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Some Properties of the Erlang Loss Function

By D. L. JAGERMAN

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This paper develops the properties of the Erlang loss function, $B(N, a)$, used in telephone traffic engineering. The extension to a transcendental function of two complex variables is constructed, thus permitting the methods of complex analysis to be employed for the further study of its properties. Exact representations, Rodrigues formulas, and addition theorems are given both for the loss function and for the related Poisson-Charlier polynomials. Asymptotic formulas and approximations are developed for the loss function and also for its derivatives. A table of coefficients is included which, together with one of the asymptotic formulas, permits computation of $B(N, a)$ by simple means even when the number of trunks, N, is very large. This same table is used to obtain $\partial B(x, a)/\partial x$.

I. INTRODUCTION

The Erlang loss function

$$B(N, a) = \frac{a^N}{N!} \Big/ \sum_{j=0}^{N} \frac{a^j}{j!} \tag{1}$$

is fundamental to the study of telephone trunking problems. A. K. Erlang[1] used $B(N, a)$ to express the probability that a call, which is a member of a Poisson stream of parameter a, arriving at a group of N telephone trunks will be rejected. Later studies of trunking problems have shown the desirability of enlarging the scope of applications of the loss function. For example, the consideration of trunk groups with nonintegral number of trunks arises in determining the equivalent number of trunks in Wilkinson's "equivalent random method."[2] Methods for accomplishing the computation by interpolation are given by Rapp[3] while continued fraction procedures for accurate computation are given by Levy-Soussan[4] and Burke.[5] Derivatives with respect to N and a arise in optimal trunk group size apportionment problems. See, for example, Akimaru and Nishimura[6,7] who studied such models and prepared tables of derivatives. In some investigations, rapid and accurate approximate computations of $B(N, a)$ for very large trunk groups are needed. This occurred in the study of certain satellite telephonic communication problems.[8,9] The need thus arises of enlarging the definition of $B(N, a)$ as given in (1). Of course, that is done implicitly in the above investigations. It has been customary to extend the definition of $B(N, a)$ by use of an integral formula (Theorem 3) ascribed to Fortet. This integral formula is used in (23) to define a transcendent, $B(z, \alpha)$, for complex z and α. The extension to the complex plane in both z and α permits the powerful methods of complex analysis to be applied for obtaining exact, asymptotic, and approximate representations.

It is the purpose of this paper to provide an investigation into the properties of $B(z, \alpha)$ with the object of generalizing known results, obtaining new results, and presenting practical methods for application to the class of problems outlined above.

Part II derives exact relations satisfied by $B(z, \alpha)$. Similar relations for the related Poisson-Charlier polynomials, $G_j(z, \alpha)$, are derived in the appendix. These relations provide efficient means for exact computation; thus, Theorems 1 and 2 constitute a practical method of computing $B(N, a)$ to a prescribed accuracy for isolated computations. Similarly, the use of Theorem 5 enables one to compute $B(z, \alpha)$ even for nonintegral number of trunks. Theorem 6 may be similarly employed. The relationship of $B(z, \alpha)$ and $G_j(z, \alpha)$ to Whittaker functions as given in Theorems 7 and 24 is the key for linking up these functions with the more well-known functions of applied mathematics, i.e., hypergeometric functions and Laguerre polynomials. The Rodrigues Theorems 8 and 22 are useful for the evaluation of integrals of the form

$$\int f^{(r)}(a)a^{-1}e^{-a}B(N, a)^{-1}da, \qquad \int f^{(r)}(a)a^{z}e^{-a}G_j(z, a)da \tag{2}$$

and, as in the case of Theorem 22, for obtaining an integral representation. The addition Theorems 9, 10, 26, and 27 yield group-theoretic structure information which is useful for simplifying formulas containing these functions, and for the evaluation of integrals. The evaluation of an integral, by means of generating functions and Theorem 10, was done in Part IV to obtain ultimately an approximate formula for $\partial B(x, a)/\partial x$. A general use of the exact relations is to serve as a springboard for asymptotic and approximate results and also for their error estimations. This is well illustrated in Part III of the paper.

The asymptotic expansions of Part III are also representations of $B(z, \alpha)$ but, unlike those of Part II, when used as approximate formulas for computation they cannot yield results of arbitrarily high accuracy, i.e., the accuracy depends on specific values of parameters. Theorem 11 is particularly useful for computation when $|z/\alpha|$ is small. It may be used for the computation of $B(z,\alpha)$ for fractional number of trunks by computing $B(z, \alpha)$ for $0 < z < 1$ and then using the recurrence formula of Theorem 4. Theorem 11 includes well-known asymptotic results, e.g.,

$$B\left(-\frac{1}{2}, a\right)^{-1} = \sqrt{\pi a} e^{a}(1 - \operatorname{erf}\sqrt{a}) \sim 1 - \frac{1}{2} a^{-1} + \frac{1.3}{2^2} a^{-2} - \frac{1.3.5}{2^3} a^{-3} + \cdots, \qquad a \to \infty, \tag{3}$$

$$B(-1, a)^{-1} = -a e^{a} E_i(-a) \sim 1 - a^{-1} + 2!\, a^{-2} - 3!\, a^{-3} + \cdots, \qquad a \to \infty.$$

An undesirable feature of many methods of computing $B(x, a)$ is the dependence of the computational effort, e.g., time of computation, on the value of x; thus, the larger the value of x the greater the computational effort. Theorem 14 overcomes this defect; the computational effort is independent of the size of x. Theorem 14 is easily usable even with a desk machine regardless of how large x is. The accuracy, however, depends on x and a parameter c. For fixed c the accuracy improves with increasing x. When x is fixed, the accuracy deteriorates when c is large and negative but greatly improves as c is increased. To facilitate the use of Theorem 14, Table I gives required coefficients, namely, $a_0(c)$, $a_1(c)$, $a_2(c)$. To use the table, one computes

$$c = \frac{a - x}{\sqrt{x}}, \tag{4}$$

then

$$B(x, a)^{-1} \cong a_0(c)\sqrt{x} + a_1(c) + \frac{a_2(c)}{\sqrt{x}}. \tag{5}$$

Possibly, one should comment that the range of values of x, c for which (5) is accurate is not as important as the fact that it is accurate over a wide range of values of $B(x, a)$, that is, values encompassing the ranges of most applications. For quantitative limitations, see Fig. 1. A method of obtaining $\partial B(x, a)/\partial x$ based on Theorem 14 is given in Part IV. This uses the formula

$$\frac{\partial B(x, a)}{\partial x} \cong -\frac{B(x, a)^2}{2\sqrt{x}}\left(a_0 - \frac{a_2}{x}\right) - \frac{x + a}{2x} B(x, a)\left\{\frac{x}{a} - 1 + B(x, a)\right\}. \tag{6}$$

