega-2\pi)\tau}e^{j\omega n}d\omega\right], \tag{2}$$

which can be written as

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega(n-\tau)}d\omega \tag{3}$$

$$= \frac{\cos[\pi(n-\tau)]}{(n-\tau)} - \frac{\sin[\pi(n-\tau)]}{\pi(n-\tau)^2}. \tag{4}$$

For $\tau = 0$, eq. (4) gives

$$\begin{aligned} h_d(n) &= 0 & n = 0 \\ &= \frac{\cos(\pi n)}{n} & n \neq 0, \end{aligned} \tag{5}$$

whereas for $\tau = -\frac{1}{2}$ (i.e., a half-sample advance) eq. (4) gives

$$h_d(n) = \frac{-\cos(\pi n)}{\pi(n+\frac{1}{2})^2} = \frac{-4}{\pi}\,\frac{\cos(\pi n)}{(2n+1)^2}. \tag{6}$$

The impulse response of eq. (5), which corresponds to an ideal differentiator with zero delay, is of infinite duration and obeys the symmetry condition

$$h_d(n) = -h_d(-n) \qquad n = 1, 2, \cdots. \tag{7}$$

The impulse response of eq. (6), which corresponds to an ideal differentiator with one-half-sample advance, is of infinite duration and obeys the symmetry condition

$$h_d(n) = -h_d(-n-1) \qquad n = 0, 1, \cdots. \tag{8}$$

In Ref. 2 it was shown that the frequency response of an ideal differentiator with zero delay had a discontinuity at half the sampling frequency [i.e., $\omega = \pi$ in eq. (1) with $\tau = 0$], whereas the frequency response of an ideal differentiator with a one-half-sample advance had no discontinuity at $\omega = \pi$. The frequency response of a half-sample-advance differentiator has a slope discontinuity at $\omega = \pi$ but, as we will see, slope discontinuities are much easier to approximate than function discontinuities. It should be noted that the output of a one-half-sample-advance differentiator is the derivative of the input signal evaluated midway between input samples. For numerical analysis applications where one desires the derivative at the sample point rather than midway between samples, the use of differentiators with zero delay is required. For most signal processing applications, either type of differentiator is generally appropriate.

We have only considered $\tau = 0$ and $\tau = -\frac{1}{2}$ as possible delays for the ideal differentiator. It can be seen from eq. (4) that these are the only values of $(-1 < \tau \leqq 0)$ such that the impulse response has desirable symmetry properties.

In order to obtain a causal approximation to the ideal differentiator which has no phase distortion (other than that corresponding to delay), it can be shown that an FIR approximation is required.[5] Therefore, consider a causal FIR filter with impulse response $h(n)$, $0 \leqq n \leqq N-1$, and frequency response

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}. \tag{9}$$

For this system to have exactly linear phase, the impulse response sequence must satisfy the condition

$$h(n) = -h(N-1-n) \qquad n = 0, 1, \cdots, N-1. \tag{10}$$

For N an odd integer, this means that $h(n)$ has odd symmetry about the sample at $n = (N-1)/2$. [This case corresponds to the $\tau = 0$ case above with an additional delay of $(N-1)/2$ samples.] For N

even, $h(n)$ has odd symmetry about a point halfway between the samples at $n = N/2$ and $n = (N/2) + 1$. (This case corresponds to the $\tau = -\frac{1}{2}$ case above with an additional delay of $N/2$ samples.) This implies that the frequency response of a filter satisfying eq. (10) can be expressed as

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}[jH^*(e^{j\omega})], \tag{11}$$

where $H^*(e^{j\omega})$ is a purely real function of ω. In particular, for N odd, $H^*(e^{j\omega})$ is

$$H^*(e^{j\omega}) = \sum_{n=1}^{(N-1)/2} a(n) \sin(\omega n), \tag{12a}$$

where

$$a(n) = 2h\left(\frac{N-1}{2} - n\right) \qquad n = 1, 2, \cdots, \left(\frac{N-1}{2}\right). \tag{12b}$$

Also, from eq. (10),

$$h\left(\frac{N-1}{2}\right) = 0. \tag{12c}$$

For N even, the expression for $H^*(e^{j\omega})$ is

$$H^*(e^{j\omega}) = \sum_{n=1}^{N/2} b(n) \sin[\omega(n - \tfrac{1}{2})], \tag{13a}$$

where

$$b(n) = 2h\left(\frac{N}{2} - n\right) \qquad n = 1, 2, \cdots, N/2. \tag{13b}$$

The factor $e^{-j\omega(N-1)/2}$ in eq. (11) represents a delay of $(N-1)/2$ samples, which may be accounted for when necessary. Therefore, in approximating the differentiator, the coefficients $a(n)$ and $b(n)$ must be chosen so that $jH^*(e^{j\omega})$ approximates the ideal differentiator frequency response of eq. (1) (with τ either 0 or $-\frac{1}{2}$). In many cases it is not required, and often it is not desirable, that the approximation be carried out over the entire band $0 \leqq \omega \leqq \pi$. Thus, in general, the real function $H^*(e^{j\omega})$ must approximate

$$\begin{aligned} D(e^{j\omega}) &= \omega & 0 \leqq \omega \leqq 2\pi F_p \\ &= (\omega - 2\pi) & 2\pi(1 - F_p) \leqq \omega \leqq 2\pi, \end{aligned} \tag{14}$$

where F_p is the highest frequency where a good approximation to a differentiator is desired. In the interval $2\pi F_p < \omega < 2\pi(1 - F_p)$, the frequency response of the differentiator is generally left unconstrained —although it certainly could be constrained to approximate zero over

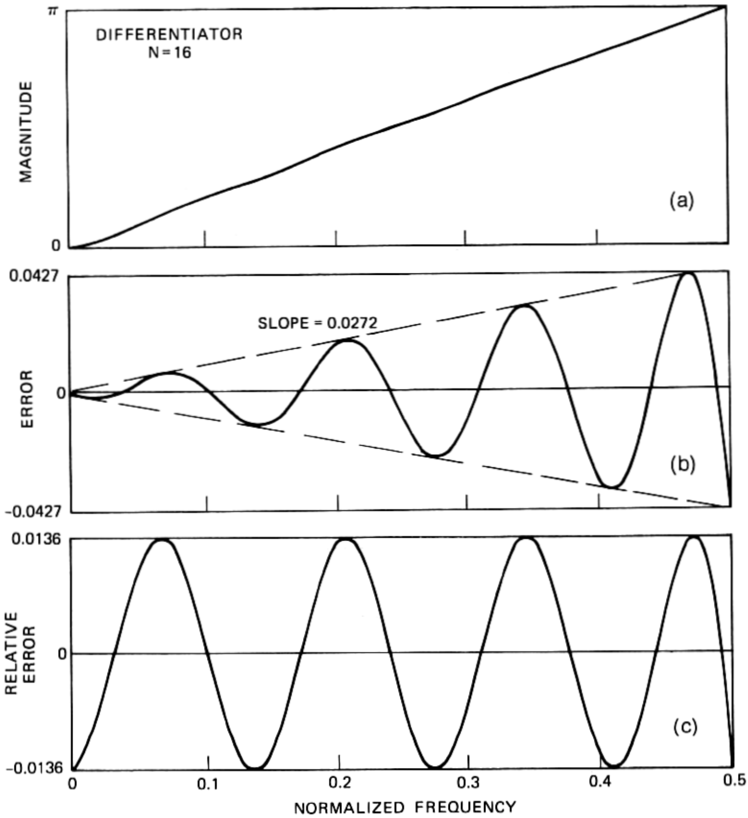

Fig. 1—The frequency response and error curves of an $N = 16$ wideband differentiator.

this interval. The frequency F_p is called the bandwidth of the differentiator. When $F_p = 0.5$, eqs. (1) and (14) are identical (to within the delay τ). This case is called a full-band differentiator.

The iterative Remez algorithm of McClellan, Parks, and Rabiner can be used to choose the values of $a(n)$ or $b(n)$ that minimize the peak relative error of approximation

$$\delta = \max_{0 \leqq \omega \leqq 2\pi F_p} \left[\frac{D(e^{j\omega}) - H^*(e^{j\omega})}{D(e^{j\omega})} \right]. \tag{15}$$

Our present concern is the general properties of the resulting approximations rather than the details of the approximation algorithm which are available in Ref. 1. The general properties of differentiators designed by the optimization procedure are illustrated by examples given in the next section.

