THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 53, No. 2, February 1974
Printed in U.S.A.

On the Behavior of Minimax FIR Digital Hilbert Transformers

By L. R. RABINER and R. W. SCHAFER

(Manuscript received July 9, 1973)

Optimum (in a minimax sense) linear phase FIR Hilbert transformers can be designed efficiently using a Remez optimization procedure. This paper presents useful design data on wideband Hilbert transformers with even and odd values of N, the impulse response duration (in samples) of the filter. Based on these data, the following observations can be made:

(i) In the case of equal lower and upper transition regions, Hilbert transformers with odd values of N can be realized more efficiently than those with even values of N, assuming the same peak errors of approximation for both cases. This is because every other impulse response sample is exactly zero for odd values of N.

(ii) The peak approximation error for Hilbert transformers with odd values of N is determined primarily by the minimum of the values of the lower and upper transition widths.

(iii) The peak approximation error for Hilbert transformers with even values of N is determined primarily by the lower transition width of the filter.

(iv) The smaller the bandwidth of the Hilbert transformer, the faster the decrease of peak error of approximation with decreasing bandwidth of the Hilbert transformer.

(v) The larger the value of N, the faster the decrease of peak approximation error with decreasing bandwidth of the Hilbert transformer.

These observations lead to the conclusion that the bandwidth of the Hilbert transformer should be made as small as possible, and that odd values of N should be used, whenever possible, for efficient direct form realizations. A set of tables of values of the impulse response coefficients is included for several different bandwidth Hilbert transformers and for both even and odd values of N.

I. INTRODUCTION

Hilbert transformers have been used in a variety of applications including radar systems, speech processing systems, modulation systems, and schemes for efficient sampling of a real bandpass signal.[1,2] Although the theory of discrete Hilbert transformer systems is well understood, it is only recently that design techniques have become available for obtaining optimum (in a minimax or Chebyshev sense) finite-duration impulse response (FIR) approximations to the ideal Hilbert transformer.[3,4] It is the purpose of this paper to present new design data on minimax FIR Hilbert transformers to aid in selecting the most efficient network to meet given design specifications. Herrmann[3] has already given a small table of data on FIR Hilbert transformers; however, the generality and widespread applicability of the data presented here justify this more complete elaboration of the properties of the minimax Hilbert transformer approximations.

II. DISCRETE-TIME HILBERT TRANSFORMERS

Most applications of Hilbert transformers involve the representation of a real bandpass signal in terms of a complex signal. For continuous-time signals, such complex signals are analytic functions of time and thus are called *analytic signals*. Although the concept of analyticity is meaningless for discrete-time signals, complex representation of real, discrete-time, bandpass signals can be used in a similar manner to analytic signals. Therefore, consider a real sequence $x(n)$ with Fourier transform $X(e^{j\omega})$. Corresponding to this sequence, we can construct the complex sequence

$$\tilde{x}(n) = x(n) + j\hat{x}(n) \tag{1}$$

whose Fourier transform, $\tilde{X}(e^{j\omega})$, is

$$\begin{aligned} \tilde{X}(e^{j\omega}) &= 2X(e^{j\omega}) \qquad 0 \leqq \omega < \pi \\ &= 0 \qquad \pi \leqq \omega < 2\pi. \end{aligned} \tag{2}$$

From eq. (1), the Fourier transform of $\tilde{x}(n)$ is

$$\tilde{X}(e^{j\omega}) = X(e^{j\omega}) + j\hat{X}(e^{j\omega})$$

and from eq. (2) it follows that

$$X(e^{j\omega}) + j\hat{X}(e^{j\omega}) = 0 \qquad \pi \leqq \omega < 2\pi$$

and

$$\tilde{X}(e^{j\omega}) = 2X(e^{j\omega}) \qquad 0 \leqq \omega < \pi.$$

These relationships are satisfied if

$$\hat{X}(e^{j\omega}) = H_d(e^{j\omega})X(e^{j\omega}), \tag{3}$$

where

$$H_d(e^{j\omega}) = -j \qquad 0 < \omega < \pi$$
$$= +j \qquad \pi < \omega < 2\pi. \tag{4}$$

Thus $\hat{x}(n)$ can be obtained by linear filtering the sequence $x(n)$ with a system having a frequency response, $H_d(e^{j\omega})$, as in eq. (4). Such a system is called an ideal Hilbert transformer and $\hat{x}(n)$ is the Hilbert transform of $x(n)$. If one allows the addition of a linear phase term in the definition of the frequency response of the "ideal" Hilbert transformer, eq. (4) becomes

$$H_d(e^{j\omega}) = -je^{-j\omega\tau} \qquad 0 < \omega < \pi$$
$$= +je^{-j(\omega-2\pi)\tau} \qquad \pi < \omega < 2\pi, \tag{5}$$

where τ is the delay in samples. The impulse response of the ideal Hilbert transformer of eq. (5) can be shown to be

$$h_d(n) = \frac{2}{\pi} \frac{\sin^2\left[\frac{\pi}{2}(n-\tau)\right]}{(n-\tau)}. \tag{6}$$

For $\tau = 0$, eq. (6) gives

$$h_d(n) = \frac{2}{\pi} \frac{\sin^2\left[\frac{\pi n}{2}\right]}{n} \qquad n \neq 0$$
$$= 0 \qquad n = 0, \tag{7}$$

whereas for $\tau = -\frac{1}{2}$ (i.e., a half-sample advance) eq. (6) gives

$$h_d(n) = \frac{1}{\pi(n+\frac{1}{2})}. \tag{8}$$

The frequency response [eq. (5)] corresponding to $\tau = 0$ has discontinuities at both $\omega = 0$ and $\omega = \pi$, whereas the frequency response corresponding to $\tau = -\frac{1}{2}$ only has a discontinuity at $\omega = 0$. The impulse response of eq. (7), corresponding to $\tau = 0$, has odd symmetry, is noncausal, and is of infinite duration. In addition, all even-numbered samples of the impulse response are exactly zero—i.e., $h_d(2n) = 0$, $n = 0, \pm1, \pm2, \cdots$. The impulse response of eq. (8), corresponding to $\tau = -\frac{1}{2}$, is also noncausal, is of infinite duration, and obeys the symmetry condition

$$h(n) = -h(-n-1) \qquad n = 0, 1, \cdots. \tag{9}$$

This impulse response does *not* have the property that even-numbered (or odd-numbered) impulse response samples are zero.

We have only considered $\tau = 0$ and $\tau = -\frac{1}{2}$ for determining the impulse response of the ideal Hilbert transformer. All other values of $(-1 < \tau \leqq 0)$ can be shown to yield impulse responses without desirable symmetry properties and hence are not suitable candidates for approximation or implementation.

To obtain a causal approximation to the ideal Hilbert transformers with no phase distortion (except for a delay), it can be shown that an FIR approximation is required.[5] Since there are a number of subtle distinctions between FIR systems of even-length and of odd-length, the next section deals with some of the general properties of FIR Hilbert transformer approximations.

III. PROPERTIES OF FIR HILBERT TRANSFORMERS

Consider a causal FIR system with impulse response $h(n)$, $0 \leqq n \leqq N-1$, and frequency response

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n}. \tag{10}$$

Since the desired frequency response $H_d(e^{j\omega})$ is purely imaginary and $h(n)$ is real, a linear phase approximation to the ideal frequency response is obtained only when $h(n)$ satisfies the symmetry condition

$$h(n) = -h(N-1-n) \qquad n = 0, 1, \cdots, N-1. \tag{11}$$

For N an odd integer, this means that $h(n)$ has odd symmetry about the sample at $n = (N-1)/2$. (This case corresponds to the $\tau = 0$ case above with an additional delay of $(N-1)/2$ samples.) For N even, $h(n)$ has odd symmetry about a point halfway between the samples at $n = N/2$ and $n = (N/2) + 1$. (This case corresponds to the $\tau = -\frac{1}{2}$ case above with an additional delay of $N/2$ samples.) This implies that the frequency response of a filter satisfying eq. (11) can be expressed as

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}[jH^*(e^{j\omega})], \tag{12}$$

where $H^*(e^{j\omega})$ is a purely real function of ω. In particular, for N odd, $H^*(e^{j\omega})$ is

$$H^*(e^{j\omega}) = \sum_{n=1}^{(N-1)/2} a(n) \sin(\omega n), \tag{13a}$$

where

$$a(n) = 2h\left(\frac{N-1}{2} - n\right) \qquad n = 1, 2, \cdots, \left(\frac{N-1}{2}\right). \tag{13b}$$

Also, from eq. (11),

$$h\left(\frac{N-1}{2}\right) = 0. \tag{13c}$$

For N even, the expression for $H^*(e^{j\omega})$ is

$$H^*(e^{j\omega}) = \sum_{n=1}^{N/2} b(n) \sin\left[\omega(n - \tfrac{1}{2})\right], \tag{14a}$$

where

$$b(n) = 2h\left(\frac{N}{2} - n\right) \qquad n = 1, 2, \cdots, N/2. \tag{14b}$$

The factor $e^{-j\omega(N-1)/2}$ in eq. (12) represents a delay of $(N - 1)/2$ samples. When N is odd, this delay is an integer number of sample intervals, but when N is even, it is an integer plus one-half of a sample interval. In approximating the Hilbert transformer, the coefficients $a(n)$ and $b(n)$ must be chosen so that $jH^*(e^{j\omega})$ approximates the ideal frequency response of eq. (5). In many cases it is not required, and often it is not desirable, that the approximation be carried out over the entire band $0 \leqq \omega \leqq 2\pi$. Thus, in general, the real function $H^*(e^{j\omega})$ must approximate

$$\begin{aligned} D(e^{j\omega}) &= -1 \qquad 2\pi F_L \leqq \omega \leqq 2\pi F_H \\ &= +1 \qquad 2\pi(1 - F_H) \leqq \omega \leqq 2\pi F_L, \end{aligned} \tag{15}$$

where F_L and F_H define the lower and upper cutoff frequencies respectively. Alternatively, F_L and $0.5 - F_H$ define the lower and upper transition bandwidths respectively. It can be seen from eqs. (13a) and (14a) that $H^*(e^{j\omega})$ is constrained to be zero at $\omega = 0$, and π when N is odd, and when N is even, $H^*(e^{j\omega})$ is constrained to be zero at $\omega = 0$. Thus, fullband approximations are impossible since F_L cannot be zero, and F_H can be 0.5 only when N is even. In the transition regions the desired frequency response is left undefined; hence, in these regions, $H^*(e^{j\omega})$ is unconstrained except at the end points. In the next section it will be shown by examples that leaving these regions unconstrained can lead to unsatisfactory FIR approximations.