Table I — Coefficients for evaluation of B(x, a) and ∂B(x, a)/∂x

c*	a_0	a_1	a_2	c*	a_0	a_1	a_2
−3.0	225.3	2032	13726	0.6	0.8230	0.7274	0.1011
−2.9	167.7	1367	8536	0.7	0.7749	0.7414	0.0985
−2.8	126.0	925.4	5334	0.8	0.7313	0.7552	0.0954
−2.7	95.63	630.5	3348	0.9	0.6917	0.7686	0.0920
−2.6	73.28	432.2	2111	1.0	0.6557	0.7814	0.0883
−2.5	56.70	298.0	1336	1.1	0.6227	0.7937	0.0845
−2.4	44.29	206.7	848.1	1.2	0.5926	0.8053	0.0806
−2.3	34.92	144.1	540.2	1.3	0.5649	0.8163	0.0767
−2.2	27.80	100.9	345.0	1.4	0.5394	0.8267	0.0729
−2.1	22.33	71.07	220.7	1.5	0.5158	0.8364	0.0691
−2.0	18.10	50.27	141.4	1.6	0.4940	0.8455	0.0654
−1.9	14.80	35.71	90.70	1.7	0.4739	0.8540	0.0619
−1.8	12.21	25.49	58.17	1.8	0.4551	0.8619	0.0585
−1.7	10.16	18.27	37.28	1.9	0.4376	0.8694	0.0552
−1.6	8.521	13.15	23.86	2.0	0.4214	0.8763	0.0521
−1.5	7.205	9.522	15.23	2.1	0.4062	0.8828	0.0492
−1.4	6.139	6.936	9.692	2.2	0.3919	0.8889	0.0464
−1.3	5.271	5.090	6.141	2.3	0.3786	0.8946	0.0438
−1.2	4.557	3.772	3.872	2.4	0.3661	0.8999	0.0413
−1.1	3.968	2.830	2.430	2.5	0.3543	0.9049	0.0390
−1.0	3.477	2.159	1.519	2.6	0.3432	0.9095	0.0368
−0.9	3.066	1.682	0.9486	2.7	0.3327	0.9139	0.0347
−0.8	2.721	1.344	0.5960	2.8	0.3228	0.9179	0.0328
−0.7	2.428	1.108	0.3816	2.9	0.3134	0.9218	0.0309
−0.6	2.178	0.9435	0.2540	3.0	0.3046	0.9254	0.0292
−0.5	1.964	0.8318	0.1804	3.1	0.2962	0.9287	0.0276
−0.4	1.780	0.7580	0.1398	3.2	0.2882	0.9319	0.0261
−0.3	1.620	0.7112	0.1187	3.3	0.2806	0.9349	0.0247
−0.2	1.481	0.6840	0.1089	3.4	0.2734	0.9377	0.0234
−0.1	1.360	0.6705	0.1052	3.5	0.2666	0.9403	0.0222
0	1.253	0.6667	0.1044	3.6	0.2600	0.9428	0.0210
0.1	1.159	0.6696	0.1048	3.7	0.2538	0.9451	0.0199
0.2	1.076	0.6771	0.1052	3.8	0.2478	0.9473	0.0189
0.3	1.002	0.6877	0.1052	3.9	0.2421	0.9494	0.0179
0.4	0.9357	0.7000	0.1045	4.0	0.2367	0.9514	0.0170
0.5	0.8764	0.7135	0.1031				

Calculate $c = \dfrac{a - x}{\sqrt{x}}$, then

$$B(x, a)^{-1} \simeq a_0\sqrt{x} + a_1 + a_2/\sqrt{x},$$

$$\frac{\partial B(x, a)}{\partial x} \simeq -\frac{B(x, a)^2}{2\sqrt{x}}\left(a_0 - \frac{a_2}{x}\right) - \frac{x + a}{2x} B(x, a)\left\{\frac{x}{a} - 1 + B(x, a)\right\}.$$

* Standardized offered load.

It is appropriate to mention, at this point, another method of approximating $B(x, a)$ by means of a formula whose computational effort is also independent of x and which, similarly, is applicable over a wide

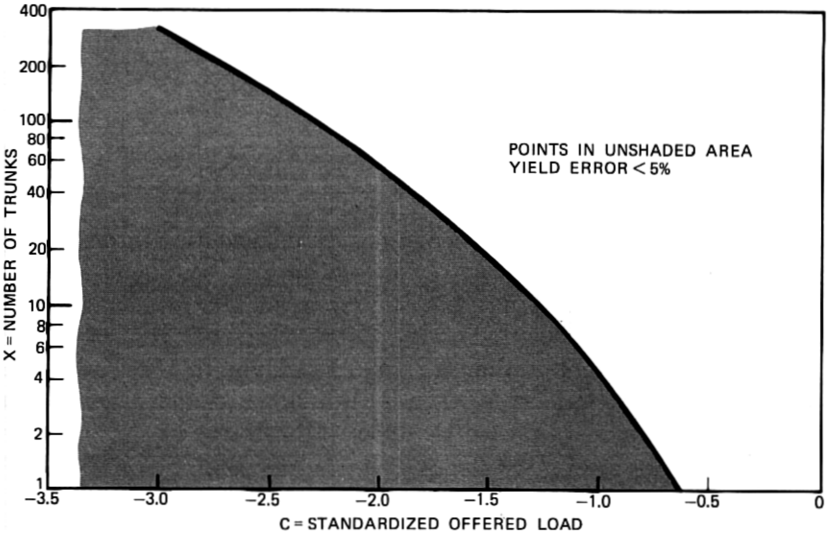

Fig. 1—Five-percent-error contour.

range of values of $B(x, a)$. This method is described in Refs. 10 and 11. A comparison of this method with that of Theorems 1 and 2 is given at length in a report by S. Miller.[9]

Derivatives and inequalities on derivatives are given in Part IV. Theorem 15 extends the well-known derivative formula for $B(x, a)$ with respect to the real variable a. Theorem 17 provides an accurate approximation for $\partial B(x, a)/\partial x$. Empirically, the accuracy seems to hold to four significant figures or better over a very wide range of values of x and a. Of significance is the corollary which shows that the approximate value obtained is always too small. If a quick appraisal of the derivative is desired, Theorem 18 may be used. The logarithmic convexity properties of $B(x, a)$ given in Theorem 19 provide the useful bounds of the corollary on the second derivatives. Also an application is given to the logarithmic interpolation of Theorem 20. This is very useful when, for example, one wishes to compute $B(x, a)$ for x between consecutive integers, say N, $N + 1$, and for which $B(N, a)$, $B(N + 1, a)$ are known. An extension of this idea is provided by Theorem 21 which permits accurate computation of $B(x, a)$.

It may be remarked that generally relations, representations, and asymptotics for $B(z, \alpha)^{-1}$ are simpler in structure than those for $B(z, \alpha)$ and may provide greater numerical accuracy in computations.

II. CONVERGENT REPRESENTATIONS

The study of telephone trunking problems, whether in equilibrium or transient condition, or even nonstationary,[12] engenders the Erlang loss function, $B(N, a)$, which initially arises in the form[13]

$$B(N, a) = \frac{a^N}{N!} \Big/ \sum_{j=0}^{N} \frac{a^j}{j!}, \qquad N \geqq 0 \text{ (integral)}, \quad a > 0. \tag{7}$$

For these reasons and for the purposes of studying certain forms arising in queuing theory related to $B(N, a)$ and also for the facilitation of numerical evaluation, it is useful to represent the loss function in diverse ways.

The numerical computation of $B(N, a)$ as given in (7) is awkward when a and N are large since then both numerator and denominator are large. A form well adapted to numerical work is

$$\begin{gathered} B(N, a)^{-1} = \sum_{j=0}^{N} N^{(j)} a^{-j}, \\ N^{(0)} = 1, \qquad N^{(j)} = N(N-1)\cdots(N-j+1) \quad (j > 0), \end{gathered} \tag{8}$$

which follows from

$$B(N, a)^{-1} = \sum_{j=0}^{N} \frac{N!}{j!} a^{j-N} = \sum_{j=0}^{N} \frac{N!}{(N-j)!} a^{-j} = \sum_{j=0}^{N} N^{(j)} a^{-j}. \tag{9}$$

A modified form of (8) is given in Theorem 1.