III. CHARACTERISTICS OF OPTIMUM DIFFERENTIATORS

The approximation to the ideal differentiator is characterized by a relative error function that is equiripple over the approximation band $0 \leqq \omega \leqq 2\pi F_p$. This is illustrated by Figs. 1 through 4 which are plots of the frequency responses of several wideband differentiators. Figure 1 shows the magnitude response, the approximation error $[D(e^{j\omega}) - H^*(e^{j\omega})]$, and the relative approximation error $[D(e^{j\omega}) - H^*(e^{j\omega})]/[D(e^{j\omega})]$ as a function of frequency for $N = 16$ and $F_p = 0.5$. The relative error curve can be seen to be equiripple

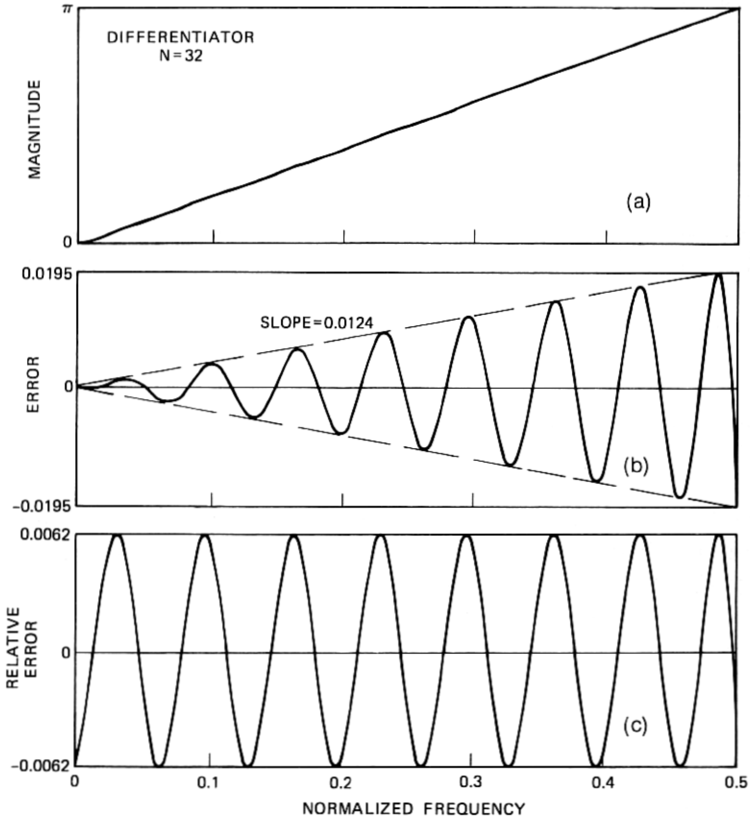

Fig. 2—The frequency response and error curves of an $N = 32$ wideband differentiator.

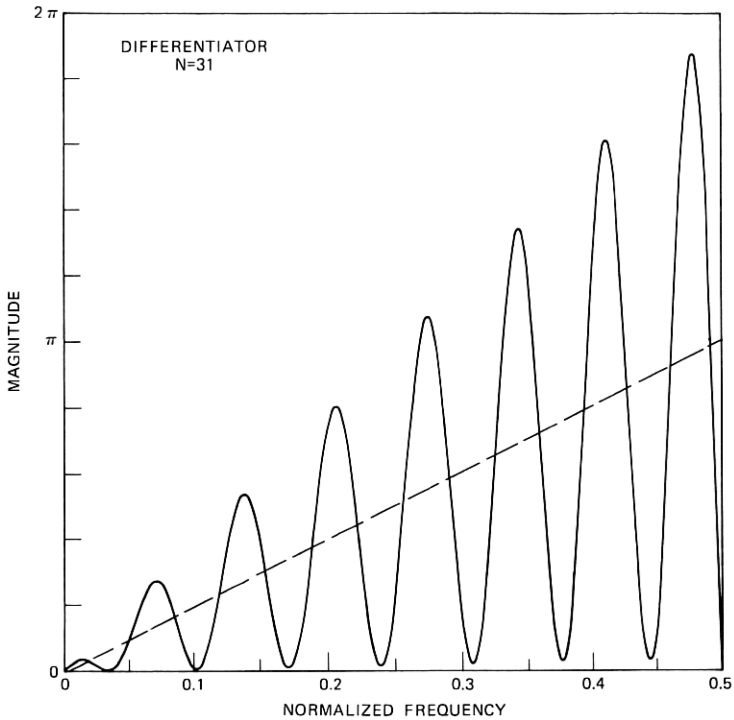

Fig. 3—The frequency response of an $N = 31$ wideband differentiator.

with a peak relative error of 0.0136. That is, the maximum error is 1.36 percent of the desired frequency response over the entire band $0 \leqq \omega \leqq \pi$. Figure 2 shows the same curves for $N = 32$ and $F_p = 0.5$. By doubling N, the peak relative error is reduced to 0.0062. Figure 3 shows the magnitude response for $N = 31$ and $F_p = 0.5$. In this case, the relative error at $f = 0.5$ is 1.0 since the desired value is π and the approximation is zero. The reason for the undesirable behavior of the approximation in this case is apparent from eqs. (12a) and (13a). When N is odd, $H^*(e^{j\omega})$ is exactly zero at $\omega = 0$ and $\omega = \pi$, independent of the choice of $a(n)$ in eq. (12a). When N is even, $H^*(e^{j\omega})$ is exactly zero only at $\omega = 0$, independent of the choice of $b(n)$ in eq. (13a). It is quite desirable to have a zero of $H^*(e^{j\omega})$ at $\omega = 0$ since this is the desired response; however, a zero of $H^*(e^{j\omega})$ at $\omega = \pi$ is at odds with the desired response at $\omega = \pi$. Thus, the inherent zero at $\omega = \pi$ for N odd is a fundamental limitation to the design of extremely wideband differentiators.[2] However, if F_p is less than 0.5, the resulting

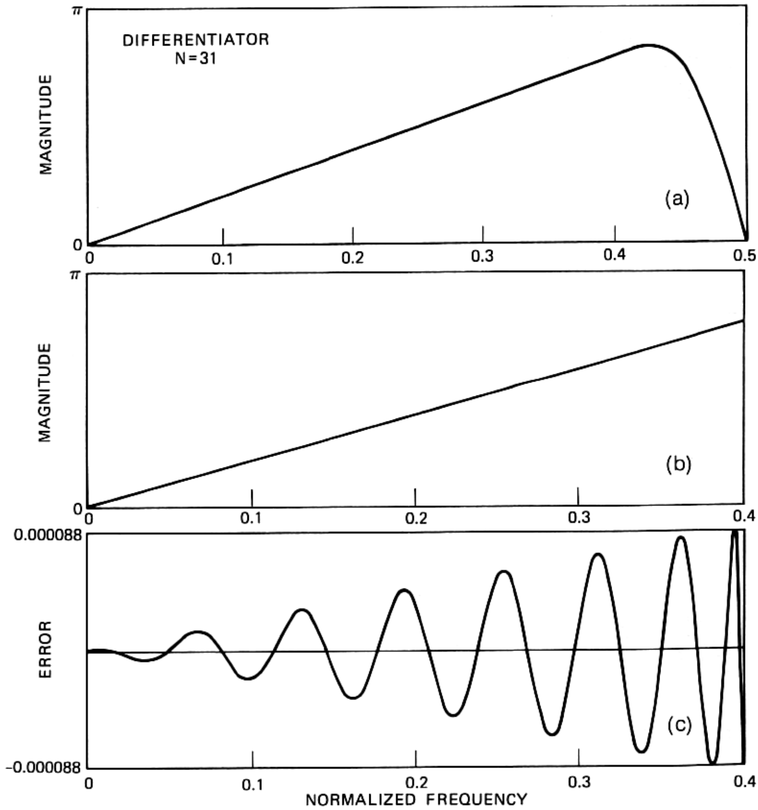

Fig. 4—The frequency response and error curve of an $N = 31$, $F_p = 0.4$ wideband differentiator.

approximation can be quite acceptable as seen in Fig. 4a for the case $N = 31$ and $F_p = 0.4$. Figure 4b shows the magnitude response plotted from $f = 0$ to $f = 0.4$, the cutoff frequency, and Fig. 4c shows the error function over this same range. The peak relative error is 0.000088 over the approximation interval.

As is clear from these examples, the basic parameters that characterize these differentiator approximations are N, F_p, and δ, the peak relative error of approximation. The examples suggest that δ can be reduced by increasing N, and by decreasing F_p. Also, there seems to be a distinct advantage in choosing even values of N provided the half-sample delay is not undesirable. To substantiate these observations, a large set of measurements of δ as a function of F_p and N were

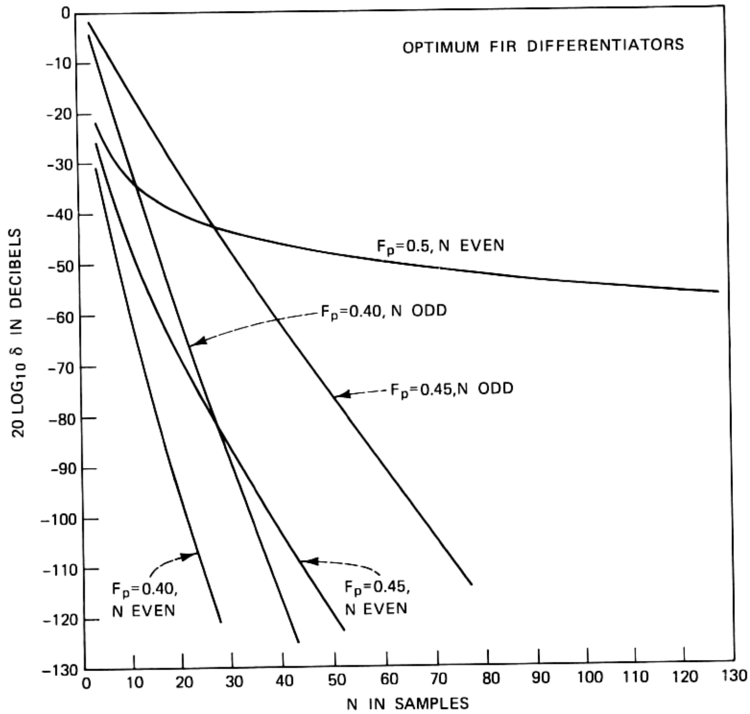

Fig. 5—The curves of 20 $\log_{10} \delta$ versus N for $F_p = 0.5, 0.45, 0.40$ for even and odd values of N.

made. The results are shown in Figs. 5 through 8* and in Tables I through VIII. Figure 5 shows the dependence of 20 $\log_{10} \delta$ upon N for F_p equal to 0.5, 0.45, and 0.40 and for even and odd values of N in the range $3 \leqq N \leqq 128$. The curve for N odd, $F_p = 0.5$, is not included since $\delta = 1.0$ independent of N. Figure 6 shows the same data as Fig. 5 with a logarithmic horizontal scale. From Figs. 5 and 6 it is seen that for the same value of F_p the values of δ for even values of N are approximately 1 to 2 orders of magnitude (20–40 dB) smaller than the values of δ for comparable odd values of N. This difference between even and odd values of N is due to the frequency response discontinuity for odd N, which is considerably more difficult to approximate than the slope discontinuity in the frequency response for even N. To substantiate this claim, the curve for $F_p = 0.5$, N even, was subtracted (on a log scale) from the $F_p = 0.45$, N even, and the

* The curves in Figs. 5 through 8 are straight-line connections of measured data points and do not represent any smoothing or fitting of the data.