The impulse response of the ideal Hilbert transformer (with $\tau = 0$) has the property

$$h_d(n) = 0 \qquad n = 0, \pm 2, \pm 4, \cdots.$$

This can be shown to be a direct result of the fact that $h_d(n)$ is real

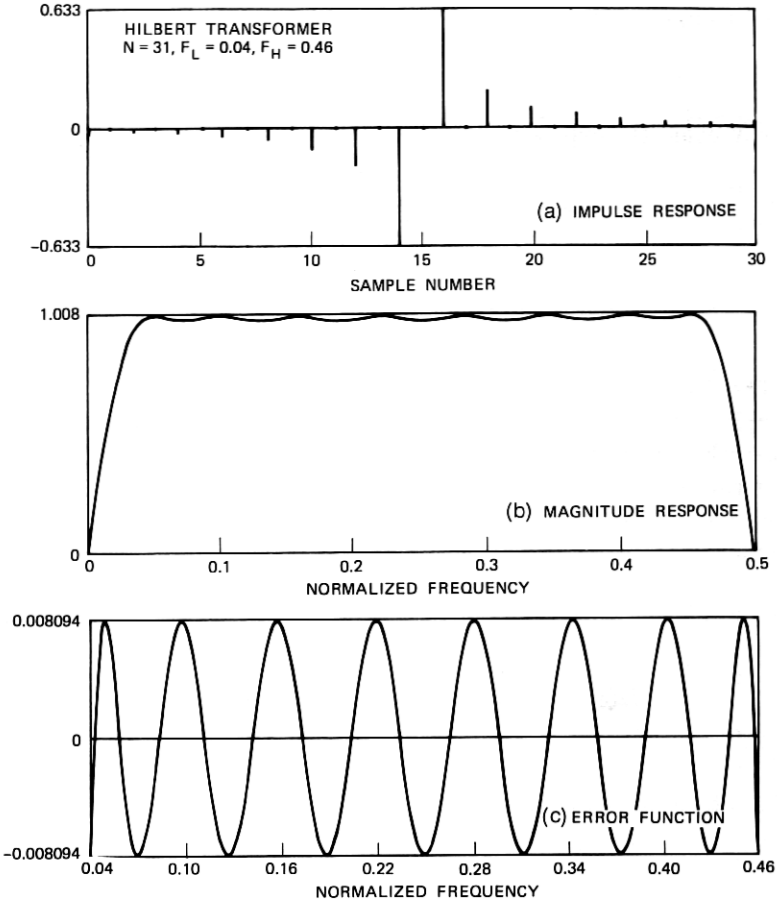

Fig. 1—The impulse response, magnitude response, and error function of an $N = 31$ Hilbert transformer with $F_L = 0.04$ and $F_H = 0.46$.

and the frequency response is an imaginary, odd, periodic function and

$$H_d(e^{j\omega}) = H_d(e^{j(\pi-\omega)}). \tag{16}$$

Similar properties can be achieved for the FIR Hilbert transformers if N is odd and $F_L = 0.5 - F_H$. To see this, assume that

$$H^*(e^{j\omega}) = H^*(e^{j(\pi-\omega)}). \tag{17}$$

Clearly this requires that $F_L = 0.5 - F_H$. Substituting eq. (13a) into eq. (17), it follows that

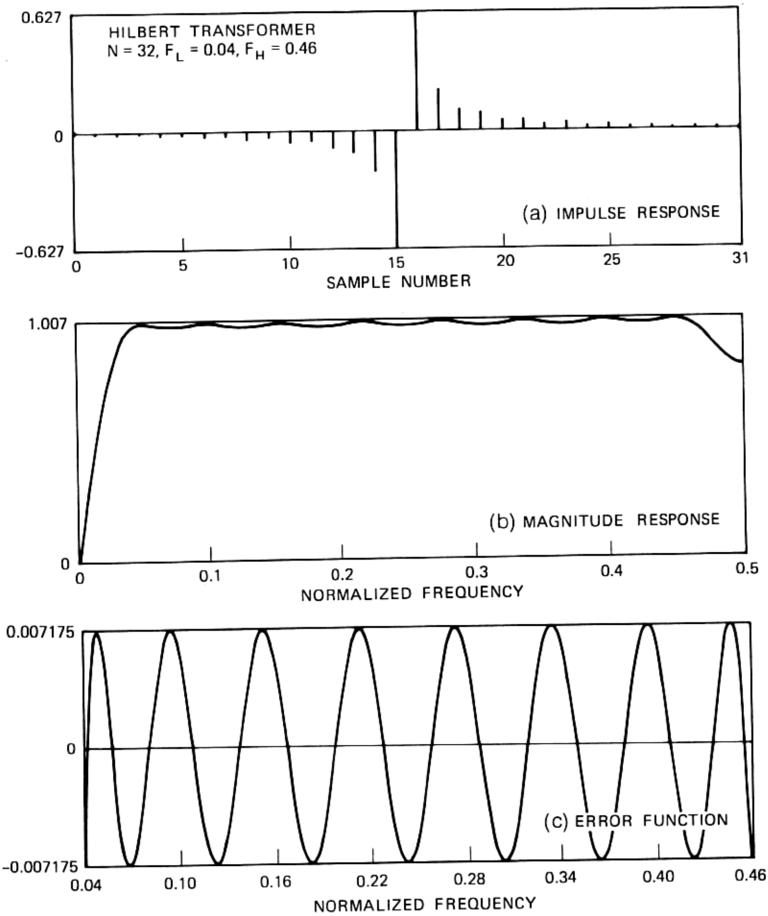

Fig. 2—The impulse response, magnitude response, and error function of an $N = 32$ Hilbert transformer with $F_L = 0.04$ and $F_H = 0.46$.

$$\sum_{n=1}^{(N-1)/2} a(n) \sin (\omega n) = \sum_{n=1}^{(N-1)/2} a(n) \sin [(\pi - \omega)n]$$

$$= \sum_{n=1}^{(N-1)/2} a(n)(-1)^{n+1} \sin (\omega n)$$

or

$$\sum_{n=1}^{(N-1)/2} a(n) \sin (\omega n)[1 - (-1)^{n+1}] = 0$$

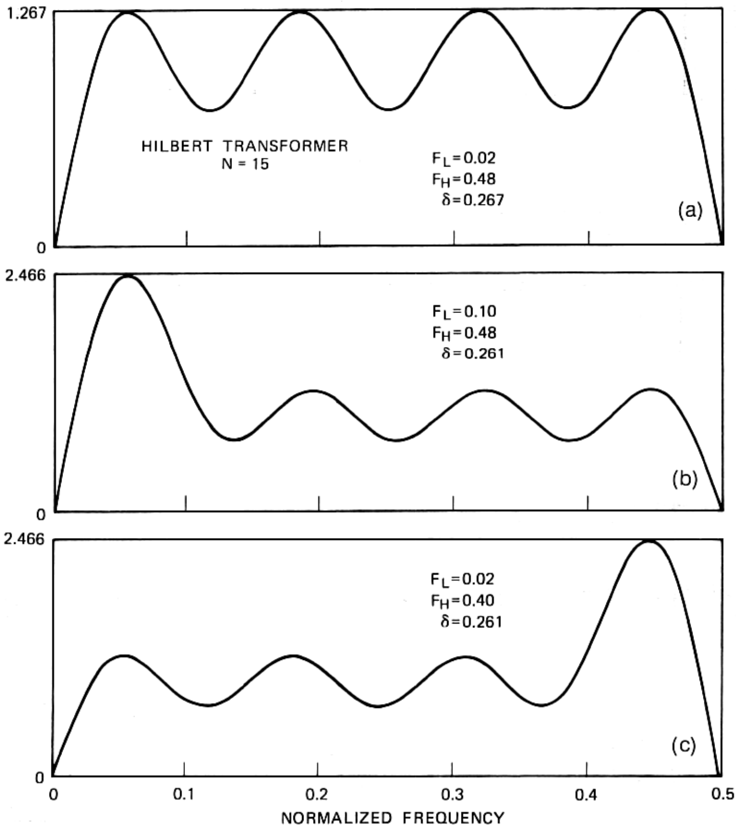

Fig. 3—The magnitude responses of three $N = 15$ Hilbert transformers with different upper and lower cutoff frequencies.