Theorem 1:

$$B(N, a)^{-1} = \sum_{j=0}^{\nu-1} N^{(j)} a^{-j} + N^{(\nu)} a^{-\nu} B(N-\nu, a)^{-1}, \qquad \nu \geqq 0.$$

Proof: Since

$$N^{(j+\nu)} = N^{(\nu)} (N-\nu)^{(j)}, \tag{10}$$

one has, from

$$B(N, a)^{-1} = \sum_{j=0}^{\nu-1} N^{(j)} a^{-j} + \sum_{j=\nu}^{N} N^{(j)} a^{-j}, \tag{11}$$

$$\begin{aligned} \sum_{j=\nu}^{N} N^{(j)} a^{-j} &= \sum_{j=0}^{N-\nu} N^{(j+\nu)} a^{-j-\nu} = N^{(\nu)} a^{-\nu} \sum_{j=0}^{N-\nu} (N-\nu)^{(j)} a^{-j} \\ &= N^{(\nu)} a^{-\nu} B(N-\nu, a)^{-1}. \end{aligned} \tag{12}$$

The formula of the theorem follows from (11) and (12).

Corollary: The case $\nu = 1$ implies the known[14] *difference equation*

$$B(N, a) = \frac{1}{1 + \dfrac{N}{aB(N-1, a)}}.$$

R. Franks suggested using the value of $\tilde{B}_\nu(N, a)$ defined by

$$\tilde{B}_\nu(N, a) = 1 \Big/ \sum_{j=0}^{\nu-1} N^{(j)} a^{-j} \tag{13}$$

to approximate $B(N, a)$ in which, for any small number $\eta > 0$, the index ν is chosen so that

$$N^{(\nu)} a^{-\nu} \leqq \eta. \tag{14}$$

Theorem 2 bounds the error of the method.

Theorem 2:

$$\tilde{B}_\nu(N, a)(1 - \eta) \leqq B(N, a) \leqq \tilde{B}_\nu(N, a).$$

Proof: From Theorem 1 one has

$$\frac{1}{\tilde{B}_\nu(N, a)^{-1} + N^{(\nu)} a^{-\nu} B(N - \nu, a)^{-1}} = B(N, a) \leqq \tilde{B}_\nu(N, a). \tag{15}$$

Thus

$$\frac{B(N, a)}{\tilde{B}_\nu(N, a)} = \frac{1}{1 + N^{(\nu)} a^{-\nu} \tilde{B}_\nu(N, a) B(N - \nu, a)^{-1}}. \tag{16}$$

Since $N^{(\nu)} a^{-\nu}$ is strictly monotone increasing as a function of N, (8) shows that

$$B(N + 1, a) < B(N, a) \tag{17}$$

for all $N \geqq 0$; thus

$$\frac{B(N, a)}{\tilde{B}_\nu(N, a)} \geqq \frac{1}{1 + \eta \tilde{B}_\nu(N, a) B(N, a)^{-1}}, \tag{18}$$

and hence

$$\frac{B(N, a)}{\tilde{B}_\nu(N, a)} \geqq 1 - \eta. \tag{19}$$

The theorem follows from (15) and (19).

An integral representation, ascribed to Fortet,[15] may be obtained for $B(N, a)$.

Theorem 3:

$$B(N, a)^{-1} = a \int_0^\infty e^{-ay} (1 + y)^N dy.$$

Proof: From the Eulerian integral

$$\int_0^\infty e^{-ay} y^l dy = \Gamma(l+1) a^{-l-1}, \qquad l > -1, \tag{20}$$

one obtains

$$N^{(j)} a^{-j} = a \binom{N}{j} \int_0^\infty e^{-ay} y^j dy. \tag{21}$$

Use of (8) now yields

$$B(N, a)^{-1} = a \int_0^\infty e^{-ay} \sum_{j=0}^{N} \binom{N}{j} y^j dy = a \int_0^\infty e^{-ay} (1+y)^N dy. \tag{22}$$

The integral representation now permits extending $B(N, a)$ into the complex plane with respect to both N and a. One defines

$$B(z, \alpha)^{-1} = \alpha \int_0^\infty e^{-\alpha y} (1+y)^z dy \tag{23}$$

in which z, α may both be complex. Clearly, $B(z, \alpha)^{-1}$ is an entire function of z for $\operatorname{Re} \alpha > 0$ (Re designates "real part"). The symbols N, a will be used for nonnegative integers and positive reals, respectively.

A generalization of Theorem 1 is given in Theorem 4.

Theorem 4:

$$B(z, \alpha)^{-1} = \sum_{j=0}^{\nu-1} z^{(j)} \alpha^{-j} + z^{(\nu)} \alpha^{-\nu} B(z-\nu, \alpha)^{-1}, \qquad \operatorname{Re} \alpha > 0.$$

Proof: Integration by parts of (23).

It is of interest to investigate the relationship of $B(z, \alpha)$ to the function

$$\psi(z, \alpha) = e^{-\alpha} \frac{\alpha^z}{\Gamma(z+1)}, \tag{24}$$

which is an extension of the Poisson distribution function, $\psi(N, a)$, with parameter a. The function $\psi(N, a)$ is a good approximation to $B(N, a)$ when a is much less than N. Exact relations between $B(z, \alpha)$ and $\psi(z, \alpha)$ are given in Theorems 5 and 6. These relations provide convenient means of calculation of $B(z, \alpha)$ for general z, α; e.g., in trunk group blocking problems when a nonintegral number of trunks is considered.

Theorem 5:

$$B(z, \alpha)^{-1} = \psi(z, \alpha)^{-1} - \sum_{s=1}^{\infty} \frac{\alpha^S}{(z+1)\cdots(z+s)}.$$

The series converges uniformly everywhere in Re $z > -1$, Re $\alpha > 0$.

Proof: Let $u = 1 + y$ in (23), then

$$B(z, \alpha)^{-1} = \alpha e^{\alpha} \int_1^{\infty} e^{-\alpha u} u^z du; \tag{25}$$

hence,

$$B(z, \alpha)^{-1} = \alpha e^{\alpha} \int_0^{\infty} e^{-\alpha u} u^z du - \alpha \int_0^1 e^{\alpha(1-u)} u^z du, \tag{26}$$

and

$$B(z, \alpha)^{-1} = \psi(z, \alpha)^{-1} - \alpha \int_0^1 e^{\alpha(1-u)} u^z du. \tag{27}$$

To exhibit the integral in (27) as an inverse factorial series, consider the beta function integral

$$\int_0^1 u^{x-1}(1 - u)^{y-1} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \tag{28}$$

which yields the special case ($s \geqq 0$ integral)

$$\int_0^1 u^z(1 - u)^s du = \frac{s!}{(z + 1)\cdots(z + 1 + s)}. \tag{29}$$

Use of the expansion

$$e^{\alpha(1-u)} = \sum_{s=0}^{\infty} \frac{\alpha^s}{s!} (1 - u)^s \tag{30}$$

in (27) and subsequent use of (29) yield the result of the theorem.

The Mittag-Leffler expansion for the integral of (27) leads to

Theorem 6:

$$B(z, \alpha)^{-1} = \psi(z, \alpha)^{-1} + e^{\alpha} \sum_{s=1}^{\infty} (-1)^s \frac{\alpha^s}{(s - 1)!(s + z)}.$$

Conditions of convergence are the same as in Theorem 5.

Proof: The expansion

$$e^{-\alpha u} = \sum_{s=0}^{\infty} (-1)^s \frac{\alpha^s u^s}{s!} \tag{31}$$

used in (27) leads immediately to the required result.

Whittaker functions,[16] $W_{k,m}(z)$, play a useful role in the discussion of $B(z, \alpha)$ and of Poisson-Charlier polynomials to be introduced later.