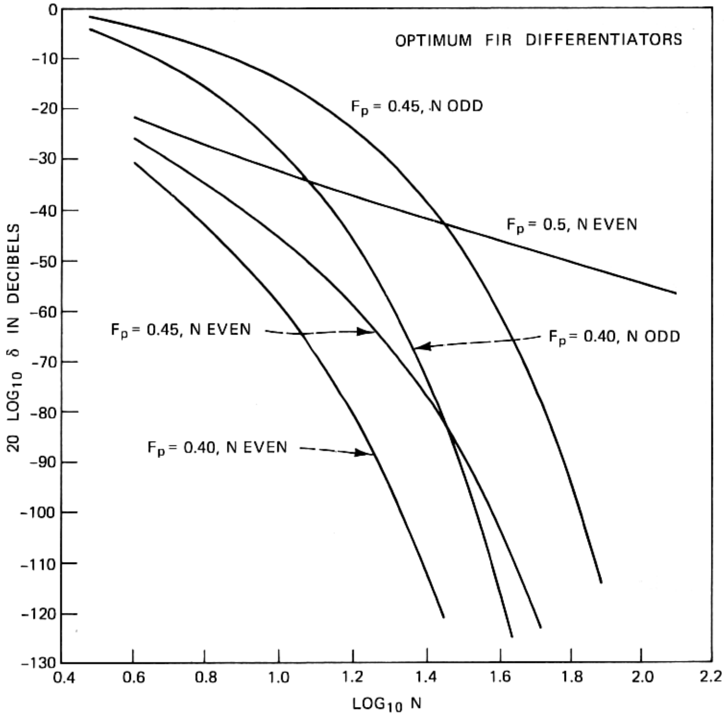

Fig. 6—The curves of 20 $\log_{10} \delta$ versus $\log_{10} N$ for $F_p = 0.5$, 0.45, and 0.40 for even and odd values of N.

$F_p = 0.40$, N even, curves and the resulting curves were replotted on the same scales as Fig. 5. It was then seen that the difference between the N even and N odd curves for fixed values of F_p was small (on the order of 2 dB) and essentially independent of N. Thus the 1 to 2 order-of-magnitude difference in the deltas is almost entirely accounted for by the frequency response discontinuity for odd values of N.

Another observation from Figs. 5 and 6 is that the smaller the bandwidth (F_p) of the differentiator, the faster the peak relative error decreases with increasing N. Thus for $F_p = 0.5$, the value of 20 $\log_{10} \delta$ decreases by only about 30 dB as N varies from 4 to 128; whereas for $F_p = 0.45$, the value of 20 $\log_{10} \delta$ decreases by about 98 dB as N varies from 4 to 52. For the cases when N is odd, the relation

$$(N - 1)(0.5 - F_p) \approx \alpha \log_{10} \delta \tag{16}$$

appears to work for δ smaller than approximately 0.01, where $\alpha \approx -0.5$. Thus, for $F_p = 0.45$, a value of $N = 41$ gives a δ of 0.000764, whereas for $F_p = 0.40$, a value of $N = 21$ gives a value of δ of 0.000878. This

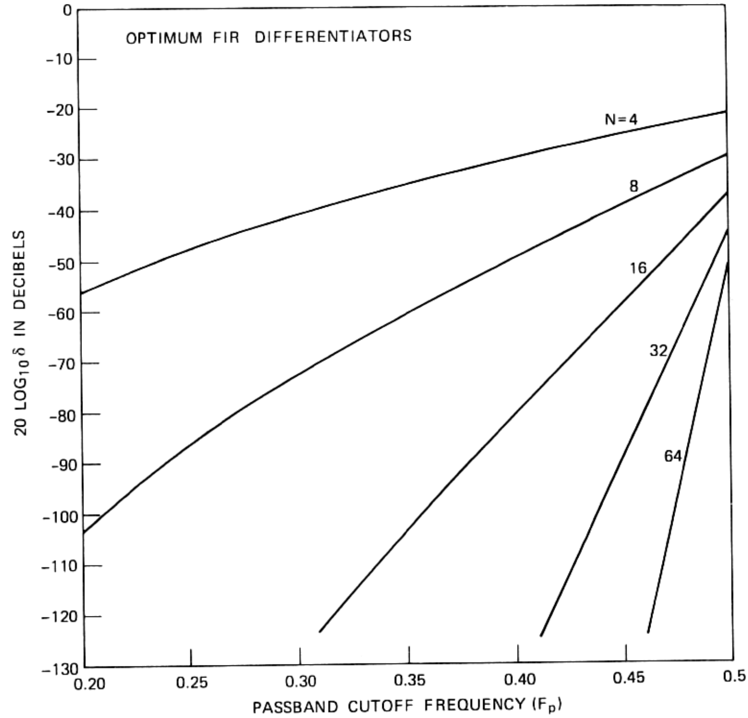

Fig. 7—The curves of 20 $\log_{10} \delta$ versus F_p for $N = 4, 8, 16, 32$, and 64.

inverse proportionality between transition bandwidth and filter order for fixed value of δ was originally noted by Kaiser.[6]

Figures 7 and 8 show the dependence of 20 $\log_{10} \delta$ upon F_p for even values of N ($N = 4, 8, 16, 32, 64$) and odd values of N ($N = 5, 9, 17, 33, 65$) respectively. Note that since $h((N-1)/2) = 0$ when N is odd, there are the same number of nonzero impulse response coefficients for differentiators of length 4 and 5, 8 and 9, etc. The data for even and odd values of N are presented on different figures because of the different nature of the solution in the two cases. As seen in Fig. 8, where N is odd, as F_p approaches 0.5, 20 $\log_{10} \delta$ approaches 0, independent of N. As previously discussed, this is because of the constrained zero of $H^*(e^{j\omega})$ at $\omega = \pi$ when N is odd. For even values of N, the curves are spaced apart for all values of F_p. The main observation from these figures is that the larger the value of N, the faster the peak relative error decreases with decreasing differentiator bandwidth.

The curves in Figs. 5 through 8 can be used to determine the length of impulse response required to meet a given specification of approxi-

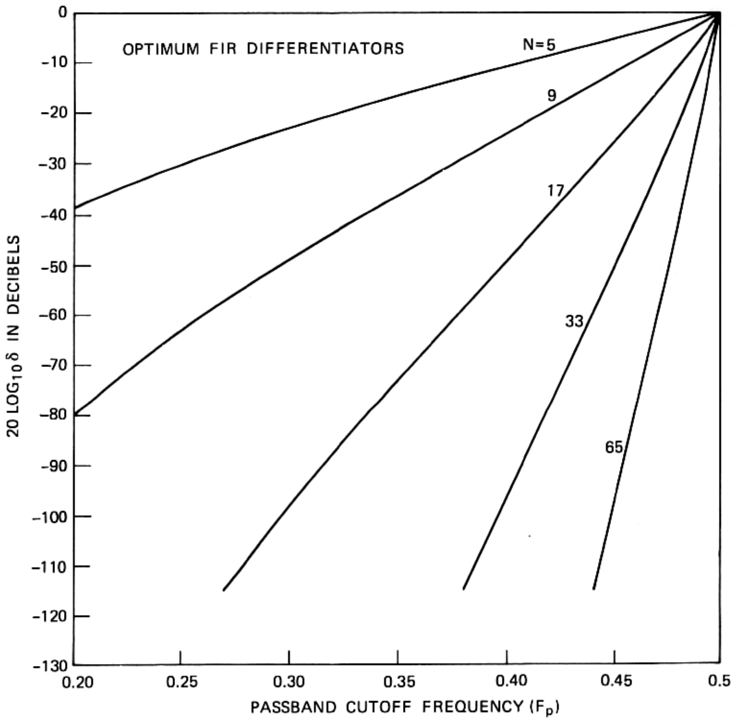

Fig. 8—The curves of 20 $\log_{10} \delta$ versus F_p for $N = 5, 9, 17, 33$, and 65.

mation error. For example, to obtain a peak relative error less than 1 percent (−40 dB) requires the following values of N (as a function of F_p):

F_p	N (odd)	N (even)
0.5	impossible	22
0.45	27	10
0.40	15	6

Similarly, to obtain a peak relative error less than 0.1 percent requires

F_p	N (odd)	N (even)
0.5	impossible	>128
0.45	41	18
0.40	21	12

These examples indicate the substantial reductions in N that result when F_p is reduced and when N is changed from odd to even.