which implies

$$a(n) = 0 \qquad n \text{ even}$$

$$= \text{unconstrained} \qquad n \text{ odd.}$$

Therefore, from eqs. (11), (13b), and (13c), it can be shown that when $(N-1)/2$ is even, $h(n) = 0$ for $n = 0, 2, 4, \cdots$ and when $(N-1)/2$ is odd, $h(n) = 0$ for $n = 1, 3, 5, \cdots$. When N is odd and $(N-1)/2$ is even, $h(0)$ and $h(N-1)$ are both zero so that the actual length of the impulse response is $N-2$ samples. Thus, for symmetrical approximations [i.e., satisfying eq. (17)], it is only necessary to consider

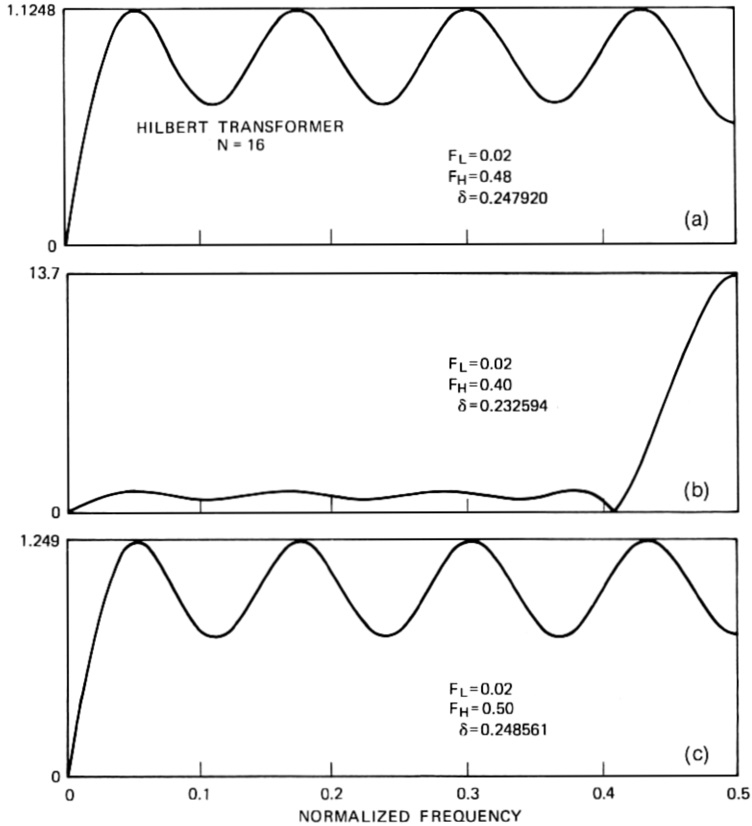

Fig. 4—The magnitude responses of three $N = 16$ Hilbert transformers with different upper and lower cutoff frequencies.

values of N such that $(N - 1)/2$ is odd, since the FIR approximations to be discussed in the next section are unique.

In a manner similar to above, it can be shown that when N is even, alternate coefficients are not zero even if $F_L = 0.5 - F_H$.

This distinction between even- and odd-length impulse responses is important in direct convolutional realizations of such systems. For this realization, the convolution summation

$$y(n) = \hat{x}(n) = \sum_{k=0}^{N-1} h(k)x(n - k)$$

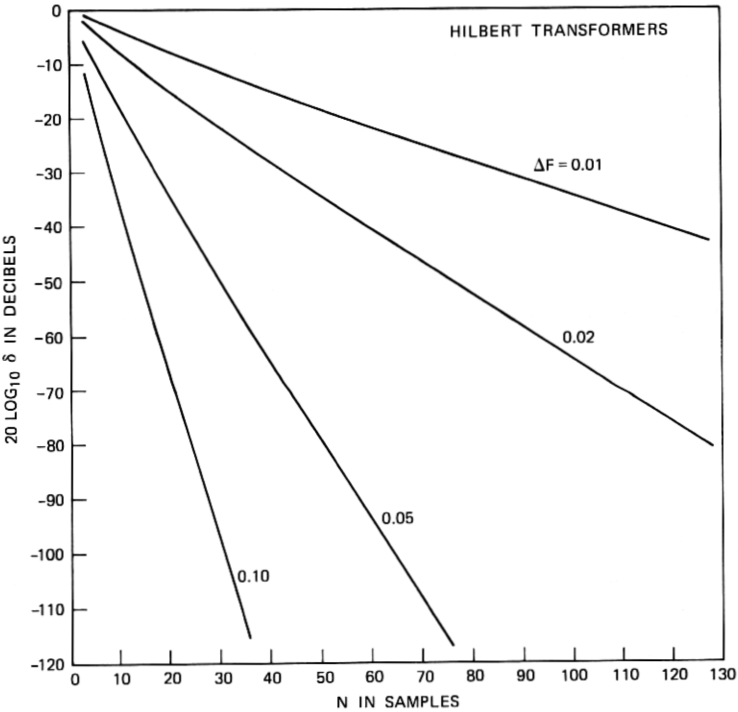

Fig. 5—The curves of $20 \log_{10} \delta$ versus N for $\Delta F = 0.01, 0.02, 0.05$, and 0.10 for even and odd values of N.

requires $N/2$ multiplications per output sample for N even, whereas only $(N + 1)/4$ multiplications per output sample are required for N odd when alternate samples of $h(n)$ are exactly zero. Thus an odd-length filter requires about half the computation required by an even-length filter one sample shorter.

IV. PROPERTIES OF OPTIMUM FIR HILBERT TRANSFORMERS

The iterative Remez algorithm of McClellan, Parks, and Rabiner[4] can be used to choose the values of $a(n)$ or $b(n)$ that minimize the peak approximation error.

$$\delta = \max_{2\pi F_L \leq \omega \leq 2\pi F_H} [D(e^{j\omega}) - H^*(e^{j\omega})]. \tag{18}$$

The resulting approximation is the unique best approximation (in the minimax sense) to $D(e^{j\omega})$ for a given choice of N, F_L, and F_H. The

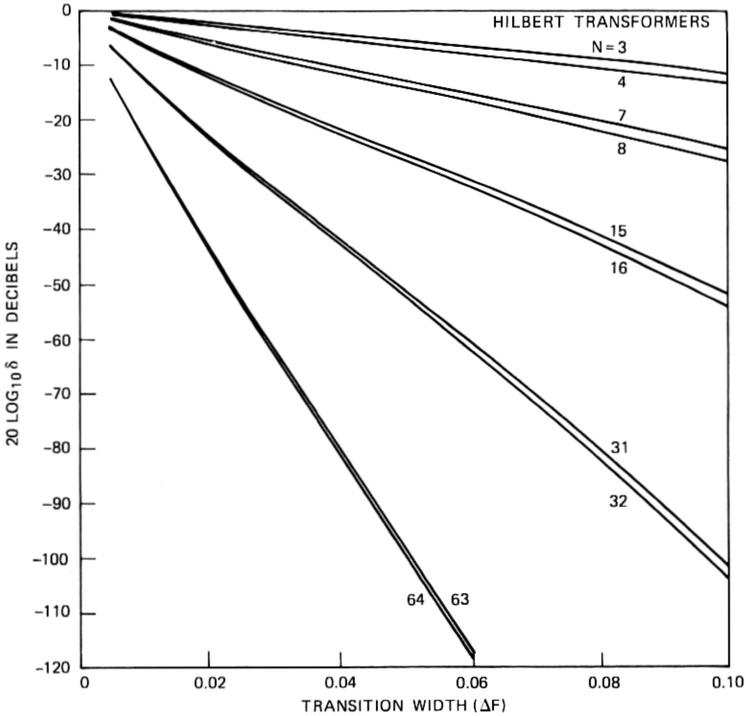

Fig. 6—The curves of 20 $\log_{10} \delta$ versus ΔF for $N = 3, 4, 7, 8, 15, 16, 31, 32, 63,$ and 64.

details of the approximation algorithm are given in Ref. 4. Our present concern is the general properties of the minimax FIR Hilbert transformer approximations obtained using the McClellan, et al., algorithm.