They may be introduced by

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty e^{-t}\left(1 + \frac{t}{z}\right)^{k-\frac{1}{2}+m} t^{-k-\frac{1}{2}+m}dt. \quad (32)$$

$$\operatorname{Re}\left(\tfrac{1}{2} - k + m\right) > 0, \qquad |\arg z| < \pi.$$

Theorem 7:

$$B(z, \alpha)^{-1} = e^{-z/2}e^{\frac{1}{2}\alpha}W_{z/2,(z+1)/2}(\alpha).$$

Proof: Let $t = \alpha y$ in (23), then

$$B(z, \alpha)^{-1} = \int_0^\infty e^{-t}\left(1 + \frac{t}{\alpha}\right)^z dt. \quad (33)$$

The required result follows on comparison with (32).

A Rodrigues type of relation for $B(N, a)^{-1}$ may be obtained from Theorem 3.

Theorem 8:

$$B(N + M, a)^{-1} = (-1)^M ae^a \frac{d^M}{da^M}[e^{-a}a^{-1}B(N, a)^{-1}].$$

Proof: From Theorem 3, one has

$$e^{-a}a^{-1}B(N, a)^{-1} = \int_0^\infty e^{-a(1+y)}(1 + y)^N dy; \quad (34)$$

hence,

$$(-1)^M \frac{d^M}{da^M}[e^{-a}a^{-1}B(N, a)^{-1}] = \int_0^\infty e^{-a(1+y)}(1 + y)^{N+M}dy. \quad (35)$$

The formula for $B(N + M, a)^{-1}$ now follows on multiplication by ae^a.

Corollary:

$$B(N, a)^{-1} = (-1)^N ae^a \frac{d^N}{da^N}[e^{-a}a^{-1}].$$

Additional formulas for $B(z, \alpha)^{-1}$ (as a function of α) provide convenient means of computation for values of α near some fixed point. Two such formulas are given in Theorems 9 and 10.

Theorem 9:

$$B(n, \alpha + t)^{-1} = \left(1 + \frac{t}{\alpha}\right)^{-n} \sum_{\nu=0}^{n} \binom{n}{\nu} B(n - \nu, \alpha)^{-1}\left(\frac{t}{\alpha}\right)^\nu.$$

Proof: The function $S_n(\alpha)$ given by

$$S_n(\alpha) = \sum_{\tau=0}^{n} \frac{\alpha^\tau}{\tau!} \tag{36}$$

is an Appell polynomial, that is,

$$\frac{dS_n(\alpha)}{d\alpha} = S_{n-1}(\alpha). \tag{37}$$

Thus the Taylor expansion for $S_n(\alpha + t)$ can be written in the form

$$S_n(\alpha + t) = \sum_{\nu=0}^{n} \frac{1}{\nu!} S_{n-\nu}(\alpha) t^\nu. \tag{38}$$

One obtains from (7)

$$B(n,\alpha)^{-1} = n!\,\alpha^{-n} S_n(\alpha) \tag{39}$$

and hence the theorem follows on substitution into (38).

Theorem 10:

$$B(z, \alpha + t)^{-1} = \left(1 + \frac{t}{\alpha}\right) e^t \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} B(z + \nu, \alpha)^{-1} t^\nu,$$
$$\operatorname{Re}\alpha > 0, \qquad |t| < \operatorname{Re}\alpha.$$

Proof: Let

$$l(z, \alpha) = e^{iz\pi - \alpha} \alpha^{-1} B(z, \alpha)^{-1}, \tag{40}$$

then, from (23),

$$\begin{aligned} \frac{d}{d\alpha} l(z, \alpha) &= \frac{d}{d\alpha} e^{iz\pi} \int_0^\infty e^{-\alpha(1+y)} (1 + y)^z dy, \\ &= -\, e^{iz\pi} \int_0^\infty e^{-\alpha(1+y)} (1 + y)^{z+1} dy, \\ &= l(z + 1, \alpha), \end{aligned} \tag{41}$$

and hence

$$\frac{d^\nu}{d\alpha^\nu} l(z, \alpha) = l(z + \nu, \alpha). \tag{42}$$

Thus, by Taylor's formula,

$$l(z, \alpha + t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} l(z + \nu, \alpha). \tag{43}$$

Substitution of (40) into (43) yields the required result. One has

$$l(z + \nu, \alpha) = e^{i(z+\nu)\pi} \int_0^\infty e^{-\alpha(1+y)} (1 + y)^{z+\nu} dy, \tag{44}$$

hence the terms of (43) are $0[(t/\mathrm{Re}\,\alpha)^{\nu}\nu^{\mathrm{Re}\,z}]$. The stated convergence criterion now follows.

III. ASYMPTOTIC EXPANSIONS

Particularly simple and convenient forms for theoretical and numerical applications may be obtained by examining asymptotic expansions.

Theorem 11:

$$B(z, \alpha)^{-1} \sim \sum_{\nu=0}^{\infty} z^{(\nu)}\alpha^{-\nu}, \qquad \alpha \to \infty, \quad |\arg \alpha| < \pi.$$

Proof: The asymptotic expansion for $W_{k,m}(z)$ is[16]

$$W_{k,m}(z) \sim e^{-\frac{1}{2}z}z^k \times \left\{1 + \sum_{\nu=1}^{\infty} \frac{[m^2-(k-\frac{1}{2})^2][m^2-(k-\frac{3}{2})^2]\cdots[m^2-(k-\nu+\frac{1}{2})^2]}{\nu!\,z^{\nu}}\right\},$$
$$z \to \infty, \quad |\arg z| < \pi. \tag{45}$$

Substitution of the parameter values given by Theorem 7 establishes the result.

It should be remarked that the error, when using the partial sum $\sum_{\nu=0}^{k-1} z^{(\nu)}\alpha^{-\nu}$ to approximate $B(z, \alpha)^{-1}$, does not exceed $|\alpha|\,|z^{(k)}\alpha^{-k}|/\mathrm{Re}\,\alpha$ provided $\mathrm{Re}\,z \leqq k$, $\mathrm{Re}\,\alpha > 0$. This follows directly from Theorem 4 and (23).

For large z, one has

Theorem 12:

$$B(z, \alpha)^{-1} \sim \psi(z, \alpha)^{-1} - \sum_{s=1}^{\infty} \frac{\alpha^s}{(z+1)\cdots(z+s)},$$

$z \to \infty$, $|\arg z| < \pi/2$, *uniformly in any bounded region of the α-plane for which* $\mathrm{Re}\,\alpha > 0$.

Proof: The representation of Theorem 5 is used. One must show

$$B(z, \alpha)^{-1} - \psi(z, \alpha)^{-1} - \sum_{s=1}^{n} \frac{\alpha^s}{(z+1)\cdots(z+s)} = o\left(\frac{\alpha^n}{(z+1)\cdots(z+n)}\right); \tag{46}$$

that is,

$$\lim_{z\to\infty} \sum_{s>n} \frac{\alpha^{s-n}}{(z+n+1)\cdots(z+s)} = 0. \tag{47}$$

Let Re $z = x$, then one has

$$\left|\sum_{s>n} \frac{\alpha^{s-n}}{(z+n+1)\cdots(z+s)}\right| \leqq \sum_{s>n} |\alpha|^{s-n} \frac{1}{(x+n+1)\cdots(x+s)}. \quad (48)$$

Let $v = x + n$ and $l = s - n$, then the dexter of (48) is

$$\sum_{l=1}^{\infty} \frac{|\alpha|^l}{(v+1)\cdots(v+l)}. \quad (49)$$

Use of (29) and (30) on (49) yields

$$\left|\sum_{s>n} \frac{\alpha^{s-n}}{(z+n+1)\cdots(z+s)}\right| \leqq |\alpha| \int_0^1 e^{|\alpha|(1-u)} u^v du; \quad (50)$$

thus

$$\left|\sum_{s>n} \frac{\alpha^{s-n}}{(z+n+1)\cdots(z+s)}\right| \leqq \frac{|\alpha| e^{|\alpha|}}{v+1} \to 0, \qquad v \to \infty. \quad (51)$$

The theorem is proved.