IV. APPLICATION OF FIR DIFFERENTIATORS

The data presented above indicate that the most efficient FIR differentiators (i.e., having the smallest value of N for a given value of peak relative approximation error) are obtained when the bandwidth, F_p, is as small as possible and when N is even. These design considerations must be viewed in the light of the intended application.

In processing sequences obtained by sampling an analog signal, it is generally true that a fullband differentiator is not required, since the sampling rate is generally set somewhat higher than twice the Nyquist frequency of the analog signal. Thus, differentiator bandwidths on the order of 0.40 to 0.49 are quite reasonable for most applications.

Since even values of N result in better approximations than comparable odd values, it is generally desirable to choose N even. However, even values of N are sometimes undesirable because the delay is a nonintegral number of samples. In situations when the differentiator is part of a larger signal processing system it may be important to be able to equalize signal delays in different parts of the system. This may be more difficult to accomplish when the delay of the differentiator is not an integral number of samples. When N is large and the differentiator is to be realized as a general-purpose computer program, it may be desirable to use an FFT realization instead of a direct discrete convolution realization. In such cases, the odd symmetry of the impulse response about $(N - 1)/2$ permits the frequency response of the differentiator to be purely imaginary only when N is odd, thus reducing the storage requirements for the transform of the impulse response and the number of intermediate multiplications by a factor of two.

These two practical limitations, however, are often of little consequence in digital signal processing applications, where the overriding concern is often simply the reduction of system complexity while maintaining a prescribed performance. In such cases, it is clear from previous discussion that even values of N provide the most efficient approximations to the differentiator.

A subset of the differentiators designed in this study is given in Tables I through VIII. Included in these tables are wideband differentiators (F_p = 0.5, 0.49, 0.48, 0.45, and 0.40) for both even and odd values from $N = 3$ to $N = 50$.* The value of peak relative error,

* Only those differentiators for which $\delta < 0.1$ are included in these tables.

denoted D, is given for each case as well as the impulse response coefficients for the filter. Note that only the first half of the impulse response is given in the table; i.e., $h(n)$ for $n = 0, 1, \cdots, (N/2) - 1$. The remainder of the impulse response can be obtained using eq. (10). These data should be adequate for most design applications.

V. ACKNOWLEDGMENTS

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Table I—Wideband differentiators ($F_p = 0.5$, N even)

	$N = 4$, $D = 0.0831040$	$N = 6$, $D = 0.0470970$	$N = 8$, $D = 0.0320840$	$N = 10$, $D = 0.0240945$
0	−0.1310637	0.0508099	−0.0289328	0.0196350
1	1.3091933	−0.1630958	0.0668126	−0.0386111
2		1.2829105	−0.1471745	0.0549917
3			1.2774790	−0.1441674
4				1.2755435

	$N = 12$, $D = 0.0192165$	$N = 14$, $D = 0.0159235$	$N = 16$, $D = 0.0135730$	$N = 18$, $D = 0.0118460$
0	−0.0146791	0.0116236	−0.0095995	0.0081685
1	0.0261968	−0.0194562	0.0153294	−0.0125786
2	−0.0291034	0.0182665	−0.0126660	0.0093698
3	0.0528837	−0.0275156	0.0169834	−0.0115683
4	−0.1430612	0.0521278	−0.0269451	0.0165044
5	1.2746868	−0.1425363	0.0517493	−0.0266517
6		1.2742573	−0.1422334	0.0515315
7			1.2739692	−0.1420377
8				1.2737782

	$N = 20$, $D = 0.0104985$	$N = 22$, $D = 0.0094195$	$N = 24$, $D = 0.0085365$	$N = 26$, $D = 0.0078050$
0	−0.0071000	0.0062722	−0.0056118	0.0050796
1	0.0106302	−0.0091800	0.0080620	−0.0071883
2	−0.0072703	0.0058390	−0.0048205	0.0040624
3	0.0084292	−0.0064393	0.0051079	−0.0041422
4	−0.0111803	0.0081015	−0.0061625	0.0048465
5	0.0162637	−0.0109978	0.0079432	−0.0060168
6	−0.0264660	0.0161349	−0.0108840	0.0078424
7	0.0513867	−0.0263595	0.0160438	−0.0108017
8	−0.1419117	0.0512852	−0.0262854	0.0159807
9	1.2736673	−0.1418134	0.0512146	−0.0262332
10		1.2735768	−0.1417436	0.0511684
11			1.2735077	−0.1417041
12				1.2734707

	$N = 28$, $D = 0.0071890$	$N = 30$, $D = 0.0066600$	$N = 32$, $D = 0.0062025$	$N = 34$, $D = 0.0058025$
0	−0.0046291	0.0042600	−0.0039405	0.0036653
1	0.0064623	−0.0058789	0.0053806	−0.0049612
2	−0.0034847	0.0030260	−0.0026653	0.0023703
3	0.0034495	−0.0029082	0.0025070	−0.0021762
4	−0.0039226	0.0032396	−0.0027291	0.0023323
5	0.0047240	−0.0038139	0.0031397	−0.0026465
6	−0.0059332	0.0046549	−0.0037470	0.0030907
7	0.0077802	−0.0058820	0.0046040	−0.0037036
8	−0.0107505	0.0077387	−0.0058437	0.0045698
9	0.0159345	−0.0107147	0.0077095	−0.0058145
10	−0.0261921	0.0159024	−0.0106896	0.0076837
11	0.0511322	−0.0261619	0.0158792	−0.0106654
12	−0.1416711	0.0511027	−0.0261387	0.0158559
13	1.2734383	−0.1416415	0.0510788	−0.0261186
14		1.2734088	−0.1416173	0.0510609
15			1.2733839	−0.1416001
16				1.2733660

Table I—continued

	$N = 36$ $D = 0.0054505$	$N = 38$ $D = 0.0051375$	$N = 40$ $D = 0.0048585$	$N = 42$ $D = 0.0046080$
0	−0.0034253	0.0032145	−0.0030272	0.0028607
1	0.0046009	−0.0042886	0.0040140	−0.0037721
2	−0.0021287	0.0019267	−0.0017546	0.0016072
3	0.0019148	−0.0017005	0.0015212	−0.0013694
4	−0.0020200	0.0017690	−0.0015623	0.0013908
5	0.0022528	−0.0019478	0.0017009	−0.0014989
6	−0.0025912	0.0022079	−0.0019054	0.0016619
7	0.0030565	−0.0025573	0.0021778	−0.0018777
8	−0.0036810	0.0030238	−0.0025346	0.0021567
9	0.0045437	−0.0036587	0.0030046	−0.0025174
10	−0.0057900	0.0045295	−0.0036367	0.0029902
11	0.0076598	−0.0057714	0.0045142	−0.0036235
12	−0.0106443	0.0076422	−0.0057629	0.0044975
13	0.0158387	−0.0106280	0.0076300	−0.0057519
14	−0.0261035	0.0158233	−0.0106158	0.0076262
15	0.0510459	−0.0260878	0.0158104	−0.0106082
16	−0.1415850	0.0510323	−0.0260762	0.0158013
17	1.2733532	−0.1415734	0.0510217	−0.0260652
18		1.2733444	−0.1415624	0.0510100
19			1.2733312	−0.1415514
20				1.2733199

	$N = 44$ $D = 0.0043825$	$N = 46$ $D = 0.0041785$	$N = 48$ $D = 0.0039920$	$N = 50$ $D = 0.0038215$
0	−0.0027118	0.0025770	−0.0024552	0.0023443
1	0.0035572	−0.0033646	0.0031912	−0.0030348
2	−0.0014794	0.0013682	−0.0012698	0.0011841
3	0.0012416	−0.0011307	0.0010345	−0.0009519
4	−0.0012475	0.0011244	−0.0010198	0.0009299
5	0.0013308	−0.0011900	0.0010713	−0.0009698
6	−0.0014615	0.0012959	−0.0011583	0.0010418
7	0.0016346	−0.0014370	0.0012745	−0.0011376
8	−0.0018561	0.0016167	−0.0014213	0.0012579
9	0.0021391	−0.0018419	0.0016028	−0.0014081
10	−0.0025029	0.0021275	−0.0018293	0.0015925
11	0.0029785	−0.0024922	0.0021165	−0.0018212
12	−0.0036153	0.0029682	−0.0024834	0.0021086
13	0.0044897	−0.0036056	0.0029610	−0.0024759
14	−0.0057381	0.0044821	−0.0035984	0.0029537
15	0.0076162	−0.0057321	0.0044752	−0.0035915
16	−0.0106048	0.0076042	−0.0057262	0.0044686
17	0.0157953	−0.0105953	0.0075986	−0.0057202
18	−0.0260586	0.0157918	−0.0105840	0.0075939
19	0.0510019	−0.0260529	0.0157830	−0.0105799
20	−0.1415416	0.0509959	−0.0260495	0.0157733
21	1.2733098	−0.1415357	0.0509909	−0.0260416
22		1.2733032	−0.1415313	0.0509884
23			1.2732988	−0.1415278
24				1.2732960

Table II—Wideband differentiators ($F_p = 0.49$, N even)

	$N = 4$ $D = 0.0756235$	$N = 6$ $D = 0.0402290$	$N = 8$ $D = 0.0255765$	$N = 10$ $D = 0.0178930$
0	−0.1249465	0.0458333	−0.0244765	0.0155119
1	1.2992160	−0.1556062	0.0604056	−0.0328369
2		1.2778984	−0.1435042	0.0519839
3			1.2742206	−0.1416396
4				1.2731675