The minimax approximation to the ideal Hilbert transformer is characterized by an error function that is equiripple over the approximation band $2\pi F_L \leqq \omega \leqq 2\pi F_H$. This is illustrated in Figs. 1 through 4 which show the responses of several optimum Hilbert transformer approximations. Figure 1 shows the impulse response, magnitude response, and the error function of an optimum Hilbert transformer having $N = 31$, $F_L = 0.04$, and $F_H = 0.46$. The peak approximation error is 0.008094 and the error curve is seen to be equiripple. It can be seen that $H(e^{j\omega})$ has the symmetry property of eq. (17) and therefore alternate samples of the impulse response are zero. To see that the minimax solution must satisfy eq. (17), assume that

$$H^*(e^{j\omega}) \neq H^*(e^{j(\pi-\omega)});$$

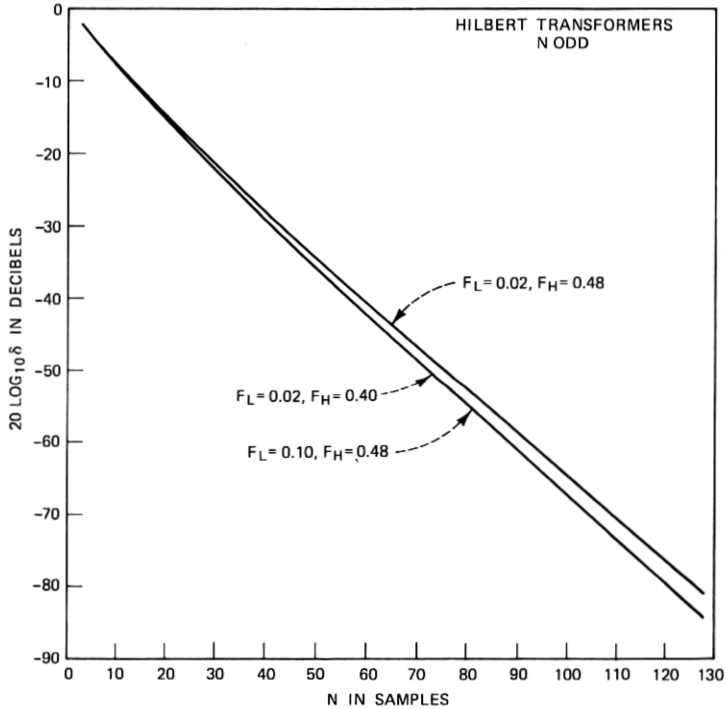

Fig. 7—The curves of 20 $\log_{10} \delta$ versus N for three sets of upper and lower cutoff frequencies with N odd.

then because $H^*(e^{j\omega}) = -H^*(e^{-j\omega})$ and because $F_L = 0.5 - F_H$, both $H^*(e^{j\omega})$ and $H^*(e^{j(\pi-\omega)})$ would satisfy the conditions for optimality, thus contradicting the uniqueness of the optimum approximation.

Figure 2 shows the same responses as in Fig. 1 for the case $N = 32$, $F_L = 0.04$, and $F_H = 0.46$. As previously noted, even though the upper and lower transition widths are equal, all the impulse response coefficients are nonzero because the frequency response cannot have the required symmetry when N is even. As seen in Fig. 2b, the magnitude response is not constrained to be zero at $f = 0.5$. Figure 2c shows the error curve to again be equiripple over the band of approximation. The peak approximation error for this case is 0.007175.

Figures 3 and 4 show the effects of making the upper and lower transition bandwidths unequal. Figure 3 shows three sets of conditions for $N = 15$; i.e., $F_L = 0.02$, $F_H = 0.48$ in Fig. 3a, $F_L = 0.10$, $F_H = 0.48$ in Fig. 3b, and $F_L = 0.02$, $F_H = 0.40$ in Fig. 3c. The peak approxi-

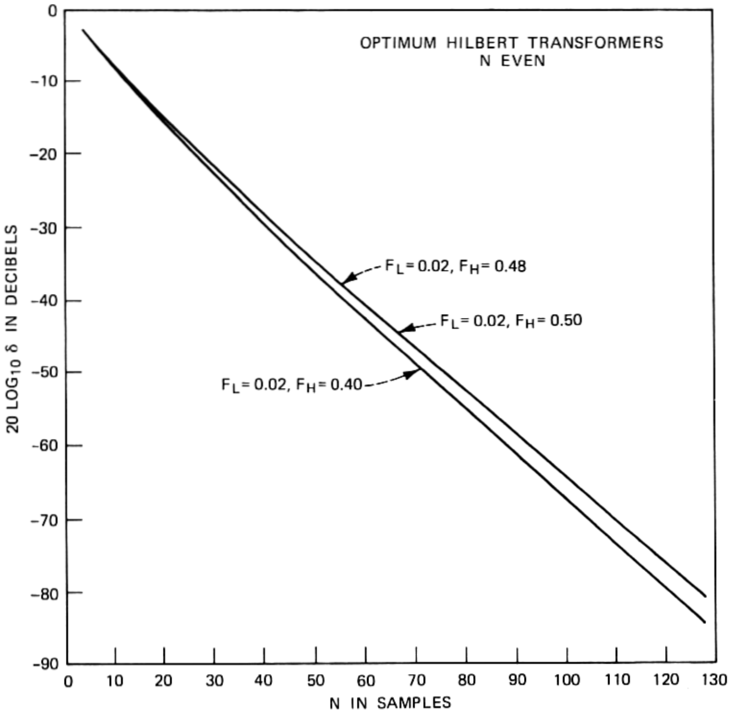

Fig. 8—The curves of $20 \log_{10} \delta$ versus N for three sets of upper and lower cutoff frequencies with N even.

mation errors are 0.267 for 3a, 0.261 for 3b, and 0.261 for 3c. Thus, even though one of the transition widths changes by a factor of 5:1 (i.e., from 0.02 to 0.10), the change in peak error is on the order of 2 percent. Furthermore, as seen in Figs. 2b and 2c, the magnitude response of the filter peaks up significantly in the wide transition band—a generally undesirable result. Also, when the transition bandwidths are unequal, the symmetry property of the frequency response no longer holds and all the impulse response coefficients are nonzero. In conclusion, the negligible decrease in peak error obtained by using unequal transition bandwidths is more than offset by the undesirable effects in the magnitude and impulse responses. Furthermore, based on this and other similar examples, it is seen that the peak error of approximation is determined primarily by the smaller of the two transition widths when the filter impulse response duration is odd.

Figure 4 shows similar examples of unequal transition widths for

Table I—Wideband Hilbert transformers ($F_L = 0.01$, N odd)

	$N = 55$, $D = 0.0954980$	$N = 59$, $D = 0.0819670$	$N = 63$, $D = 0.0704360$	$N = 67$, $D = 0.0606110$
0	−0.0555675	−0.0477809	−0.0411397	−0.0354857
2	−0.0167258	−0.0145478	−0.0126851	−0.0110945
4	−0.0194725	−0.0169056	−0.0147109	−0.0129141
6	−0.0226148	−0.0196199	−0.0171191	−0.0149160
8	−0.0264560	−0.0228180	−0.0197933	−0.0173168
10	−0.0310395	−0.0266103	−0.0229817	−0.0200179
12	−0.0366783	−0.0311604	−0.0267924	−0.0231739
14	−0.0439134	−0.0368259	−0.0313469	−0.0269273
16	−0.0535532	−0.0440473	−0.0369689	−0.0314908
18	−0.0671782	−0.0536627	−0.0441620	−0.0371318
20	−0.0881528	−0.0672589	−0.0537578	−0.0443213
22	−0.1253143	−0.0882199	−0.0673427	−0.0538996
24	−0.2109951	−0.1253639	−0.0882996	−0.0674568
26	−0.6362148	−0.2110201	−0.1254324	−0.0883853
28		−0.6362213	−0.2110732	−0.1254904
30			−0.6362430	−0.2111059
32				−0.6362516

	$N = 71$, $D = 0.0522480$	$N = 75$, $D = 0.0450690$	$N = 79$, $D = 0.0388830$	$N = 83$, $D = 0.0336160$
0	−0.0306629	−0.0265190	−0.0229388	−0.0198861
2	−0.0097354	−0.0085562	−0.0075151	−0.0066273
4	−0.0113179	−0.0099546	−0.0087784	−0.0077367
6	−0.0131328	−0.0115623	−0.0101565	−0.0090091
8	−0.0151642	−0.0133373	−0.0117808	−0.0103953
10	−0.0175162	−0.0153870	−0.0135612	−0.0119879
12	−0.0202240	−0.0177321	−0.0155902	−0.0137883
14	−0.0233958	−0.0204320	−0.0179182	−0.0158166
16	−0.0271460	−0.0235782	−0.0206260	−0.0181339
18	−0.0316681	−0.0273088	−0.0237742	−0.0208124
20	−0.0372851	−0.0318335	−0.0274953	−0.0239473
22	−0.0444587	−0.0374340	−0.0319865	−0.0276626
24	−0.0540183	−0.0445843	−0.0375627	−0.0321489
26	−0.0675540	−0.0541292	−0.0447012	−0.0377091
28	−0.0884541	−0.0676461	−0.0542333	−0.0448244
30	−0.1255376	−0.0885235	−0.0677331	−0.0543328
32	−0.2111303	−0.1255838	−0.0885965	−0.0678187
34	−0.6362585	−0.2111601	−0.1256401	−0.0886665
36		−0.6362709	−0.2111964	−0.1256905
38			−0.6362830	−0.2112252
40				−0.6362926

$N = 16$. The lower and upper cutoff frequencies for these examples are: $F_L = 0.02$, $F_H = 0.48$ for Fig. 4a, $F_L = 0.02$, $F_H = 0.40$ for Fig. 4b, and $F_L = 0.02$, $F_H = 0.50$ for Fig. 4c. The resulting peak approximation errors are 0.247920 for 4a, 0.232594 for 4b, and 0.248561 for 4c. For the data of Fig. 4b (where the upper transition width is 5 times the lower transition width) the magnitude response becomes