Useful asymptotic formulas are obtained when both α and z have infinite limits but approach infinity in a fixed ratio, that is, $\alpha = cz$, c fixed. The cases $c > 1$, $c = 1$ are discussed by A. Descloux[17] for large real z. Theorem 13 generalizes the result for $c > 1$ to complex z and provides the structure of the coefficients for the complete expansion. The case $c = 1$ is obtained as a corollary to Theorem 14 where the result is also generalized to complex z.

Theorem 13:

$$B(z, cz)^{-1} \sim \sum_{l=0}^{\infty} g_l z^{-l},$$

$$z \to \infty, \quad |\arg z| < \frac{\pi}{2}, \quad c > 1,$$

$$g_l = \left(\frac{c}{c-1}\frac{d}{dc}\right)^l \frac{c}{c-1}.$$

Proof:* One has, from (23)

$$B(z, cz)^{-1} = cz \int_0^{\infty} e^{-czy}(1+y)^z dy. \quad (52)$$

* The author wishes to thank C. L. Mallows for this proof, which replaces a much longer proof originally supplied by the author.

Defining the function $h(y)$ by

$$h(y) = cy - \ln (1 + y), \tag{53}$$

one may write, since $h(0) = 0$, $h(\infty) = \infty$, and $h(y)$ is monotonic increasing,

$$B(z, cz)^{-1} = z \int_0^\infty e^{-zh(y)} \frac{c(1 + y)}{c(1 + y) - 1} dh. \tag{54}$$

The factor $c(1 + y)/[c(1 + y) - 1]$ is now expanded in powers of h as follows:

$$\frac{c(1 + y)}{c(1 + y) - 1} = \sum_{l=0}^\infty \frac{h^l}{l!} g_l. \tag{55}$$

A theorem on Abelian asymptotics for Laplace transforms[18] and (54), (55) yield the asymptotic behavior of $B(z, cz)^{-1}$ for $z \to \infty$, $|\arg z| < \pi/2$; thus,

$$B(z, cz)^{-1} \sim \sum_{l=0}^\infty g_l z^{-l}. \tag{56}$$

The coefficients g_l may be evaluated as follows. Let

$$w = c - \ln c, \tag{57}$$

and

$$k(w) = \frac{c}{c - 1}, \tag{58}$$

then

$$k(w + h) = k[c(1 + y) - \ln c(1 + y)] = \frac{c(1 + y)}{c(1 + y) - 1}. \tag{59}$$

Thus, Taylor expansion yields

$$\frac{c(1 + y)}{c(1 + y) - 1} = \sum_{l=0}^\infty \frac{h^l}{l!} \left(\frac{d}{dw} \right)^l k(w). \tag{60}$$

One has

$$\frac{d}{dw} = \frac{c}{c - 1} \frac{d}{dc}, \tag{61}$$

hence

$$\left(\frac{d}{dw} \right)^l k(w) = \left(\frac{c}{c - 1} \frac{d}{dc} \right)^l \frac{c}{c - 1} = g_l. \tag{62}$$

The following formula is obtained directly from Theorem 13.

$$B(z, cz)^{-1} \sim \frac{c}{c - 1} - \frac{c}{(c - 1)^3} \frac{1}{z} + \frac{2c^2 + c}{(c - 1)^5} \frac{1}{z^2}. \tag{63}$$

The evaluation and behavior of $B(z, \alpha)$ for α in a neighborhood of z

is often of interest; accordingly, the function $B(z, z + c\sqrt{z})^{-1}$ will be considered for $z \to \infty$; c is a fixed real number.

Theorem 14: There exists a representation of the form

$$B(z, z + c\sqrt{z}) \sim \sum_{j=0}^{\infty} a_j(c) z^{-(j-1)/2},$$

$$z \to \infty, \quad |\arg z| < \frac{\pi}{2}, \quad c \text{ real},$$

in which

$$\begin{aligned} a_0(c) &= e^{\frac{1}{2}c^2} \int_c^\infty e^{-\frac{1}{2}u^2} du, \\ a_1(c) &= \frac{2}{3} + \frac{1}{3} c^2 - \frac{1}{3} c^3 a_0(c), \\ a_2(c) &= -\frac{1}{18} c^5 - \frac{7}{36} c^3 + \frac{1}{12} c + \left(\frac{1}{18} c^6 + \frac{1}{4} c^4 + \frac{1}{12}\right) a_0(c). \end{aligned}$$

Proof: From (23), one has

$$B(z, z + c\sqrt{z})^{-1} = (z + c\sqrt{z}) \int_0^\infty e^{-(z+c\sqrt{z})u} (1 + u)^z du, \tag{64}$$

$$|\arg z| < \frac{\pi}{2}.$$

Let $u = v/\sqrt{z}$, then

$$\begin{aligned} B(z, z + c\sqrt{z})^{-1} &= \int_0^\infty e^{-(\frac{1}{2}v^2 + cv)} h(v, z) dv, \\ h(v, z) &= e^{\frac{1}{2}v^2 - \sqrt{z}v} \left(1 + \frac{v}{\sqrt{z}}\right)^z (\sqrt{z} + c). \end{aligned} \tag{65}$$

Let K be a positive constant, then, for $|v| \leqq K$, $h(v, z)$ clearly possesses an asymptotic development in $\sqrt{z}$ uniformly in v; thus,

$$h(v, z) \sim \sum_{j=0}^{\infty} b_j(v, c) z^{-(j-1)/2}, \qquad z \to \infty, \tag{66}$$

in which the coefficients $b_j(v, c)$ are polynomials in v. In particular,

$$\begin{aligned} b_0(v, c) &= 1, \\ b_1(v, c) &= \frac{1}{3} v^3 + c, \\ b_2(v, c) &= \frac{1}{3} c v^3 - \frac{1}{4} v^4 + \frac{1}{18} v^6. \end{aligned} \tag{67}$$

Since

$$e^{-(\frac{1}{2}v^2+cv)}v^k \epsilon L(0, \infty) \tag{68}$$

for each $k > 0$ and any c, termwise integration of (66) leads to the required asymptotic expansion. Thus, letting

$$a_j(c) = \int_0^\infty e^{-(\frac{1}{2}v^2+cv)}b_j(v, c)dv, \tag{69}$$

one has

$$B(z, z + c\sqrt{z})^{-1} \sim \sum_{j=0}^{\infty} a_j(c)z^{-(j-1)/2}. \tag{70}$$

The formulas for $a_0(c)$, $a_1(c)$, $a_2(c)$ stated in the theorem are obtained by evaluation of (69) using $b_j(v, c)$ as given in (67).

Corollary:

$$B(z, z)^{-1} \sim \sqrt{\frac{\pi z}{2}} + \frac{2}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2z}},$$

$$z \to \infty, \qquad |\arg z| < \frac{\pi}{2}.$$

Proof: The result is obtained from Theorem 14 with $c = 0$.

This theorem helps explain the phenomenon of the efficiency of large trunk groups since even when $a > x$, $B(x, a)$ is small as long as a is in a small neighborhood of x; thus, Theorem 14 shows that $B(x, x + c\sqrt{x}) \sim 1/a_0\sqrt{x}$, $x \to \infty$.

Theorem 14 shows that the parameter c may be viewed as a standardized offered load measuring the deviation of a from x in units of $\sqrt{x}$. The value of this viewpoint is derived from the very simple approximating form for $B(x, a)$; thus,

$$B(x, a)^{-1} \simeq a_0\sqrt{x} + a_1 + \frac{a_2}{\sqrt{x}}. \tag{71}$$

An application of this is to the computation of $\partial B(x, a)/\partial x$ given in (92). Another advantage is the capability of computing $B(x, a)$ by means of a single-entry table against the standardized offered load c rather than the usual double-entry table against x and a.