	$N = 12$ $D = 0.0132725$	$N = 14$ $D = 0.0102355$	$N = 16$ $D = 0.0081120$	$N = 18$ $D = 0.0065675$
0	−0.0108071	0.0079847	−0.0061368	0.0048544
1	0.0208881	−0.0145299	0.0107000	−0.0081958
2	−0.0265471	0.0159989	−0.0106296	0.0075254
3	0.0507873	−0.0256995	0.0153658	−0.0101326
4	−0.1411075	0.0504637	−0.0254815	0.0152063
5	1.2727987	−0.1409274	0.0503597	−0.0254177
6		1.2726648	−0.1408696	0.0503343
7			1.2726240	−0.1408602
8				1.2726199

	$N = 20$ $D = 0.0054065$	$N = 22$ $D = 0.0045055$	$N = 24$ $D = 0.0037945$	$N = 26$ $D = 0.0032230$
0	−0.0039242	0.0032252	−0.0026848	0.0022597
1	0.0064632	−0.0052075	0.0042657	−0.0035440
2	−0.0055757	0.0042694	−0.0033533	0.0026892
3	0.0071314	−0.0052518	0.0040021	−0.0031297
4	−0.0100173	0.0070397	−0.0051824	0.0039440
5	0.0151654	−0.0099887	0.0070237	−0.0051698
6	−0.0254039	0.0151620	−0.0099918	0.0070290
7	0.0503330	−0.0254130	0.0151745	−0.0100057
8	−0.1408656	0.0503509	−0.0254312	0.0151940
9	1.2726300	−0.1408860	0.0503717	−0.0254529
10		1.2726507	−0.1409083	0.0503940
11			1.2726749	−0.1409325
12				1.2726997

	$N = 28$ $D = 0.0027515$	$N = 30$ $D = 0.0023760$	$N = 32$ $D = 0.0020535$	$N = 34$ $D = 0.0017890$
0	−0.0019148	0.0016418	−0.0014118	0.0012240
1	0.0029716	−0.0025230	0.0021529	−0.0018542
2	−0.0021894	0.0018089	−0.0015120	0.0012767
3	0.0024976	−0.0020279	0.0016726	−0.0013946
4	−0.0030847	0.0024627	−0.0020003	0.0016484
5	0.0039389	−0.0030810	0.0024618	−0.0019999
6	−0.0051770	0.0039458	−0.0030894	0.0024683
7	0.0070419	−0.0051893	0.0039581	−0.0031014
8	−0.0100233	0.0070579	−0.0052056	0.0039744
9	0.0152150	−0.0100424	0.0070768	−0.0052210
10	−0.0254777	0.0152355	−0.0100622	0.0070931
11	0.0504213	−0.0254978	0.0152556	−0.0100801
12	−0.1409620	0.0504402	−0.0255185	0.0152744
13	1.2727299	−0.1409799	0.0504625	−0.0255380
14		1.2727475	−0.1410032	0.0504816
15			1.2727717	−0.1410217
16				1.2727889

Table II—continued

	$N = 36$ $D = 0.0015625$	$N = 38$ $D = 0.0013680$	$N = 40$ $D = 0.0012035$	$N = 42$ $D = 0.0010655$
0	−0.0010650	0.0009290	−0.0008146	0.0007191
1	0.0016041	−0.0013927	0.0012161	−0.0010685
2	−0.0010867	0.0009299	−0.0008017	0.0006955
3	0.0011746	−0.0009968	0.0008520	−0.0007339
4	−0.0013744	0.0011567	−0.0009814	0.0008401
5	0.0016487	−0.0013748	0.0011577	−0.0009836
6	−0.0020081	0.0016562	−0.0013820	0.0011649
7	0.0024812	−0.0020191	0.0016666	−0.0013917
8	−0.0031146	0.0024941	−0.0020314	0.0016776
9	0.0039898	−0.0031300	0.0025076	−0.0020436
10	−0.0052395	0.0040046	−0.0031441	0.0025208
11	0.0071107	−0.0052559	0.0040200	−0.0031576
12	−0.0100974	0.0071289	−0.0052706	0.0040335
13	0.0152926	−0.0101144	0.0071443	−0.0052845
14	−0.0255565	0.0153093	−0.0101310	0.0071569
15	0.0504998	−0.0255738	0.0153250	−0.0101445
16	−0.1410399	0.0505181	−0.0255892	0.0153398
17	1.2728072	−0.1410588	0.0505341	−0.0256034
18		1.2728270	−0.1410757	0.0505482
19			1.2728442	−0.1410899
20				1.2728587

	$N = 44$ $D = 0.0009435$	$N = 46$ $D = 0.0008355$	$N = 48$ $D = 0.0007420$	$N = 50$ $D = 0.0006635$
0	−0.0006352	0.0005614	−0.0004979	0.0004439
1	0.0009406	−0.0008294	0.0007336	−0.0006528
2	−0.0006060	0.0005300	−0.0004653	0.0004109
3	0.0006355	−0.0005529	0.0004835	−0.0004247
4	−0.0007235	0.0006264	−0.0005457	0.0004778
5	0.0008423	−0.0007260	0.0006293	−0.0005488
6	−0.0009909	0.0008492	−0.0007323	0.0006359
7	0.0011743	−0.0009997	0.0008573	−0.0007405
8	−0.0014027	0.0011844	−0.0010091	0.0008665
9	0.0016892	−0.0014134	0.0011947	−0.0010188
10	−0.0020555	0.0017009	−0.0014241	0.0012045
11	0.0025331	−0.0020675	0.0017119	−0.0014341
12	−0.0031702	0.0025453	−0.0020788	0.0017219
13	0.0040464	−0.0031824	0.0025566	−0.0020892
14	−0.0052977	0.0040593	−0.0031941	0.0025670
15	0.0071710	−0.0053102	0.0040709	−0.0032047
16	−0.0101577	0.0071839	−0.0053222	0.0040816
17	0.0153533	−0.0101712	0.0071961	−0.0053332
18	−0.0256184	0.0153665	−0.0101838	0.0072068
19	0.0505624	−0.0256316	0.0153794	−0.0101945
20	−0.1411034	0.0505765	−0.0256436	0.0153900
21	1.2728719	−0.1411169	0.0505881	−0.0256542
22		1.2728844	−0.1411291	0.0505982
23			1.2728967	−0.1411395
24				1.2729083

Table III—Wideband differentiators
($F_p = 0.48$, N even)

	$N = 4$ $D = 0.0685890$	$N = 6$ $D = 0.0342610$	$N = 8$ $D = 0.0203665$	$N = 10$ $D = 0.0132955$
0	−0.1194462	0.0415099	−0.0208464	0.0123838
1	1.2897494	−0.1488110	0.0549279	−0.0282322
2		1.2731593	−0.1400594	0.0492614
3			1.2710796	−0.1392143
4				1.2708226

	$N = 12$ $D = 0.0091960$	$N = 14$ $D = 0.0066075$	$N = 16$ $D = 0.0048850$	$N = 18$ $D = 0.0036880$
0	−0.0080758	0.0055801	−0.0040121	0.0029697
1	0.0169344	−0.0110911	0.0076843	−0.0055383
2	−0.0243059	0.0141089	−0.0090057	0.0061179
3	0.0488103	−0.0240382	0.0139418	−0.0088998
4	−0.1392103	0.0488703	−0.0241146	0.0140184
5	1.2709260	−0.1393617	0.0490132	−0.0242418
6		1.2711088	−0.1395291	0.0491596
7			1.2712857	−0.1396846
8				1.2714456

	$N = 20$ $D = 0.0028315$	$N = 22$ $D = 0.0022025$	$N = 24$ $D = 0.0017315$	$N = 26$ $D = 0.0013725$
0	−0.0022456	0.0017272	−0.0013462	0.0010612
1	0.0041061	−0.0031111	0.0023970	−0.0018733
2	−0.0043398	0.0031799	−0.0023882	0.0018287
3	0.0060476	−0.0042958	0.0031532	−0.0023725
4	−0.0089705	0.0061132	−0.0043558	0.0032063
5	0.0141281	−0.0090663	0.0061971	−0.0044281
6	−0.0243671	0.0142374	−0.0091603	0.0062788
7	0.0492947	−0.0244843	0.0143370	−0.0092466
8	−0.1398254	0.0494151	−0.0245867	0.0144256
9	1.2715885	−0.1399466	0.0495200	−0.0246775
10		1.2717111	−0.1400541	0.0496127
11			1.2718185	−0.1401477
12				1.2719131

	$N = 28$ $D = 0.0010945$	$N = 30$ $D = 0.0008810$	$N = 32$ $D = 0.0007135$	$N = 34$ $D = 0.0005780$
0	−0.0008419	0.0006751	−0.0005457	0.0004414
1	0.0014784	−0.0011790	0.0009488	−0.0007659
2	−0.0014238	0.0011215	−0.0008944	0.0007185
3	0.0018215	−0.0014209	0.0011234	−0.0008972
4	−0.0024200	0.0018636	−0.0014586	0.0011558
5	0.0032688	−0.0024753	0.0019117	−0.0014998
6	−0.0044975	0.0033307	−0.0025277	0.0019575
7	0.0063517	−0.0045625	0.0033857	−0.0025767
8	−0.0093227	0.0064189	−0.0046203	0.0034366
9	0.0145047	−0.0093921	0.0064786	−0.0046731
10	−0.0247589	0.0145754	−0.0094534	0.0065333
11	0.0496959	−0.0248305	0.0146379	−0.0095087
12	−0.1402316	0.0497675	−0.0248937	0.0146945
13	1.2719969	−0.1403032	0.0498316	−0.0249515
14		1.2720686	−0.1403686	0.0498901
15			1.2721348	−0.1404270
16				1.2721930