Table I—continued

	$N = 87$ $D = 0.0290440$	$N = 91$ $D = 0.0251650$	$N = 95$ $D = 0.0217910$
0	−0.0172371	−0.0149759	−0.0130099
2	−0.0058530	−0.0051648	−0.0045718
4	−0.0068525	−0.0060643	−0.0053689
6	−0.0079733	−0.0070506	−0.0062800
8	−0.0092241	−0.0081894	−0.0072616
10	−0.0106323	−0.0094332	−0.0083873
12	−0.0122073	−0.0108306	−0.0096455
14	−0.0139949	−0.0124102	−0.0110350
16	−0.0160152	−0.0141947	−0.0126022
18	−0.0183265	−0.0162129	−0.0143770
20	−0.0210006	−0.0185127	−0.0163895
22	−0.0241199	−0.0211705	−0.0186922
24	−0.0278194	−0.0242786	−0.0213465
26	−0.0322920	−0.0279699	−0.0244424
28	−0.0378402	−0.0324331	−0.0281175
30	−0.0449441	−0.0379675	−0.0325665
32	−0.0544359	−0.0450542	−0.0380864
34	−0.0679017	−0.0545312	−0.0451608
36	−0.0887289	−0.0679795	−0.0546233
38	−0.1257349	−0.0887891	−0.0680547
40	−0.2112513	−0.1257759	−0.0888468
42	−0.6363005	−0.2112740	−0.1258168
44		−0.6363082	−0.2112989
46			−0.6363167

very large in the upper transition band. For this case, the peak approximation error is about 6 percent smaller than the peak error for equal transition widths. Thus the slight decrease in peak error does not compensate for the undesirable peaking in the transition region of the frequency response. On the other hand, Fig. 4c shows that setting the upper transition width to 0, i.e., letting $F_H = 0.50$ produces a negligible change in the peak error. Based on these and other examples, it has been found that for even values of N, the peak approximation error depends almost entirely on the lower transition width (because of the zero at $\omega = 0$). Thus the upper transition width should be made less than or equal to the lower transition width to minimize the peaking in the upper transition band.

V. DESIGN DATA FOR OPTIMUM HILBERT TRANSFORMERS

The basic Hilbert transformer parameters are N, F_L, F_H, and δ, the peak approximation error (or the ripple) of the filter. It is assumed that $F_H = 0.5 - F_L$, i.e., the upper and lower transition widths are equal, for the data to be presented in this section. Thus there are only three parameters: N, δ, and $\Delta F = F_L = 0.5 - F_H$. A large set of

Table II—Wideband Hilbert transformers ($F_L = 0.02$, N odd)

	$N = 31$	$N = 35$	$N = 39$	$N = 43$
	$D = 0.0750950$	$D = 0.0554920$	$D = 0.0413090$	$D = 0.0308560$
0	−0.0510610	−0.0380993	−0.0286598	−0.0216528
2	−0.0315637	−0.0241883	−0.0187406	−0.0146215
4	−0.0425110	−0.0324289	−0.0250998	−0.0196329
6	−0.0577251	−0.0433438	−0.0332813	−0.0259590
8	−0.0805394	−0.0584460	−0.0441044	−0.0340917
10	−0.1196632	−0.0811443	−0.0591324	−0.0448233
12	−0.2075778	−0.1201457	−0.0816868	−0.0597557
14	−0.6350674	−0.2078690	−0.1206017	−0.0821807
16		−0.6351797	−0.2081174	−0.1209598
18			−0.6352463	−0.2083547
20				−0.6353344

	$N = 47$	$N = 51$	$N = 55$	$N = 59$
	$D = 0.0231120$	$D = 0.0172930$	$D = 0.0130540$	$D = 0.0098300$
0	−0.0164094	−0.0124427	−0.0095117	−0.0072606
2	−0.0114516	−0.0090077	−0.0071168	−0.0056269
4	−0.0154645	−0.0122483	−0.0097423	−0.0077659
6	−0.0204740	−0.0162677	−0.0129935	−0.0104170
8	−0.0267802	−0.0212626	−0.0170107	−0.0136817
10	−0.0348570	−0.0275401	−0.0219890	−0.0176945
12	−0.0455267	−0.0355514	−0.0282298	−0.0226586
14	−0.0603428	−0.0461607	−0.0361989	−0.0288540
16	−0.0826818	−0.0608791	−0.0467260	−0.0367817
18	−0.1213191	−0.0831083	−0.0613745	−0.0472445
20	−0.2085553	−0.1216348	−0.0835102	−0.0618163
22	−0.6353986	−0.2087518	−0.1219297	−0.0838702
24		−0.6354638	−0.2089363	−0.1221934
26			−0.6355268	−0.2090940
28				−0.6355770

	$N = 63$	$N = 67$	$N = 71$	$N = 75$
	$D = 0.0074430$	$D = 0.0056250$	$D = 0.0042570$	$D = 0.0032330$
0	−0.0055706	−0.0042726	−0.0032801	−0.0025265
2	−0.0044618	−0.0035440	−0.0028140	−0.0022373
4	−0.0062078	−0.0049725	−0.0039850	−0.0031981
6	−0.0083775	−0.0067529	−0.0054510	−0.0044073
8	−0.0110475	−0.0089470	−0.0072626	−0.0059065
10	−0.0143195	−0.0116331	−0.0094805	−0.0077446
12	−0.0183266	−0.0149084	−0.0121819	−0.0099820
14	−0.0232716	−0.0189147	−0.0154612	−0.0126922
16	−0.0294397	−0.0238353	−0.0194595	−0.0159751
18	−0.0373177	−0.0299778	−0.0243626	−0.0199615
20	−0.0477262	−0.0378112	−0.0304737	−0.0248494
22	−0.0622273	−0.0481616	−0.0382687	−0.0309326
24	−0.0841979	−0.0626016	−0.0485700	−0.0386871
26	−0.1224340	−0.0845011	−0.0629480	−0.0489418
28	−0.2092445	−0.1226550	−0.0847785	−0.0632654
30	−0.6356280	−0.2093771	−0.1228583	−0.0850334
32		−0.6356716	−0.2095022	−0.1230441
34			−0.6357140	−0.2096158
36				−0.6357524

Table II—continued

	$N = 79$ $D = 0.0024390$	$N = 83$ $D = 0.0018650$	$N = 87$ $D = 0.0014170$	$N = 91$ $D = 0.0010810$
0	−0.0019358	−0.0015010	−0.0011577	−0.0008961
2	−0.0017746	−0.0014164	−0.0011264	−0.0008980
4	−0.0025624	−0.0020630	−0.0016565	−0.0013331
6	−0.0035600	−0.0028866	−0.0023363	−0.0018946
8	−0.0048021	−0.0039174	−0.0031921	−0.0026054
10	−0.0063300	−0.0051899	−0.0042528	−0.0034906
12	−0.0081910	−0.0067435	−0.0055519	−0.0045785
14	−0.0104453	−0.0086254	−0.0071273	−0.0059016
16	−0.0131630	−0.0108928	−0.0090270	−0.0074976
18	−0.0164470	−0.0136187	−0.0113048	−0.0094123
20	−0.0204251	−0.0168997	−0.0140366	−0.0117008
22	−0.0252943	−0.0208693	−0.0173154	−0.0144355
24	−0.0313515	−0.0257215	−0.0212753	−0.0177116
26	−0.0390711	−0.0317515	−0.0261107	−0.0216618
28	−0.0492818	−0.0394368	−0.0321163	−0.0264799
30	−0.0635544	−0.0496063	−0.0397698	−0.0324618
32	−0.0852651	−0.0638299	−0.0499010	−0.0400852
34	−0.1232135	−0.0854855	−0.0640805	−0.0501796
36	−0.2097186	−0.1233750	−0.0856865	−0.0643163
38	−0.6357869	−0.2098177	−0.1235215	−0.0858747
40		−0.6358204	−0.2099064	−0.1236593
42			−0.6358499	−0.2099905
44				−0.6358782

	$N = 95$ $D = 0.0008240$
0	−0.0006935
2	−0.0007156
4	−0.0010724
6	−0.0015362
8	−0.0021265
10	−0.0028655
12	−0.0037775
14	−0.0048903
16	−0.0062348
18	−0.0078492
20	−0.0097764
22	−0.0120734
24	−0.0148104
26	−0.0180826
28	−0.0220234
30	−0.0268256
32	−0.0327847
34	−0.0403791
36	−0.0504390
38	−0.0645363
40	−0.0860506
42	−0.1237874
44	−0.2100684
46	−0.6359043

Table III—Wideband Hilbert transformers ($F_L = 0.05$, N odd)

	$N = 15$ $D = 0.0475550$	$N = 19$ $D = 0.0227350$	$N = 23$ $D = 0.0110710$	$N = 27$ $D = 0.0054480$
0	−0.0529897	−0.0272752	−0.0144218	−0.0077528
2	−0.0882059	−0.0478756	−0.0272241	−0.0158203
4	−0.1868274	−0.0931810	−0.0525858	−0.0311403
6	−0.6278288	−0.1902395	−0.0971984	−0.0564206
8		−0.6290423	−0.1929460	−0.1004278
10			−0.6299931	−0.1950992
12				−0.6307509