Table I gives the values of $a_0(c)$, $a_1(c)$, $a_2(c)$ for $-3 \leqq c \leqq 4$ in steps of 0.1 with the intention of covering a practical range of values of $B(x, a)$. As an illustration, it is desired to compute $B(400, 378)$. Use of (71) with $c = -1.1$ gives the result 0.0122 correct to the last figure. If c does not appear in the table, then interpolation is used. For example, to compute $B(400, 377.6)$ for which $c = -1.12$ linear interpolation in the table of coefficients and use of (71) yields 0.0118 correct to the last figure. The method, of course, is valid even when the number of trunks is nonintegral. Consider, for example, $B(400.34, 420)$ for which $c = 0.98463$. The result obtained by linear interpolation in the table is 0.0713 correct to half a unit of the last figure.

The accuracy deteriorates when x is decreased or when c is large and negative. Thus, for $B(10, 8)$, one obtains 0.12144 as against the correct value 0.12166. In this case $c = -0.6325$ is not too disadvantageous. The case $B(10, 5)$ for which $c = -1.58114$ yields a much greater error, namely, 0.0256 as against the correct value 0.0184. This error occurs, however, for a small trunk group where exact calculation is quite feasible. To aid the delineation of suitable regions of (c, x) for which the table is accurate, a curve is given in Fig. 1 defining 5-percent error. When a computation is made from the table using any point (c, x) in the unbounded, unshaded region, the error incurred will be less than 5 percent of the true value of $B(x, a)$.

IV. DERIVATIVES AND INEQUALITIES

It is desired to obtain formulas for the derivatives of $B(z, \alpha)$, with respect to z and α.

Theorem 15:

$$\frac{\partial B(z, \alpha)}{\partial \alpha} = \left\{ \frac{z}{\alpha} - 1 + B(z, \alpha) \right\} B(z, \alpha), \qquad \operatorname{Re} \alpha > 0.$$

Proof: From (23), one has

$$\frac{\partial B(z, \alpha)^{-1}}{\partial \alpha} = \int_0^\infty e^{-\alpha u}(1 + u)^z du - \alpha \int_0^\infty e^{-\alpha u}(1 + u)^z u du; \tag{72}$$

hence,

$$\frac{\partial B(z, \alpha)^{-1}}{\partial \alpha} = \frac{1}{\alpha} B(z, \alpha)^{-1} - B(z + 1, \alpha)^{-1} + B(z, \alpha)^{-1}. \tag{73}$$

Use of Theorem 4 provides the relation

$$\frac{\partial B(z, \alpha)^{-1}}{\partial \alpha} = -\frac{z}{\alpha} B(z, \alpha)^{-1} - 1 + B(z, \alpha)^{-1}. \tag{74}$$

Since

$$\frac{\partial B(z, \alpha)^{-1}}{\partial \alpha} = -B(z, \alpha)^{-2} \frac{\partial B(z, \alpha)}{\partial \alpha}, \tag{75}$$

the result of the theorem follows from (74).

For the purpose of obtaining an approximate formula for the derivative with respect to z, consider

$$f(u) = aB(x, a)e^{-au}(1 + u)^x \tag{76}$$

in which $a > 0$, $x > 0$, and for which, by (23),

$$\int_0^\infty f(u)du = 1. \tag{77}$$

It is convenient to introduce the random variable ξ with density function $f(u)$. The power moments μ_r defined by

$$\mu_r = E\xi^r, \qquad r > 0 \text{ (integral)}, \tag{78}$$

are given in the following theorem.

Theorem 16:

$$\mu_r = B(x, a) \sum_{l=0}^{r} (-1)^{r-l} \binom{r}{l} B(x + l, a)^{-1}.$$

Proof: Define a generating function $\phi(t)$ by

$$\phi(t) = Ee^{t\xi} = aB(x, a) \int_0^\infty e^{-(a-t)u}(1 + u)^x du, \tag{79}$$

then, since

$$(a - t)B(x, a - t) \int_0^\infty e^{-(a-t)u}(1 + u)^x du = 1, \tag{80}$$

one has

$$\phi(t) = \frac{aB(x, a)}{(a - t)B(x, a - t)}. \tag{81}$$

Use of Theorem 10 in (81) provides the expansion

$$\phi(t) = B(x, a)e^{-t} \sum_{r=0}^{\infty} \frac{B(x + r, a)^{-1}}{r!} t^r. \tag{82}$$

Since

$$\phi(t) = \sum_{r=0}^{\infty} \frac{\mu_r}{r!} t^r, \tag{83}$$

the coefficient of t^r in the expansion of (82) in powers of t yields the required result. Thus

$$\mu_r = B(x, a)r! \sum_{l=0}^{r} \frac{(-1)^{r-l}}{(r - l)!} \frac{B(x + l, a)^{-1}}{l!} \tag{84}$$

and the formula of the theorem follows.

Corollary: *The central moments* α_r *are given by*

$$\alpha_r = E(\xi - \mu_1)^r = B(x, a) \sum_{l=0}^{r} (-1)^{r-l} \binom{r}{l} (\mu_1 + 1)^{r-l} B(x + l, a)^{-1}.$$

Proof: The same as for Theorem 16 but considering the function $e^{-\mu_1 t}\phi(t)$ instead of $\phi(t)$.

An approximation to $\partial B(x, a)/\partial x$ may now be obtained.

Theorem 17:

$$-B(x, a)^{-1} \frac{\partial B(x, a)}{\partial x} = \ln(1 + \mu_1) - \frac{1}{2} \frac{\alpha_2}{(1 + \mu_1)^2} + \frac{1}{3} \frac{\alpha_3}{(1 + \mu_1)^3} - \frac{1}{4} \alpha_4 \theta, \qquad 0 < \theta < 1.$$

Proof: From (23) and (76), one obtains

$$-B(x, a)^{-1} \frac{\partial B(x, a)}{\partial x} = E \ln(1 + \xi). \tag{85}$$

Since, by use of the mean value formula,

$$\ln(1 + \xi) = \ln(1 + \mu_1) + \frac{1}{1 + \mu_1} (\xi - \mu_1) - \frac{1}{2} \frac{1}{(1 + \mu_1)^2} (\xi - \mu_1)^2 + \frac{1}{3} \frac{1}{(1 + \mu_1)^3} (\xi - \mu_1)^3 - \frac{1}{4} \theta (\xi - \mu_1)^4, \qquad 0 < \theta < 1, \tag{86}$$

one has, from (85) and the corollary to Theorem 16, the required result.

Corollary:

$$-B(x, a)^{-1} \frac{\partial B(x, a)}{\partial x} < \ln(1 + \mu_1) - \frac{1}{2} \frac{\alpha_2}{(1 + \mu_1)^2} + \frac{1}{3} \frac{\alpha_3}{(1 + \mu_1)^3}.$$

Proof: The error term of Theorem 17 is omitted.

For ready reference the following formulas are given in which $B = B(x, a)$, $B_1 = B(x + 1, a)$, $B_2 = B(x + 2, a)$, $B_3 = B(x + 3, a)$, $B_4 = B(x + 4, a)$.

$$\begin{aligned}
\mu_1 &= -1 + BB_1^{-1}, \\
\alpha_2 &= (\mu_1 + 1)^2 - 2(\mu_1 + 1)BB_1^{-1} + BB_2^{-1}, \\
\alpha_3 &= -(\mu_1 + 1)^3 + 3(\mu_1 + 1)^2 BB_1^{-1} - 3(\mu_1 + 1)BB_2^{-1} + BB_3^{-1}, \\
\alpha_4 &= (\mu_1 + 1)^4 - 4(\mu_1 + 1)^3 BB_1^{-1} + 6(\mu_1 + 1)^2 BB_2^{-1} \\
&\qquad - 4(\mu_1 + 1)BB_3^{-1} + BB_4^{-1}.
\end{aligned} \tag{87}$$

The evaluation of B_1^{-1}, B_2^{-1}, B_3^{-1}, B_4^{-1} is facilitated by successive use of Theorem 4.