Table III—continued

	$N = 36$ $D = 0.0004745$	$N = 38$ $D = 0.0003865$	$N = 40$ $D = 0.0003195$	$N = 42$ $D = 0.0002630$
0	−0.0003616	0.0002947	−0.0002435	0.0002004
1	0.0006258	−0.0005102	0.0004213	−0.0003471
2	−0.0005831	0.0004750	−0.0003908	0.0003220
3	0.0007241	−0.0005881	0.0004825	−0.0003971
4	−0.0009268	0.0007493	−0.0006120	0.0005023
5	0.0011925	−0.0009582	0.0007785	−0.0006362
6	−0.0015400	0.0012274	−0.0009902	0.0008052
7	0.0019999	−0.0015771	0.0012610	−0.0010185
8	−0.0026210	0.0020389	−0.0016123	0.0012909
9	0.0034828	−0.0026616	0.0020753	−0.0016434
10	−0.0047202	0.0035246	−0.0026993	0.0021077
11	0.0065816	−0.0047636	0.0035632	−0.0027329
12	−0.0095586	0.0066259	−0.0048032	0.0035981
13	0.0147451	−0.0096042	0.0066668	−0.0048390
14	−0.0250027	0.0147916	−0.0096453	0.0067035
15	0.0499419	−0.0250492	0.0148333	−0.0096830
16	−0.1404791	0.0499890	−0.0250919	0.0148714
17	1.2722451	−0.1405266	0.0500314	−0.0251305
18		1.2722929	−0.1405696	0.0500707
19			1.2723365	−0.1406089
20				1.2723761

	$N = 44$ $D = 0.0002185$	$N = 46$ $D = 0.0001810$	$N = 48$ $D = 0.0001510$	$N = 50$ $D = 0.0001260$
0	−0.0001665	0.0001382	−0.0001153	0.0000961
1	0.0002884	−0.0002394	0.0002001	−0.0001671
2	−0.0002673	0.0002224	−0.0001860	0.0001561
3	0.0003292	−0.0002736	0.0002287	−0.0001920
4	−0.0004153	0.0003449	−0.0002881	0.0002416
5	0.0005243	−0.0004342	0.0003619	−0.0003032
6	−0.0006607	0.0005454	−0.0004530	0.0003786
7	0.0008310	−0.0006830	0.0005652	−0.0004709
8	−0.0010458	0.0008545	−0.0007040	0.0005843
9	0.0013195	−0.0010707	0.0008768	−0.0007241
10	−0.0016732	0.0013455	−0.0010939	0.0008979
11	0.0021385	−0.0017002	0.0013697	−0.0011156
12	−0.0027646	0.0021664	−0.0017254	0.0013924
13	0.0036307	−0.0027935	0.0021925	−0.0017489
14	−0.0048726	0.0036606	−0.0028205	0.0022170
15	0.0067378	−0.0049031	0.0036879	−0.0028453
16	−0.0097176	0.0067686	−0.0049314	0.0037137
17	0.0149065	−0.0097490	0.0067975	−0.0049574
18	−0.0251657	0.0149386	−0.0097785	0.0068242
19	0.0501065	−0.0251984	0.0149684	−0.0098059
20	−0.1406450	0.0501395	−0.0252289	0.0149961
21	1.2724119	−0.1406780	0.0501700	−0.0252568
22		1.2724452	−0.1407088	0.0501979
23			1.2724760	−0.1407371
24				1.2725043

Table IV—Wideband differentiators
($F_p = 0.45$, N even)

	$N = 4$ $D = 0.0507220$	$N = 6$ $D = 0.0209865$	$N = 8$ $D = 0.0102350$	$N = 10$ $D = 0.0054640$
0	−0.1049330	0.0313204	−0.0132691	0.0066049
1	1.2640770	−0.1318473	0.0425774	−0.0188979
2		1.2599360	−0.1307889	0.0423179
3			1.2621213	−0.1324320
4				1.2640211

	$N = 12$ $D = 0.0030870$	$N = 14$ $D = 0.0018135$	$N = 16$ $D = 0.0010970$	$N = 18$ $D = 0.0006780$
0	−0.0036000	0.0020785	−0.0012494	0.0007731
1	0.0097311	−0.0054557	0.0032324	−0.0019893
2	−0.0189385	0.0098781	−0.0056282	0.0033951
3	0.0434812	−0.0197685	0.0104785	−0.0060664
4	−0.1337806	0.0444447	−0.0204678	0.0109918
5	1.2654301	−0.1348054	0.0452000	−0.0210311
6		1.2664828	−0.1355959	0.0458000
7			1.2672905	−0.1362207
8				1.2679283

	$N = 20$ $D = 0.0004255$	$N = 22$ $D = 0.0002715$	$N = 24$ $D = 0.0001750$	$N = 26$ $D = 0.0001140$
0	−0.0004882	0.0003148	−0.0002055	0.0001357
1	0.0012579	−0.0008143	0.0005353	−0.0003569
2	−0.0021291	0.0013757	−0.0009079	0.0006095
3	0.0037159	−0.0023684	0.0015526	−0.0010402
4	−0.0064459	0.0040014	−0.0025818	0.0017137
5	0.0114147	−0.0067686	0.0042459	−0.0027693
6	−0.0214891	0.0117693	−0.0070416	0.0044582
7	0.0462857	−0.0218711	0.0120681	−0.0072772
8	−0.1367253	0.0466885	−0.0221913	0.0123242
9	1.2684422	−0.1371425	0.0470259	−0.0224649
10		1.2688667	−0.1374915	0.0473133
11			1.2692216	−0.1377887
12				1.2695236

	$N = 28$ $D = 0.0000750$	$N = 30$ $D = 0.0000495$	$N = 32$ $D = 0.0000330$	$N = 34$ $D = 0.0000220$
0	−0.0000905	0.0000609	−0.0000412	0.0000280
1	0.0002406	−0.0001637	0.0001122	−0.0000776
2	−0.0004147	0.0002853	−0.0001979	0.0001385
3	0.0007084	−0.0004888	0.0003409	−0.0002400
4	−0.0011618	0.0008008	−0.0005589	0.0003943
5	0.0018573	−0.0012723	0.0008856	−0.0006245
6	−0.0029342	0.0019861	−0.0013722	0.0009638
7	0.0046439	−0.0030806	0.0021014	−0.0014640
8	−0.0074817	0.0048076	−0.0032113	0.0022063
9	0.0125460	−0.0076617	0.0049527	−0.0033295
10	−0.0227011	0.0127401	−0.0078204	0.0050837
11	0.0475609	−0.0229076	0.0129110	−0.0079627
12	−0.1380441	0.0477767	−0.0230885	0.0130637
13	1.2697831	−0.1382662	0.0479655	−0.0232500
14		1.2700086	−0.1384607	0.0481336
15			1.2702059	−0.1386335
16				1.2703809

Table IV—continued

	$N = 36$, $D = 0.0000145$	$N = 38$, $D = 0.0000100$	$N = 40$, $D = 0.0000065$	$N = 42$, $D = 0.0000045$
0	−0.0000192	0.0000132	−0.0000091	0.0000063
1	0.0000537	−0.0000377	0.0000264	−0.0000185
2	−0.0000974	0.0000691	−0.0000490	0.0000352
3	0.0001700	−0.0001213	0.0000870	−0.0000628
4	−0.0002805	0.0002014	−0.0001451	0.0001052
5	0.0004448	−0.0003198	0.0002312	−0.0001684
6	−0.0006855	0.0004929	−0.0003569	0.0002608
7	0.0010361	−0.0007430	0.0005378	−0.0003927
8	−0.0015475	0.0011033	−0.0007961	0.0005809
9	0.0023015	−0.0016248	0.0011652	−0.0008467
10	−0.0034360	0.0023892	−0.0016958	0.0012237
11	0.0052009	−0.0035337	0.0024690	−0.0017621
12	−0.0080899	0.0053080	−0.0036223	0.0025431
13	0.0131997	−0.0082058	0.0054048	−0.0037043
14	−0.0233932	0.0133232	−0.0083101	0.0054940
15	0.0482831	−0.0235233	0.0134344	−0.0084063
16	−0.1387868	0.0484182	−0.0236399	0.0135362
17	1.2705364	−0.1389253	0.0485392	−0.0237470
18		1.2706765	−0.1390494	0.0486498
19			1.2708019	−0.1391628
20				1.2709168

	$N = 44$, $D = 0.0000030$	$N = 46$, $D = 0.0000020$	$N = 48$, $D = 0.0000015$	$N = 50$, $D = 0.0000010$
0	−0.0000044	0.0000031	−0.0000022	0.0000016
1	0.0000132	−0.0000094	0.0000066	−0.0000047
2	−0.0000251	0.0000182	−0.0000132	0.0000094
3	0.0000452	−0.0000330	0.0000239	−0.0000176
4	−0.0000767	0.0000559	−0.0000412	0.0000302
5	0.0001232	−0.0000905	0.0000666	−0.0000493
6	−0.0001910	0.0001411	−0.0001043	0.0000776
7	0.0002887	−0.0002133	0.0001587	−0.0001181
8	−0.0004266	0.0003157	−0.0002350	0.0001756
9	0.0006211	−0.0004593	0.0003421	−0.0002560
10	−0.0008938	0.0006594	−0.0004904	0.0003673
11	0.0012777	−0.0009381	0.0006959	−0.0005202
12	−0.0018231	0.0013283	−0.0009802	0.0007304
13	0.0026113	−0.0018802	0.0013760	−0.0010198
14	−0.0037787	0.0026748	−0.0019337	0.0014209
15	0.0055751	−0.0038485	0.0027338	−0.0019836
16	−0.0084933	0.0056505	−0.0039129	0.0027888
17	0.0136282	−0.0085737	0.0057202	−0.0039732
18	−0.0238434	0.0137137	−0.0086482	0.0057849
19	0.0487497	−0.0239323	0.0137919	−0.0087170
20	−0.1392649	0.0488417	−0.0240143	0.0138645
21	1.2710199	−0.1393589	0.0489262	−0.0240897
22		1.2711151	−0.1394452	0.0490041
23			1.2712024	−0.1395247
24				1.2712825