	$N = 31$ $D = 0.0026800$	$N = 35$ $D = 0.0013490$	$N = 39$ $D = 0.0006790$	$N = 43$ $D = 0.0003430$
0	−0.0041956	−0.0023116	−0.0012787	−0.0007098
2	−0.0092821	−0.0054978	−0.0032636	−0.0019390
4	−0.0188358	−0.0115383	−0.0071031	−0.0043804
6	−0.0344010	−0.0214763	−0.0135513	−0.0085912
8	−0.0595516	−0.0371891	−0.0237704	−0.0153599
10	−0.1030376	−0.0621930	−0.0395684	−0.0257859
12	−0.1968315	−0.1052127	−0.0644154	−0.0416246
14	−0.6313536	−0.1982643	−0.1070280	−0.0663138
16		−0.6318550	−0.1994533	−0.1085658
18			−0.6322687	−0.2004560
20				−0.6326171

	$N = 47$ $D = 0.0001730$	$N = 51$ $D = 0.0000880$	$N = 55$ $D = 0.0000450$	$N = 59$ $D = 0.0000230$
0	−0.0003957	−0.0002206	−0.0001243	−0.0000703
2	−0.0011534	−0.0006843	−0.0004080	−0.0002435
4	−0.0027042	−0.0016654	−0.0010288	−0.0006354
6	−0.0054585	−0.0034627	−0.0022014	−0.0013985
8	−0.0099769	−0.0064829	−0.0042215	−0.0027472
10	−0.0170000	−0.0112537	−0.0074746	−0.0049655
12	−0.0275785	−0.0184771	−0.0124599	−0.0084200
14	−0.0434280	−0.0291657	−0.0198467	−0.0135851
16	−0.0679619	−0.0450039	−0.0306162	−0.0211031
18	−0.1098913	−0.0693880	−0.0464284	−0.0319297
20	−0.2013159	−0.1110305	−0.0706666	−0.0477053
22	−0.6329151	−0.2020517	−0.1120458	−0.0718040
24		−0.6331695	−0.2027050	−0.1129440
26			−0.6333949	−0.2032809
28				−0.6335934

measurements of δ as a function of ΔF and N were made, and the results are shown in Figs. 5 and 6. Figure 5 shows a plot of $20 \log_{10} \delta$ as a function of N for values of ΔF of 0.01, 0.02, 0.05, and 0.10, and for even and odd values of N in the range $3 \leqq N \leqq 128$. The curves for even and odd values of N, for fixed transition widths, are almost indistinguishable on the scales of Fig. 5, and hence a single curve is

Table III—continued

	N = 63 D = 0.0000120	N = 67 D = 0.0000060	N = 71 D = 0.0000030	N = 75 D = 0.0000020
0	−0.0000396	−0.0000224	−0.0000127	−0.0000072
2	−0.0001447	−0.0000860	−0.0000512	−0.0000305
4	−0.0003910	−0.0002402	−0.0001476	−0.0000907
6	−0.0008853	−0.0005592	−0.0003531	−0.0002226
8	−0.0017821	−0.0011533	−0.0007456	−0.0004811
10	−0.0032906	−0.0021759	−0.0014366	−0.0009467
12	−0.0056838	−0.0038306	−0.0025777	−0.0017311
14	−0.0093111	−0.0063793	−0.0043663	−0.0029833
16	−0.0146260	−0.0101567	−0.0070545	−0.0048946
18	−0.0222481	−0.0155978	−0.0109630	−0.0077079
20	−0.0331126	−0.0233034	−0.0165113	−0.0117315
22	−0.0488448	−0.0341916	−0.0242838	−0.0173711
24	−0.0728120	−0.0498758	−0.0351847	−0.0251972
26	−0.1137361	−0.0737183	−0.0508176	−0.0361023
28	−0.2037871	−0.1144451	−0.0745415	−0.0516821
30	−0.6337675	−0.2042389	−0.1150864	−0.0752934
32		−0.6339227	−0.2046464	−0.1156699
34			−0.6340625	−0.2050163
36				−0.6341892

	N = 79 D = 0.0000010
0	−0.0000041
2	−0.0000179
4	−0.0000550
6	−0.0001389
8	−0.0003074
10	−0.0006182
12	−0.0011532
14	−0.0020239
16	−0.0033761
18	−0.0053956
20	−0.0083167
22	−0.0124372
24	−0.0181511
26	−0.0260178
28	−0.0369200
30	−0.0524475
32	−0.0759556
34	−0.1161821
36	−0.2053402
38	−0.6343000

given for both.* Based on these curves, it is seen that the larger the transition bandwidth of the Hilbert transformer, the faster the decrease of peak error with increasing N. Thus for $\Delta F = 0.01$, the value of $20 \log_{10} \delta$ decreases by only about 42 dB as N varies from 3 to 128;

* Points on each curve are connected for convenience in plotting.

Table IV—Wideband Hilbert transformers ($F_L = 0.10$, N odd)

	$N = 7$ $D = 0.0515030$	$N = 11$ $D = 0.0111870$	$N = 15$ $D = 0.0025460$	$N = 19$ $D = 0.0005960$
0	−0.1270413	−0.0379718	−0.0125869	−0.0043760
2	−0.6012845	−0.1426690	−0.0517464	−0.0203793
4		−0.6102909	−0.1563345	−0.0622833
6			−0.6159002	−0.1655747
8				−0.6195926

	$N = 23$ $D = 0.0001420$	$N = 27$ $D = 0.0000340$	$N = 31$ $D = 0.0000080$	$N = 35$ $D = 0.0000020$
0	−0.0015643	−0.0005691	−0.0002098	−0.0000778
2	−0.0082383	−0.0033614	−0.0013764	−0.0005626
4	−0.0269557	−0.0119241	−0.0052972	−0.0023445
6	−0.0702312	−0.0324676	−0.0153339	−0.0072495
8	−0.1722057	−0.0764431	−0.0371398	−0.0184388
10	−0.6221851	−0.1771894	−0.0814346	−0.0411255
12		−0.6240992	−0.1810706	−0.0855177
14			−0.6255683	−0.1841662
16				−0.6267261

	$N = 39$ $D = 0.0000010$
0	−0.0000292
2	−0.0002304
4	−0.0010348
6	−0.0034149
8	−0.0091741
10	−0.0212775
12	−0.0445889
14	−0.0889475
16	−0.1867128
18	−0.6276691

whereas for $\Delta F = 0.05$, the value of $20 \log_{10} \delta$ decreases by about 112 dB as N varies from 3 to 76.

Figure 6 shows plots of $20 \log_{10} \delta$ as a function of ΔF for even and odd values of N. The actual values used were $N = 3, 4, 7, 8, 15, 16, 31, 32, 63$, and 64.† As seen in this figure, as ΔF tends to 0, $20 \log_{10} \delta$ tends to 0 dB or a peak error of 1.0, independent of N. It is also seen from Fig. 6 that the larger the value of N, the faster the decrease of peak error with increasing transition width.

From Figs. 5 and 6 it is seen that for a fixed value of δ the product of N and ΔF is approximately a constant. Thus a simple relation of

† The values $N = 3, 7, 15, 31, 63$ were chosen since they satisfy the condition that $(N - 1)/2$ be an even integer.

Table V—Wideband Hilbert transformers
($F_L = 0.02$, N even)

	$N = 28$ $D = 0.0951180$	$N = 30$ $D = 0.0814020$	$N = 32$ $D = 0.0701460$	$N = 34$ $D = 0.0602910$
0	−0.0583712	−0.0503043	−0.0435848	−0.0376976
1	−0.0113151	−0.0095838	−0.0080700	−0.0068443
2	−0.0250915	−0.0221816	−0.0196145	−0.0174179
3	−0.0167301	−0.0141454	−0.0119484	−0.0102336
4	−0.0325317	−0.0285499	−0.0251301	−0.0222922
5	−0.0247101	−0.0206725	−0.0174096	−0.0148471
6	−0.0431589	−0.0372953	−0.0325172	−0.0285854
7	−0.0372340	−0.0305387	−0.0254134	−0.0213430
8	−0.0603244	−0.0504644	−0.0431125	−0.0372679
9	−0.0602139	−0.0470364	−0.0378782	−0.0311972
10	−0.0952250	−0.0740018	−0.0601857	−0.0503464
11	−0.1181848	−0.0814343	−0.0608355	−0.0476850
12	−0.2181622	−0.1322336	−0.0948931	−0.0737466
13	−0.6290021	−0.2041128	−0.1187316	−0.0819751
14		−0.6431801	−0.2177788	−0.1319350
15			−0.6295038	−0.2045900
16				−0.6427692

	$N = 36$ $D = 0.0519110$	$N = 38$ $D = 0.0446700$	$N = 40$ $D = 0.0385670$	$N = 42$ $D = 0.0333770$
0	−0.0326595	−0.0282952	−0.0245988	−0.0214323
1	−0.0058524	−0.0050023	−0.0042702	−0.0036896
2	−0.0154507	−0.0137594	−0.0122671	−0.0109456
3	−0.0087342	−0.0075285	−0.0064963	−0.0055936
4	−0.0197750	−0.0176078	−0.0157127	−0.0140321
5	−0.0126789	−0.0109166	−0.0094489	−0.0081451
6	−0.0252849	−0.0224120	−0.0199636	−0.0178443
7	−0.0181526	−0.0155645	−0.0133761	−0.0115729
8	−0.0325485	−0.0286715	−0.0253906	−0.0225664
9	−0.0260945	−0.0220630	−0.0188546	−0.0161964
10	−0.0430437	−0.0372556	−0.0325934	−0.0287451
11	−0.0385775	−0.0318807	−0.0267466	−0.0227040
12	−0.0600130	−0.0502752	−0.0429692	−0.0372458
13	−0.0614526	−0.0483355	−0.0391895	−0.0324747
14	−0.0946252	−0.0735543	−0.0598556	−0.0501784
15	−0.1192901	−0.0825735	−0.0620113	−0.0488826
16	−0.2174115	−0.1316258	−0.0943740	−0.0733751
17	−0.6299570	−0.2051058	−0.1197944	−0.0830622
18		−0.6423602	−0.2170604	−0.1313650
19			−0.6303873	−0.2055264
20				−0.6420162

the form

$$N\Delta F \approx -\ 0.61 \log_{10} \delta \tag{19}$$

has been suggested by Kaiser as a reasonable approximation to most cases of interest in Figs. 5 and 6. Similar inverse proportionality between filter order N and transition with ΔF was originally noted by Kaiser.[6]