An alternative method of obtaining $\partial B(x, a)/\partial x$ is based on Theorem 14. Let

$$f(x, c) = B(x, a), \qquad a = x + c\sqrt{x}, \tag{88}$$

then, from Theorem 14,

$$\frac{\partial f(x, c)^{-1}}{\partial x} \sim - \sum_{j=0}^{\infty} \frac{j-1}{2} a_j(c) x^{-(j+1)/2}, \qquad x \to \infty; \tag{89}$$

hence,

$$\frac{\partial f(x, c)^{-1}}{\partial x} \simeq \frac{1}{2\sqrt{x}} \left\{ a_0(c) - \frac{a_2(c)}{x} \right\}. \tag{90}$$

Thus, the computation of $\partial f(x, c)/\partial x$ is easily accomplished with the help of Table I and the formula

$$\frac{\partial f(x, c)}{\partial x} = - f(x, c)^2 \frac{\partial f(x, c)^{-1}}{\partial x} \simeq - \frac{B(x, a)^2}{2\sqrt{x}} \left\{ a_0(c) - \frac{a_2(c)}{x} \right\}. \tag{91}$$

One now has

$$\frac{\partial B(x, a)}{\partial x} = \frac{\partial f(x, c)}{\partial x} - \frac{\partial B(x, a)}{\partial a} \left(1 + \frac{c}{2\sqrt{x}} \right). \tag{92}$$

A simple upper bound on $-B(x, a)^{-1}[\partial B(x, a)/\partial x]$ is given in the following theorem.

Theorem 18:

$$-B(x, a)^{-1} \frac{\partial B(x, a)}{\partial x} < \ln (1 + \mu_1).$$

Proof: Since the function $-\ln(1 + u)$ is convex for $u \geqq 0$, the required inequality follows from Jensen's inequality, namely,

$$g(E\xi) \leqq Eg(\xi) \tag{93}$$

valid for functions $g(x)$ convex over the range of the random variable ξ, and (85).

A function $g(x) > 0$ is said to be log-convex over a set if $\ln g(x)$ is convex over the set. It is known[19] that the sum of log-convex functions is log-convex and hence that the integral of a log-convex function with respect to a parameter is log-convex provided the function is log-convex for every value of the parameter. Since a necessary and sufficient condition that a twice-differentiable function be convex is the nonnegativity of its second derivative over the corresponding set, one derives the inequality

$$g''g - g'^2 \geqq 0 \tag{94}$$

as a necessary and sufficient condition that $g > 0$ be log-convex. One now has

Theorem 19: $B(x, a)^{-1}$, $[aB(x, a)]^{-1}$ *are log-convex functions of* x *and of* a, *respectively, for* $a > 0$ *and all* x.

Proof: The results are immediate from (23) and the observations that $(1 + u)^x$ is log-convex as a function of x for $u \geqq 0$, and e^{-au} is log-convex as a function of a for $u \geqq 0$.

Corollary:

$$B(x, a) \frac{\partial^2 B(x, a)}{\partial x^2} \leqq \left[\frac{\partial B(x, a)}{\partial x} \right]^2,$$

$$aB(x, a) \left[2 \frac{\partial B(x, a)}{\partial a} + a \frac{\partial^2 B(x, a)}{\partial a^2} \right] \leqq \left[B(x, a) + a \frac{\partial B(x, a)}{\partial a} \right]^2.$$

Proof: Use of (94).

An immediate application of Theorem 19 is to the logarithmic interpolation of $B(x, a)$, that is, linear interpolation of $\ln B(x, a)$.

Theorem 20: *Let* $a, b, p, q > 0$, $p + q = 1$, *then*

$$B(x, a)^p B(y, a)^q \leqq B(px + qy, a),$$
$$[aB(x, a)]^p [bB(x, b)]^q \leqq (pa + qb) B(x, pa + qb).$$

Proof: Jensen's inequality applied to $-\ln B(x, a)$ and $-\ln [aB(x, a)]$, respectively.

An extension of the result of Theorem 20, for the purpose of obtaining an approximate formula for $B(x, a)$ when x is not an integer, may be derived from the corollary to Theorem 16. Let $N = [x]$, $\delta = x - N$, and α_r be the central moments computed for the density function

$$f(u) = aB(N, a) e^{-au} (1 + u)^N, \tag{95}$$

then one has

Theorem 21:

$$B(x, a)^{-1} = B(N, a)^{-1} \sum_{r=0}^{k-1} \binom{\delta}{r} \alpha_r (1 + \mu_1)^{\delta - r} + B(N, a)^{-1} \binom{\delta}{k} \alpha_k \theta, \qquad k \text{ even}, \qquad |\theta| \leqq 1.$$

Proof: Let ξ be the random variable with density function $f(u)$, then

$$B(x, a)^{-1} = B(N, a)^{-1} E(1 + \xi)^\delta. \tag{96}$$

Since

$$(1+\xi)^\delta = \sum_{r=0}^{k-1} \binom{\delta}{r} (1+\mu_1)^{\delta-r}(\xi-\mu_1)^r + \binom{\delta}{k} (\xi-\mu_1)^k\theta, \quad (97)$$

the result follows from (96) and the corollary to Theorem 16.

A useful special case of Theorem 21 is

$$B(x, a) \simeq \frac{B^{1-\delta}B_1{}^\delta}{1-\frac{1}{2}\delta(1-\delta)\left(\frac{B_1^2}{BB_2}-1\right)} \quad (98)$$

in which

$$B = B(N, a), \qquad B_1 = B(N+1, a), \qquad B_2 = B(N+2, a). \quad (99)$$

V. CONCLUSION

Further investigations would be desirable; for example, one would like to know the contour function $g(z)$ for which $B[z, g(z)]$ is constant. Truncation error formulas for the asymptotic expansions of Theorems 13 and 14 would be useful; also, the general structure of the coefficients $a_j(c)$ of Theorem 14 should be determined. Asymptotic formulas of various types should be obtained for $G_j(z, \alpha)$ similar to those given for $B(z, \alpha)$. These formulas may then be used to obtain asymptotic results for its zeros which are needed in many transient and time-variable blocking analyses.