Table V—Wideband differentiators
($F_p = 0.40$, N even)

	$N = 4$ $D = 0.0296405$	$N = 6$ $D = 0.0089245$	$N = 8$ $D = 0.0031370$	$N = 10$ $D = 0.0012040$
0	−0.0861745	0.0205935	−0.0068427	0.0026524
1	1.2288833	−0.1115391	0.0300431	−0.0109331
2		1.2405798	−0.1179778	0.0335962
3			1.2484761	−0.1225086
4				1.2534103

	$N = 12$ $D = 0.0004890$	$N = 14$ $D = 0.0002065$	$N = 16$ $D = 0.0000900$	$N = 18$ $D = 0.0000400$
0	−0.0011231	0.0005036	−0.0002350	0.0001128
1	0.0045751	−0.0020735	0.0009893	−0.0004888
2	−0.0129299	0.0057108	−0.0027250	0.0013644
3	0.0362512	−0.0145085	0.0066583	−0.0032958
4	−0.1255952	0.0381971	−0.0157403	0.0074380
5	1.2567278	−0.1278179	0.0396824	−0.0167274
6		1.2590998	−0.1294942	0.0408526
7			1.2608785	−0.1308018
8				1.2622592

	$N = 20$ $D = 0.0000180$	$N = 22$ $D = 0.0000085$	$N = 24$ $D = 0.0000040$	$N = 26$ $D = 0.0000020$
0	−0.0000553	0.0000276	−0.0000141	0.0000072
1	0.0002476	−0.0001279	0.0000672	−0.0000358
2	−0.0007056	0.0003738	−0.0002014	0.0001100
3	0.0017093	−0.0009151	0.0005008	−0.0002787
4	−0.0037888	0.0020219	−0.0011121	0.0006255
5	0.0080905	−0.0042201	0.0023044	−0.0012978
6	−0.0175364	0.0086475	−0.0045990	0.0025623
7	0.0417986	−0.0182150	0.0091270	−0.0049367
8	−0.1318492	0.0425824	−0.0187905	0.0095470
9	1.2633610	−0.1327106	0.0432403	−0.0192881
10		1.2642636	−0.1334294	0.0438045
11			1.2650144	−0.1340417
12				1.2656525

	$N = 28$ $D = 0.0000010$	$N = 30$ $D = 0.0000005$
0	−0.0000038	0.0000019
1	0.0000192	−0.0000104
2	−0.0000606	0.0000336
3	0.0001568	−0.0000889
4	−0.0003569	0.0002058
5	0.0007458	−0.0004345
6	−0.0014706	0.0008608
7	0.0027963	−0.0016311
8	−0.0052373	0.0030087
9	0.0099149	−0.0055056
10	−0.0197195	0.0102394
11	0.0442895	−0.0200961
12	−0.1345657	0.0447099
13	1.2661969	−0.1350178
14		1.2666657

Table VI—Wideband differentiators
($F_p = 0.48$, N odd)

	$N = 31$, $D = 0.0904880$	$N = 33$, $D = 0.0775765$	$N = 35$, $D = 0.0669570$	$N = 37$, $D = 0.0574865$
0	0.0608611	−0.0522909	0.0452050	−0.0389237
1	−0.0899683	0.0775166	−0.0671563	0.0580516
2	0.0575832	−0.0499780	0.0435425	−0.0380356
3	−0.0599504	0.0521363	−0.0454862	0.0398662
4	0.0676325	−0.0587798	0.0512472	−0.0449414
5	−0.0779065	0.0675169	−0.0587481	0.0514649
6	0.0906186	−0.0781443	0.0677554	−0.0591983
7	−0.1063470	0.0910075	−0.0784716	0.0682828
8	0.1262399	−0.1068305	0.0913845	−0.0790428
9	−0.1522623	0.1267944	−0.1072511	0.0919846
10	0.1878138	−0.1527018	0.1270991	−0.1077415
11	−0.2401009	0.1881949	−0.1529673	0.1275383
12	0.3258611	−0.2405147	0.1884984	−0.1534175
13	−0.4949906	0.3261771	−0.2407258	0.1888647
14	0.9974893	−0.4951904	0.3263414	−0.2410280
15	0.	0.9975854	−0.4953164	0.3265623
16		0.	0.9976379	−0.4954722
17			0.	0.9977334
18				0.

	$N = 39$, $D = 0.0496845$	$N = 41$, $D = 0.0427980$	$N = 43$, $D = 0.0370790$	$N = 45$, $D = 0.0319585$
0	0.0337241	−0.0291326	0.0252974	−0.0218781
1	−0.0504549	0.0437514	−0.0381107	0.0331074
2	0.0333314	−0.0292008	0.0256448	−0.0225406
3	−0.0350300	0.0308071	−0.0271405	0.0239650
4	0.0395115	−0.0347953	0.0307025	−0.0271669
5	−0.0452125	0.0398250	−0.0351626	0.0311429
6	0.0519079	−0.0456920	0.0403302	−0.0357293
7	−0.0596859	0.0524473	−0.0462408	0.0409394
8	0.0687827	−0.0602598	0.0530210	−0.0468694
9	−0.0795659	0.0694016	−0.0608706	0.0536820
10	0.0924131	−0.0801166	0.0699494	−0.0614800
11	−0.1081069	0.0929299	−0.0806174	0.0705137
12	0.1279109	−0.1086479	0.0934269	−0.0811778
13	−0.1537285	0.1283931	−0.1090871	0.0939352
14	0.1891258	−0.1541598	0.1287811	−0.1095404
15	−0.2412344	0.1894996	−0.1544809	0.1291782
16	0.3267197	−0.2415476	0.1897616	−0.1548211
17	−0.4955762	0.3269657	−0.2417578	0.1900419
18	0.9977978	−0.4957493	0.3271268	−0.2419859
19	0.	0.9978660	−0.4958426	0.3272880
20		0.	0.9979125	−0.4959485
21			0.	0.9979703
22				0.

Table VI—continued

	$N = 47$ $D = 0.0277365$	$N = 49$ $D = 0.0239425$
0	0.0190371	−0.0164918
1	−0.0289080	0.0251594
2	0.0198593	−0.0174902
3	−0.0211875	0.0187459
4	0.0240589	−0.0213377
5	−0.0276055	0.0245186
6	0.0316820	−0.0281603
7	−0.0362942	0.0322648
8	0.0415174	−0.0368924
9	−0.0474776	0.0421467
10	0.0542475	−0.0480777
11	−0.0620081	0.0548148
12	0.0710446	−0.0625739
13	−0.0816707	0.0715793
14	0.0943907	−0.0821746
15	−0.1099542	0.0948557
16	0.1295429	−0.1103751
17	−0.1551425	0.1299234
18	0.1903205	−0.1554828
19	−0.2421995	0.1906061
20	0.3274491	−0.2424311
21	−0.4960631	0.3276270
22	0.9980259	−0.4961835
23	0.	0.9980887
24		0.

Table VII—Wideband differentiators
($F_p = 0.45$, N odd)

	$N = 15$, $D = 0.0735575$	$N = 17$, $D = 0.0507440$	$N = 19$, $D = 0.0351550$	$N = 21$, $D = 0.0245205$
0	0.0621149	−0.0435352	0.0307251	−0.0218457
1	−0.1208630	0.0858820	−0.0616198	0.0445927
2	0.1395483	−0.1005756	0.0736612	−0.0545490
3	−0.1970413	0.1402768	−0.1025906	0.0762772
4	0.2916922	−0.1997274	0.1436066	−0.1061155
5	−0.4713966	0.2942249	−0.2029221	0.1470256
6	0.9854400	−0.4732577	0.2968538	−0.2059085
7	0.	0.9864199	−0.4751178	0.2992430
8		0.	0.9873796	−0.4767810
9			0.	0.9882420
10				0.

	$N = 23$, $D = 0.0171585$	$N = 25$, $D = 0.0120320$	$N = 27$, $D = 0.0084550$	$N = 29$, $D = 0.0059730$
0	0.0155952	−0.0111731	0.0080321	−0.0058000
1	−0.0324316	0.0236892	−0.0173702	0.0127850
2	0.0406871	−0.0305049	0.0229594	−0.0173278
3	−0.0573479	0.0434319	−0.0330574	0.0252509
4	0.0797751	−0.0606437	0.0464403	−0.0357494
5	−0.1095875	0.0831228	−0.0637571	0.0492781
6	0.1502102	−0.1127769	0.0861795	−0.0666162
7	−0.2086526	0.1530955	−0.1156518	0.0889542
8	0.3014107	−0.2111147	0.1556885	−0.1182549
9	−0.4782695	0.3033453	−0.2133255	0.1580256
10	0.9889982	−0.4796075	0.3050898	−0.2153078
11	0.	0.9896812	−0.4808120	0.3066433
12		0.	0.9902957	−0.4818795
13			0.	0.9908395
14				0.