Table V—continued

	$N = 44$ $D = 0.0289000$	$N = 46$ $D = 0.0249860$	$N = 48$ $D = 0.0216400$	$N = 50$ $D = 0.0187460$
0	−0.0186686	−0.0162707	−0.0141858	−0.0123663
1	−0.0031491	−0.0027127	−0.0023498	−0.0020288
2	−0.0097718	−0.0087416	−0.0078077	−0.0069733
3	−0.0048431	−0.0042104	−0.0036580	−0.0031894
4	−0.0125771	−0.0112581	−0.0100901	−0.0090601
5	−0.0070753	−0.0061883	−0.0053834	−0.0047149
6	−0.0159563	−0.0143100	−0.0128560	−0.0115401
7	−0.0100472	−0.0087497	−0.0076664	−0.0067136
8	−0.0201647	−0.0180530	−0.0161867	−0.0145598
9	−0.0139985	−0.0121846	−0.0106353	−0.0093153
10	−0.0254980	−0.0227293	−0.0203292	−0.0182394
11	−0.0194634	−0.0168032	−0.0145949	−0.0127501
12	−0.0326375	−0.0288154	−0.0256046	−0.0228635
13	−0.0273381	−0.0232943	−0.0200391	−0.0173661
14	−0.0429264	−0.0372432	−0.0326723	−0.0288867
15	−0.0397492	−0.0330275	−0.0278909	−0.0238409
16	−0.0597342	−0.0500908	−0.0428880	−0.0372447
17	−0.0625157	−0.0493928	−0.0402628	−0.0335462
18	−0.0941798	−0.0731977	−0.0596122	−0.0500194
19	−0.1202296	−0.0835088	−0.0629931	−0.0498713
20	−0.2167857	−0.1311139	−0.0939759	−0.0730461
21	−0.6307415	−0.2059084	−0.1206478	−0.0839375
22		−0.6416971	−0.2165035	−0.1308803
23			−0.6311019	−0.2062839
24				−0.6413871

As discussed earlier, it has been found that for odd values of N, the peak error is determined primarily by the smaller transition width of the filter; whereas for even values of N, the peak error is determined primarily by the lower transition width. Figures 7 and 8 present data which essentially verify these claims. Figure 7 shows curves of $20 \log_{10} \delta$ as a function of N for three sets of conditions: (*i*) $F_L = 0.02$, $F_H = 0.48$, (*ii*) $F_L = 0.10$, $F_H = 0.48$, (*iii*) $F_L = 0.02$, $F_H = 0.40$ for filters where N is odd. The curves for cases (*ii*) and (*iii*) are indistinguishable on these scales. This figure shows that the maximum differences between the peak errors for these cases is about 3.4 dB for $N = 127$, 1.5 dB for $N = 63$, and 0.7 dB for $N = 31$. Figure 8 shows similar results for even values of N. For this figure the three cases were: (*i*) $F_L = 0.02$, $F_H = 0.48$, (*ii*) $F_L = 0.02$, $F_H = 0.50$, (*iii*) $F_L = 0.02$, $F_H = 0.40$. In this case the curves for cases (*i*) and (*ii*) are indistinguishable. The maximum differences between peak errors are almost identical to the errors for comparable cases when N is odd. These figures substantiate the conclusions stated previously—that the peak ripple is determined primarily by the smaller transition width for N odd, and by the lower transition width for N even.

VI. APPLICATION OF FIR HILBERT TRANSFORMERS

The above discussion suggests that, for direct realizations, the most efficient FIR Hilbert transformer (i.e., using the smallest number of multiplications per sample to obtain a desired value of peak approximation error) has as large a transition bandwidth as possible, and an odd number of impulse response samples. As an example, to obtain a peak error of less than 1 percent ($\delta \leqq 0.01$) requires the following values of N (as a function of ΔF):

ΔF	N (odd)	Number of Multiplications per Sample	N (even)	Number of Multiplications per Sample
0.01	119	30	118	59
0.02	59	15	60	30
0.05	27	7	24	12
0.10	11	3	12	6

whereas to obtain a peak error of less than 0.1 percent ($\delta \leqq 0.001$) requires:

ΔF	N (odd)	Number of Multiplications per Sample	N (even)	Number of Multiplications per Sample
0.01	>127	—	>128	—
0.02	95	24	94	47
0.05	39	10	38	19
0.10	19	5	18	9

The above tables indicate the substantial processing advantages of odd-length Hilbert transformers with symmetrical frequency responses.

Further discussion of the relative merits of even and odd values of N for signal processing applications is given in Ref. 7.

A subset of the Hilbert transformers designed in this study is given in Tables I through VII. These are symmetrical approximations ($F_L = 0.5 - F_H$), with $F_L = 0.01$, 0.02, 0.05, and 0.10 and both even

Table VI—Wideband Hilbert transformers ($F_L = 0.05$, N even)

	$N = 12$	$N = 14$	$N = 16$	$N = 18$
	$D = 0.0877240$	$D = 0.0597790$	$D = 0.0410810$	$D = 0.0283820$
0	−0.0736839	−0.0521805	−0.0373724	−0.0269588
1	−0.0363534	−0.0241948	−0.0165007	−0.0114666
2	−0.0903566	−0.0652477	−0.0485333	−0.0367081
3	−0.0984140	−0.0611815	−0.0408488	−0.0284029
4	−0.2224184	−0.1316266	−0.0902308	−0.0658999
5	−0.6161179	−0.1891560	−0.1026126	−0.0652968
6		−0.6508407	−0.2207546	−0.1308795
7			−0.6192659	−0.1925465
8				−0.6486609

	$N = 20$	$N = 22$	$N = 24$	$N = 26$
	$D = 0.0196660$	$D = 0.0136990$	$D = 0.0095770$	$D = 0.0067230$
0	−0.0195468	−0.0142530	−0.0104407	−0.0076828
1	−0.0080974	−0.0057586	−0.0041350	−0.0029765
2	−0.0280261	−0.0215342	−0.0166115	−0.0128555
3	−0.0202798	−0.0147091	−0.0107983	−0.0079913
4	−0.0496954	−0.0381941	−0.0296899	−0.0232496
5	−0.0447385	−0.0319145	−0.0233406	−0.0173478
6	−0.0902215	−0.0664464	−0.0506653	−0.0394210
7	−0.1060595	−0.0686265	−0.0478081	−0.0346831
8	−0.2194255	−0.1302985	−0.0902502	−0.0669352
9	−0.6218219	−0.1952413	−0.1087422	−0.0712032
10		−0.6469252	−0.2184181	−0.1298971
11			−0.6237772	−0.1973125
12				−0.6456253

	$N = 28$	$N = 30$	$N = 32$	$N = 34$
	$D = 0.0047080$	$D = 0.0033280$	$D = 0.0023350$	$D = 0.0016500$
0	−0.0056474	−0.0041824	−0.0030839	−0.0022879
1	−0.0021606	−0.0015760	−0.0011563	−0.0008479
2	−0.0099450	−0.0077198	−0.0059708	−0.0046343
3	−0.0059617	−0.0044628	−0.0033653	−0.0025388
4	−0.0182731	−0.0144233	−0.0113703	−0.0089951
5	−0.0130602	−0.0098963	−0.0075683	−0.0058006
6	−0.0310326	−0.0246460	−0.0196285	−0.0157095
7	−0.0258302	−0.0195324	−0.0149782	−0.0115559
8	−0.0514404	−0.0404127	−0.0321135	−0.0257649
9	−0.0502551	−0.0369370	−0.0278948	−0.0213933
10	−0.0902847	−0.0673263	−0.0520567	−0.0411844
11	−0.1108639	−0.0732746	−0.0522566	−0.0388270
12	−0.2176401	−0.1295790	−0.0903025	−0.0676060
13	−0.6253137	−0.1989583	−0.1125885	−0.0750001
14		−0.6445903	−0.2170031	−0.1292842
15			−0.6265553	−0.2003315
16				−0.6437163