VI. ACKNOWLEDGMENTS

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APPENDIX

The function $B(z, \alpha)$ is related to the Poisson-Charlier polynomials[20–22] much used in telephone traffic studies. Let

$$\begin{aligned} \psi_0(z, \alpha) &= \psi(z, \alpha), \\ \psi_j(z, \alpha) &= \frac{d^j}{d\alpha^j}\psi(z, \alpha), \end{aligned} \quad (100)$$

then the Poisson-Charlier polynomials, $G_j(z, \alpha)$, are defined by

$$\psi_j(z, \alpha) = \psi(z, \alpha)G_j(z, \alpha). \quad (101)$$

The Taylor expansion

$$\psi(z, \alpha + t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \psi_j(z, \alpha) \tag{102}$$

yields the generating function[23]

$$e^{-t}\left(1 + \frac{t}{\alpha}\right)^z = \sum_{j=0}^{\infty} G_j(z, \alpha) \frac{t^j}{j!}. \tag{103}$$

Thus, explicit formulas for $G_j(z, \alpha)$ are

$$\begin{aligned} G_j(z, \alpha) &= \frac{j!}{\alpha^j} \sum_{\nu=0}^{j} (-1)^\nu \binom{z}{j-\nu} \frac{\alpha^\nu}{\nu!} \\ &= \sum_{\nu=0}^{j} (-1)^{j-\nu} \binom{j}{\nu} \nu!\, \alpha^{-\nu} \binom{z}{\nu}. \end{aligned} \tag{104}$$

The first few polynomials are

$$\begin{aligned} G_0(z, \alpha) &= 1, \\ G_1(z, \alpha) &= \frac{1}{\alpha}(z - \alpha), \\ G_2(z, \alpha) &= \frac{1}{\alpha^2}[z^2 - (2\alpha + 1)z + \alpha^2], \\ G_3(z, \alpha) &= \frac{1}{\alpha^3}[z^3 - 3(\alpha + 1)z^2 + (3\alpha^2 + 3\alpha + 2)z - \alpha^3]. \end{aligned} \tag{105}$$

A recurrence relation derived from (103) is

$$G_{j+1}(z, \alpha) = \frac{z - j - \alpha}{\alpha} G_j(z, \alpha) - \frac{j}{\alpha} G_{j-1}(z, \alpha). \tag{106}$$

The polynomials, $G_j(z, \alpha)$, possess many properties analogous to those of $B(z, \alpha)^{-1}$. A Rodrigues formula is given in

Theorem 22:

$$G_{j+k}(z, \alpha) = \alpha^{-z} e^{\alpha} \frac{d^k}{d\alpha^k} [e^{-\alpha} \alpha^z G_j(z, \alpha)].$$

Proof: One has from (100)

$$\psi_{j+k}(z, \alpha) = \frac{d^k}{d\alpha^k} \psi_j(z, \alpha), \tag{107}$$

and hence

$$G_{j+k}(z, \alpha)\psi(z, \alpha) = \frac{d^k}{d\alpha^k} [\psi(z, \alpha) G_j(z, \alpha)]. \tag{108}$$

The result follows on use of (24).

Corollary:

$$G_j(z, \alpha) = \alpha^{-z}e^{\alpha}\frac{d^j}{d\alpha^j}[e^{-\alpha}\alpha^z].$$

Proof:

$$G_0(z, \alpha) = 1. \tag{109}$$

An integral representation for $G_j(z, \alpha)$ is given in

Theorem 23:

$$G_j(-z, \alpha) = (-1)^j\frac{\alpha^z}{\Gamma(z)}\int_0^\infty e^{-\alpha y}(1+y)^j y^{z-1}dy,$$

$$\mathrm{Re}\,\alpha > 0, \qquad \mathrm{Re}\,z > 0.$$

Proof: From (20), one has

$$e^{-\alpha}\alpha^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty e^{-\alpha(1+y)}y^{z-1}dy. \tag{110}$$

Substitution of (110) into the corollary of Theorem 22 yields the result.

Theorem 23 permits obtaining a Wittaker function representation.

Theorem 24:

$$G_j(z, \alpha) = (-1)^j\alpha^{-(z+j+1)/2}e^{\alpha/2}W_{(j+z+1)/2,(j-z)/2}(\alpha).$$

Proof: Comparison of Theorem 23 with (32) and replacement of $-z$ by z.

The representation of Theorem 24 remains valid, by analytic continuation, even outside the region of convergence of the integral of (32).

Corollary 1:

$$B(N, \alpha)^{-1} = (-1)^N G_N(-1, \alpha).$$

Proof: Comparison of Theorems 7 and 24.

Corollary 2:

$$G_j(z, \alpha) = \frac{z}{\alpha}G_{j-1}(z-1, \alpha) - G_{j-1}(z, \alpha).$$

Proof: Substitution of the representation of Theorem 24 into the recurrence relation[24]

$$W_{k,m}(z) = \sqrt{z}W_{k-\frac{1}{2},m-\frac{1}{2}}(z) + (\tfrac{1}{2} - k + m)W_{k-1,m}(z) \tag{111}$$

yields the result.

Corollary 3:

$$G_j(z, \alpha) = G_j(z - 1, \alpha) + \frac{j}{\alpha} G_{j-1}(z - 1, \alpha).$$

Proof: Same as for Corollary 2, except the following recurrence relation is used:

$$W_{k,m}(z) = \sqrt{z} W_{k-\frac{1}{2}, m+\frac{1}{2}}(z) + (\tfrac{1}{2} - k - m) W_{k-1,m}(z). \tag{112}$$

A representation of $G_j(z, \alpha)$ in terms of $B(z, \alpha)$ is given in

Theorem 25:

$$G_N(-j, \alpha) = (-1)^{N+j-1} \frac{\alpha^j}{(j-1)!} \frac{d^{j-1}}{d\alpha^{j-1}} \frac{1}{\alpha B(N, \alpha)}.$$

Proof: From (23),

$$\frac{1}{\alpha B(N, \alpha)} = \int_0^\infty e^{-\alpha y}(1 + y)^N dy, \tag{113}$$

and Theorem 23,

$$G_N(-j, \alpha) = (-1)^N \frac{\alpha^j}{(j-1)!} \int_0^\infty e^{-\alpha y}(1 + y)^N y^{j-1} dy, \tag{114}$$

one has the result on use of

$$\frac{d^{j-1}}{d\alpha^{j-1}} \int_0^\infty e^{-\alpha y}(1 + y)^N dy = (-1)^{j-1} \int_0^\infty e^{-\alpha y}(1 + y)^N y^{j-1} dy. \tag{115}$$

The Poisson-Charlier polynomials possess addition formulas similar to those of $B(z, \alpha)^{-1}$ as given in Theorems 9 and 10.

Theorem 26:

$$G_j(z, \alpha + t) = \left(1 + \frac{t}{\alpha}\right)^{-j} \sum_{\nu=0}^{j} \binom{j}{\nu} G_{j-\nu}(z, \alpha)(-1)^\nu \left(\frac{t}{\alpha}\right)^\nu.$$

Proof: Use of (103) shows that the system of functions

$$\left[\frac{(-1)^j}{j!}\right] \alpha^j G_j(z, \alpha)$$

has the generating function $e^{\alpha t}(1 - t)^z$, and hence[25] form an Appell system with respect to α, thus,

$$\frac{d}{d\alpha}\left[\frac{(-1)^j}{j!} \alpha^j G_j(z, \alpha)\right] = \frac{(-1)^{j-1}}{(j-1)!} \alpha^{j-1} G_{j-1}(z, \alpha). \tag{116}$$

The Taylor expansion of $[(-1)^j/j!](\alpha + t)^j G_j(z, \alpha, + t)$ in powers of t now yields the required result.

Theorem 27:

$$G_j(z, \alpha + t) = \left(1 + \frac{t}{\alpha}\right)^{-z} e^t \sum_{\nu=0}^{\infty} G_{j+\nu}(z, \alpha) \frac{t^\nu}{\nu!}.$$

The series is permanently convergent.

Proof: Use of (107) and Taylor's expansion yields

$$\psi_j(z, \alpha + t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \psi_{j+\nu}(z, \alpha). \tag{117}$$

The result is now obtained from (101) and (117). Since, from (104),

$$G_j(z, \alpha) \sim \left(\frac{z}{\alpha}\right)^j, \qquad j \to \infty, \tag{118}$$

the convergence is permanent.

An asymptotic expansion is given by

Theorem 28:

$$G_j(z, \alpha) \sim (-1)^j \left\{1 + \sum_{\nu=0}^{j} (-1)^\nu \frac{j^{(\nu)} z^{(\nu)}}{\nu! \alpha^\nu}\right\}, \qquad \alpha \to \infty.$$

Proof: The result is obtained directly from (104). It may also be obtained from Theorem 24 and (45).

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