	$N = 31$, $D = 0.0042255$	$N = 33$, $D = 0.0029835$	$N = 35$, $D = 0.0021205$	$N = 37$, $D = 0.0015065$
0	0.0041972	−0.0030316	0.0022041	−0.0016019
1	−0.0094342	0.0069542	−0.0051557	0.0038195
2	0.0131118	−0.0099202	0.0075348	−0.0057161
3	−0.0193447	0.0148315	−0.0114071	0.0087679
4	0.0276221	−0.0213785	0.0165958	−0.0128862
5	−0.0382866	0.0298388	−0.0233329	0.0182658
6	0.0518947	−0.0406255	0.0319384	−0.0251623
7	−0.0692328	0.0542892	−0.0428193	0.0338874
8	0.0914895	−0.0716110	0.0565198	−0.0448418
9	−0.1206189	0.0937794	−0.0738174	0.0585671
10	0.1601348	−0.1227515	0.0958981	−0.0758355
11	−0.2170932	0.1620418	−0.1247175	0.0978301
12	0.3080442	−0.2187083	0.1637913	−0.1265044
13	−0.4828449	0.3093083	−0.2201835	0.1653775
14	0.9913327	−0.4837123	0.3104607	−0.2215197
15	0.	0.9917735	−0.4845030	0.3115043
16		0.	0.9921756	−0.4852199
17			0.	0.9925410
18				0.

Table VII—continued

	$N = 39$ $D = 0.0010700$	$N = 41$ $D = 0.0007635$	$N = 43$ $D = 0.0005430$	$N = 45$ $D = 0.0003895$
0	0.0011652	−0.0008504	0.0006202	−0.0004546
1	−0.0028328	0.0021064	−0.0015651	0.0011677
2	0.0043404	−0.0033002	0.0025070	−0.0019091
3	−0.0067453	0.0051953	−0.0039977	0.0030822
4	0.0100145	−0.0077927	0.0060586	−0.0047187
5	−0.0143150	0.0112340	−0.0088112	0.0069219
6	0.0198555	−0.0156929	0.0124011	−0.0098156
7	−0.0268851	0.0213726	−0.0169985	0.0135415
8	0.0357108	−0.0285178	0.0228014	−0.0182649
9	−0.0467274	0.0374289	−0.0300468	0.0241771
10	0.0604684	−0.0484943	0.0390296	−0.0315102
11	−0.0777016	0.0622422	−0.0501326	0.0405542
12	0.0996092	−0.0794383	0.0638811	−0.0516874
13	−0.1281462	0.1012620	−0.0810377	0.0654312
14	0.1668343	−0.1296683	0.1027800	−0.0825450
15	−0.2227461	0.1681805	−0.1310632	0.1042063
16	0.3124603	−0.2238759	0.1694129	−0.1323710
17	−0.4858737	0.3133399	−0.2249101	0.1705665
18	0.9928727	−0.4864766	0.3141448	−0.2258767
19	0.	0.9931793	−0.4870273	0.3148957
20		0.	0.9934589	−0.4875406
21			0.	0.9937194
22				0.

	$N = 47$ $D = 0.0002770$	$N = 49$ $D = 0.0001985$
0	0.0003314	−0.0002432
1	−0.0008674	0.0006481
2	0.0014489	−0.0011040
3	−0.0023700	0.0018278
4	0.0036666	−0.0028563
5	−0.0054274	0.0042647
6	0.0077575	−0.0061421
7	−0.0107775	0.0085916
8	0.0146251	−0.0117291
9	−0.0194593	0.0156885
10	0.0254670	−0.0206239
11	−0.0328758	0.0267183
12	0.0419704	−0.0341940
13	−0.0531259	0.0433323
14	0.0668603	−0.0545041
15	−0.0839314	0.0682257
16	0.1055160	−0.0852521
17	−0.1335695	0.1067601
18	0.1716214	−0.1347058
19	−0.2267589	0.1726205
20	0.3155805	−0.2275936
21	−0.4880090	0.3162280
22	0.9939572	−0.4884507
23	0.	0.9941812
24		0.

Table VIII—Wideband differentiators ($F_p = 0.40$, N odd)

	$N = 9$ $D = 0.0609450$	$N = 11$ $D = 0.0292625$	$N = 13$ $D = 0.0142550$	$N = 15$ $D = 0.0070280$
0	−0.0756455	0.0391813	−0.0207637	0.0111819
1	0.2086718	−0.1121159	0.0622962	−0.0352879
2	−0.4002986	0.2123949	−0.1213132	0.0715702
3	0.9466429	−0.4116991	0.2254262	−0.1332217
4	0.	0.9534363	−0.4221135	0.2364777
5		0.	0.9591518	−0.4305757
6			0.	0.9637379
7				0.

	$N = 17$ $D = 0.0034905$	$N = 19$ $D = 0.0017425$	$N = 21$ $D = 0.0008780$	$N = 23$ $D = 0.0004415$
0	−0.0060893	0.0033408	−0.0018491	0.0010251
1	0.0202278	−0.0116764	0.0067871	−0.0039493
2	−0.0429142	0.0259414	−0.0157683	0.0095916
3	0.0813764	−0.0504920	0.0315941	−0.0198219
4	−0.1433229	0.0898511	−0.0572637	0.0367532
5	0.2455792	−0.1518244	0.0972194	−0.0632651
6	−0.4374621	0.2531345	−0.1590934	0.1036232
7	0.9674450	−0.4431314	0.2595207	−0.1653232
8	0.	0.9704860	−0.4478868	0.2649402
9		0.	0.9730241	−0.4518964
10			0.	0.9751570
11				0.

	$N = 25$ $D = 0.0002235$	$N = 27$ $D = 0.0001140$	$N = 29$ $D = 0.0000580$	$N = 31$ $D = 0.0000295$
0	−0.0005721	0.0003211	−0.0001806	0.0001015
1	0.0023072	−0.0013534	0.0007945	−0.0004659
2	−0.0058456	0.0035688	−0.0021774	0.0013261
3	0.0124605	−0.0078430	0.0049320	−0.0030942
4	−0.0236760	0.0152804	−0.0098561	0.0063438
5	0.0414998	−0.0273300	0.0180098	−0.0118517
6	−0.0686617	0.0458852	−0.0307650	0.0206299
7	0.1092853	−0.0735522	0.0499183	−0.0339823
8	−0.1707635	0.1143420	−0.0779740	0.0536245
9	0.2696310	−0.1755688	0.1188552	−0.0819780
10	−0.4553459	0.2737411	−0.1798160	0.1228947
11	0.9769854	−0.4583518	0.2773477	−0.1835837
12	0.	0.9785734	−0.4609763	0.2805263
13		0.	0.9799561	−0.4632794
14			0.	0.9811659
15				0.

Table VIII—continued

	$N = 33$, $D = 0.0000150$	$N = 35$, $D = 0.0000075$	$N = 37$, $D = 0.0000040$	$N = 39$, $D = 0.0000020$
0	−0.0000572	0.0000324	−0.0000182	0.0000104
1	0.0002730	−0.0001602	0.0000946	−0.0000556
2	−0.0008061	0.0004901	−0.0002985	0.0001813
3	0.0019374	−0.0012117	0.0007581	−0.0004734
4	−0.0040743	0.0026122	−0.0016742	0.0010703
5	0.0077839	−0.0051026	0.0033417	−0.0021825
6	−0.0138180	0.0092391	−0.0061707	0.0041092
7	0.0231441	−0.0157479	0.0107053	−0.0072577
8	−0.0370061	0.0255562	−0.0176463	0.0121583
9	0.0570510	−0.0398558	0.0278845	−0.0194879
10	−0.0856320	0.0602338	−0.0425629	0.0301031
11	0.1265440	−0.0889869	0.0632195	−0.0451067
12	−0.1869606	0.1298634	−0.0921018	0.0659923
13	0.2833588	−0.1900104	0.1329205	−0.0949672
14	−0.4653230	0.2859028	−0.1928014	0.1357111
15	0.9822372	−0.4671520	0.2882201	−0.1953339
16	0.	0.9831935	−0.4688120	0.2903130
17		0.	0.9840602	−0.4703068
18			0.	0.9848390
19				0.

	$N = 41$, $D = 0.0000010$	$N = 43$, $D = 0.0000005$	$N = 45$, $D = 0.0000005$
0	−0.0000060	0.0000035	−0.0000019
1	0.0000330	−0.0000195	0.0000113
2	−0.0001106	0.0000672	−0.0000408
3	0.0002959	−0.0001844	0.0001147
4	−0.0006846	0.0004364	−0.0002774
5	0.0014247	−0.0009274	0.0006013
6	−0.0027341	0.0018140	−0.0011982
7	0.0049144	−0.0033182	0.0022305
8	−0.0083667	0.0057416	−0.0039235
9	0.0136084	−0.0094801	0.0065776
10	−0.0212934	0.0150360	−0.0105809
11	0.0322462	−0.0230414	0.0164220
12	−0.0475332	0.0342936	−0.0247140
13	0.0686105	−0.0498257	0.0362270
14	−0.0976501	0.0710606	−0.0519676
15	0.1383067	−0.1001420	0.0733295
16	−0.1976766	0.1407019	−0.1024316
17	0.2922413	−0.1998276	0.1428897
18	−0.4716800	0.2940053	−0.2017829
19	0.9855531	−0.4729332	0.2956031
20	0.	0.9862041	−0.4740651
21		0.	0.9867909
22			0.