	$N = 36$ $D = 0.0011710$	$N = 38$ $D = 0.0008310$	$N = 40$ $D = 0.0005910$	$N = 42$ $D = 0.0004190$
0	−0.0017032	−0.0012691	−0.0009465	−0.0007052
1	−0.0006244	−0.0004591	−0.0003397	−0.0002522
2	−0.0036007	−0.0027975	−0.0021716	−0.0016832
3	−0.0019213	−0.0014545	−0.0011051	−0.0008426
4	−0.0071254	−0.0056461	−0.0044715	−0.0035365
5	−0.0044616	−0.0034385	−0.0026575	−0.0020612
6	−0.0126059	−0.0101283	−0.0081395	−0.0065367
7	−0.0089661	−0.0069853	−0.0054622	−0.0042881
8	−0.0207937	−0.0168426	−0.0136681	−0.0110982
9	−0.0165912	−0.0129692	−0.0102003	−0.0080657
10	−0.0330465	−0.0267559	−0.0217832	−0.0177909
11	−0.0295922	−0.0229337	−0.0179839	−0.0142292
12	−0.0526097	−0.0418905	−0.0338324	−0.0275664
13	−0.0538732	−0.0403563	−0.0310313	−0.0242859
14	−0.0903686	−0.0679104	−0.0530752	−0.0424512
15	−0.1139494	−0.0763652	−0.0552251	−0.0416870
16	−0.2165425	−0.1291214	−0.0904306	−0.0681302
17	−0.6275161	−0.2013870	−0.1150770	−0.0775488
18		−0.6430778	−0.2161671	−0.1289480
19			−0.6283058	−0.2023087
20				−0.6425071

	$N = 44$ $D = 0.0002970$	$N = 46$ $D = 0.0002110$	$N = 48$ $D = 0.0001500$	$N = 50$ $D = 0.0001070$
0	−0.0005261	−0.0003931	−0.0002935	−0.0002189
1	−0.0001872	−0.0001391	−0.0001037	−0.0000773
2	−0.0013044	−0.0010110	−0.0007823	−0.0006044
3	−0.0006418	−0.0004892	−0.0003738	−0.0002860
4	−0.0027965	−0.0022113	−0.0017457	−0.0013761
5	−0.0015986	−0.0012405	−0.0009647	−0.0007512
6	−0.0052500	−0.0042167	−0.0033821	−0.0027090
7	−0.0033696	−0.0026507	−0.0020894	−0.0016495
8	−0.0090193	−0.0073334	−0.0059575	−0.0048348
9	−0.0063942	−0.0050794	−0.0040451	−0.0032278
10	−0.0145676	−0.0119479	−0.0098009	−0.0080374
11	−0.0113197	−0.0090412	−0.0072487	−0.0058283
12	−0.0226020	−0.0186095	−0.0153542	−0.0126823
13	−0.0192248	−0.0153413	−0.0123211	−0.0099429
14	−0.0344657	−0.0282591	−0.0233103	−0.0193023
15	−0.0323002	−0.0254597	−0.0203150	−0.0163525
16	−0.0534295	−0.0429264	−0.0350095	−0.0288317
17	−0.0564114	−0.0428283	−0.0333997	−0.0265154
18	−0.0904417	−0.0683159	−0.0537337	−0.0433047
19	−0.1160705	−0.0785556	−0.0574287	−0.0438482
20	−0.2158094	−0.1288020	−0.0904537	−0.0684414
21	−0.6290120	−0.2030878	−0.1169158	−0.0794554
22		−0.6420267	−0.2155088	−0.1286420
23			−0.6296096	−0.2037902
24				−0.6415800

Table VII—Wideband Hilbert transformers ($F_L = 0.10$, N even)

	$N = 6$, $D = 0.0908340$	$N = 8$, $D = 0.0409350$	$N = 10$, $D = 0.0188980$	$N = 12$, $D = 0.0088710$
0	−0.1291026	−0.0685959	−0.0381031	−0.0217187
1	−0.1482639	−0.0642207	−0.0320979	−0.0171366
2	−0.6651173	−0.2171077	−0.1158537	−0.0691363
3		−0.5959958	−0.1657805	−0.0792088
4			−0.6563900	−0.2154404
5				−0.6069062

	$N = 14$, $D = 0.0042080$	$N = 16$, $D = 0.0020170$	$N = 18$, $D = 0.0009720$	$N = 20$, $D = 0.0004710$
0	−0.0125849	−0.0073860	−0.0043698	−0.0026042
1	−0.0095049	−0.0053963	−0.0031146	−0.0018192
2	−0.0429910	−0.0272417	−0.0174135	−0.0111820
3	−0.0436051	−0.0254919	−0.0153794	−0.0094478
4	−0.1183342	−0.0735061	−0.0478330	−0.0317741
5	−0.1765512	−0.0886565	−0.0513442	−0.0315749
6	−0.6513727	−0.2145760	−0.1200051	−0.0764602
7		−0.6134417	−0.1833481	−0.0950101
8			−0.6483398	−0.2140796
9				−0.6176569

	$N = 22$, $D = 0.0002300$	$N = 24$, $D = 0.0001120$	$N = 26$, $D = 0.0000550$	$N = 28$, $D = 0.0000270$
0	−0.0015625	−0.0009411	−0.0005678	−0.0003437
1	−0.0010738	−0.0006373	−0.0003815	−0.0002288
2	−0.0072008	−0.0046408	−0.0029868	−0.0019225
3	−0.0058736	−0.0036749	−0.0023155	−0.0014609
4	−0.0213129	−0.0143562	−0.0096696	−0.0065138
5	−0.0200200	−0.0129010	−0.0084074	−0.0055027
6	−0.0512512	−0.0351492	−0.0243353	−0.0169296
7	−0.0569012	−0.0362082	−0.0237777	−0.0158604
8	−0.1211777	−0.0785910	−0.0537606	−0.0377068
9	−0.1879880	−0.0995599	−0.0611011	−0.0398850
10	−0.6463321	−0.2137791	−0.1220080	−0.0801285
11		−0.6205691	−0.1913704	−0.1030239
12			−0.6448821	−0.2135097
13				−0.6227594

and odd values of N from 3 to 95.* The peak approximation error in the band $2\pi F_L \leqq \omega \leqq 2\pi F_H$, denoted D, is given for each case as well as the impulse response of the filter. Note that only the first half of the impulse response is given in the table; the last half can be obtained using eq. (11). When N is odd, only the even-indexed samples are given.

* Only those Hilbert transformers for which $\delta < 0.1$ are given in these tables.

Table VII—continued

	$N = 30$ $D = 0.0000130$	$N = 32$ $D = 0.0000070$	$N = 34$ $D = 0.0000030$	$N = 36$ $D = 0.0000020$
0	−0.0002100	−0.0001278	−0.0000778	−0.0000474
1	−0.0001373	−0.0000832	−0.0000508	−0.0000309
2	−0.0012437	−0.0007997	−0.0005133	−0.0003291
3	−0.0009210	−0.0005851	−0.0003730	−0.0002375
4	−0.0044021	−0.0029567	−0.0019805	−0.0013237
5	−0.0036033	−0.0023756	−0.0015700	−0.0010367
6	−0.0118308	−0.0082330	−0.0057153	−0.0039588
7	−0.0106521	−0.0072175	−0.0049095	−0.0033408
8	−0.0268217	−0.0191244	−0.0136464	−0.0097328
9	−0.0268094	−0.0183719	−0.0127195	−0.0088460
10	−0.0557869	−0.0398133	−0.0287628	−0.0209019
11	−0.0642873	−0.0428072	−0.0294506	−0.0206098
12	−0.1227568	−0.0814133	−0.0572528	−0.0414122
13	−0.1938141	−0.1056496	−0.0669934	−0.0453215
14	−0.6439281	−0.2134097	−0.1231647	−0.0822898
15		−0.6243396	−0.1959100	−0.1078912
16			−0.6429969	−0.2131694
17				−0.6257461

	$N = 38$ $D = 0.0000010$
0	−0.0000291
1	−0.0000187
2	−0.0002118
3	−0.0001500
4	−0.0008867
5	−0.0006795
6	−0.0027458
7	−0.0022607
8	−0.0069528
9	−0.0061396
10	−0.0152613
11	−0.0145127
12	−0.0304415
13	−0.0316100
14	−0.0585324
15	−0.0691050
16	−0.1236067
17	−0.1974694
18	−0.6423832

VII. ACKNOWLEDGMENT

The authors would like to acknowledge the valuable comments and criticisms provided by J. F. Kaiser.

REFERENCES

1. B. Gold, A. V. Oppenheim, and C. M. Rader, "Theory and Implementation of the Discrete Hilbert Transform," Proc. Symp. Comput. Process. Comm., Polytechnic Press, 1970, pp. 235–250.

2. M. R. Schroeder, J. L. Flanagan, and E. A. Lundry, "Bandwidth Compression of Speech by Analytic-Signal Rooting," Proc. IEEE, *55*, No. 3 (March 1967), pp. 396–401.
3. O. Herrmann, "Transversal Filters for Hilbert Transformation," Arch. Elek. Ubertr., *23*, No. 12 (December 1969), pp. 581–587.
4. J. H. McClellan, T. W. Parks, and L. R. Rabiner, "A Computer Program for Designing Optimum FIR Linear Phase Digital Filters," IEEE Trans. on Audio and Electroacoustics, *AU-21*, No. 6 (December 1973), pp. 506–526.
5. A. J. Gibbs, "On the Frequency-Domain Responses of Causal Digital Filters," Ph.D. Thesis, Dept. of Electrical Engineering, University of Wisconsin, 1969.
6. J. F. Kaiser, "Digital Filters," chapter 7 in *System Analysis by Digital Computer*, edited by F. F. Kuo and J. F. Kaiser, New York: John Wiley and Sons, 1966.
7. L. R. Rabiner and R. W. Schafer, "On the Behavior of Minimax Relative Error FIR Digital Differentiators," B.S.T.J., this issue, pp. 333–361.