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Transverse Coupling in Fiber Optics Part II: Coupling to Mode Sinks

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The number of modes that can propagate without radiation loss in oversized waveguides is sharply reduced if the waveguide is coupled to a structure supporting radiation modes, the loss mechanism being analogous to Cerenkov radiation. The coupling formula derived in Part I[1] is used to evaluate the loss for a specific configuration: a reactive surface (e.g., a thin dielectric slab) acting as a waveguide, coupled to a semi-infinite dielectric acting as a mode sink. The method consists in first assuming that the substrate is finite in size and lossy and adding the losses associated with each substrate mode. The substrate dimensions are subsequently made infinite and the dissipation loss is made to vanish. The expression obtained for the radiation loss coincides with an expression obtained by solving the boundary value problem. The method is then applied to the problem of mode selection for dielectric rods coupled to dielectric slabs, which is of particular importance for optical communications and integrated optics. A 2-dB/m radiation loss is calculated for the first higher order mode when the rod radius is 10 μm, $\lambda = 1$ μm, $n = 1.41$, and the rod-to-slab spacing is 0.15 μm.

I. INTRODUCTION

An expression for the coupling between lossy single-mode open waveguides was derived in Part I.[1] We now investigate the coupling of a waveguide with finite cross section with a waveguide with infinite cross section (called a substrate), the latter supporting radiation modes. Radiation losses are suffered whenever the propagation constant h of the guided mode is smaller than the highest propagation constant h_s of the radiation modes carried by the substrate. Radiation then takes place at the Cerenkov angle $\theta = \cos^{-1}(h/h_s)$. By properly choosing the dimensions and permittivities of the waveguide and those of the substrate, it is possible to reduce the number of modes that can propagate without attenuation (in the absence of dissipation and scattering losses). This arrangement is of great practical importance because optical fibers are usually highly overmoded to facilitate fabrication and splicing.[2] (For a coherent source, it is important to reduce the number of modes because different modes usually have different group velocities. If a short optical pulse is sent through the fiber, mode conversion takes place because of the imperfections of the fiber; this causes the pulse to spread in time.) The mode selection mechanism just described is also of practical importance in the microwave range for oversized waveguides such as oversized microstrips on dielectric substrates and oversized dielectric strips.* Multimoding in traveling wave tubes can also be avoided with the help of mode sinks.

We investigate the loss mechanism for two specific configurations. First, a reactive surface acting as a waveguide coupled to a semi-infinite dielectric acting as a mode sink. We show that, by adding the losses associated with each substrate mode, an expression for the total loss is obtained that coincides with an expression obtained by solving the boundary value problem. Then the method is applied to the problem of a dielectric rod coupled to a dielectric slab.[2] The case of dielectric rods coupled to dielectric cylinders supporting whispering gallery modes and acting as mode sinks[3] will be discussed in another paper.

II. RADIATION LOSSES IN SUBSTRATES—GENERAL FORMULA

To evaluate the radiation losses, let us first assume that the transverse dimensions of the substrate are finite, and let $h_{sz} = h_{sr} + ih_{si}$ be the propagation constant of a trapped mode in the substrate, with h_{sr} real and h_{si} real positive (the subscript s stands for "substrate").† If h_o denotes the propagation constant of a trapped mode of the waveguide in the absence of the substrate, the propagation constant h of the coupled wave is, from eq. (6a) in Part I,

$$h = h_o + \tfrac{1}{2}(h_{sz} - h_o) - [\tfrac{1}{4}(h_{sz} - h_o)^2 + C^2]^{\frac{1}{2}}, \tag{1}$$

where $C^2 \equiv c_a c_b / P_a P_b$ denotes the coupling coefficient defined in Part I. The minus sign before the square root has been selected because it

* In the microwave range, there are no compelling reasons for using dielectric waveguides that are large compared with the wavelength in all dimensions, but we may want to use strips (either metallic or dielectric) whose widths exceed one wavelength for improved accuracy.

† The dependence of the field on time (t) and on the axial coordinate (z) is denoted $\exp[i(hz - \omega t)]$. This term is henceforth omitted.

corresponds to the mode whose field is concentrated in the waveguide cross section rather than in the substrate (that is, we require $h = h_o$ when $C^2 = 0$).

Let us now assume that h_o is real (lossless waveguide) and that

$$h_{si} \gg C. \tag{2}$$

Using this condition, eq. (2), we can expand the r.h.s. of eq. (1) in power series of C^2 and keep only the first two terms in the expansion. The loss is given by the imaginary part h_i of h. Because the imaginary part of C^2 can be neglected in the case that we consider, we have

$$h_i \approx C^2 h_{si} [(h_{sr} - h_o)^2 + h_{si}^2]^{-1}. \tag{3}$$

The total loss $\mathcal{L}$ experienced by the waveguide is now obtained by summing over the various modes of the substrate:

$$\mathcal{L} = \sum_{\alpha} C_\alpha^2 h_{si} [(h_{sr\alpha} - h_o)^2 + h_{si}^2]^{-1}, \tag{4}$$

where the subscript α refers to the substrate modes. We have assumed, for simplicity, that h_{si} does not depend on α. It is shown in the next section for a simple configuration that in the limit of dense substrate modes eq. (4) is in agreement with an exact result, obtained from a boundary value method.

If we let the cross-section area S of the substrate tend to infinity, the substrate modes become denser and denser, and the summation in eq. (4) can be replaced by an integral

$$\begin{aligned} \mathcal{L} &= \lim_{S\to\infty} \sum_{\alpha} C_\alpha^2 h_{si} [(h_{sr\alpha} - h_o)^2 + h_{si}^2]^{-1} \\ &= \int \mathcal{C}(h_{sr}) h_{si} [(h_{sr} - h_o)^2 + h_{si}^2]^{-1} dh_{sr}, \end{aligned} \tag{5}$$

where we have defined a coupling density $\mathcal{C}$ by

$$\mathcal{C}(h_{sr}) dh_{sr} = \lim_{S\to\infty} \sum_{\alpha} C_\alpha^2,$$

the range of α being defined by the condition

$$h_{sr} < h_{sr\alpha} < h_{sr} + dh_{sr}. \tag{6}$$

This density exists because, as $S \to \infty$, the coupling coefficient C^2 decreases at least as fast as S^{-1}, the power in the substrate being proportional to S if the power density is kept a constant.

We can now let h_{si} tend to zero, the condition eq. (2) being preserved. The second factor in the integrand of eq. (5) is sharply peaked at $h_{sr} = h_o$ and behaves as a symbolic δ-function. Thus, in the limit $h_{si} \rightarrow 0$ we have

$$\mathfrak{L} = \pi \mathfrak{C}(h_o). \tag{7}$$

It should be noted that the subscript α in eqs. (4) to (6) stands for three subscripts m, n, and s, where m refers to modes in the x direction, n refers to modes in the y direction (we assume for simplicity that the substrate modes are separable in Cartesian coordinates), and s refers to the state of polarization (e.g., H or E modes).

III. COUPLING TO A SEMI-INFINITE SUBSTRATE

Consider a reactive surface coupled to a semi-infinite dielectric (Fig. 1). We consider only H modes and assume that the field is

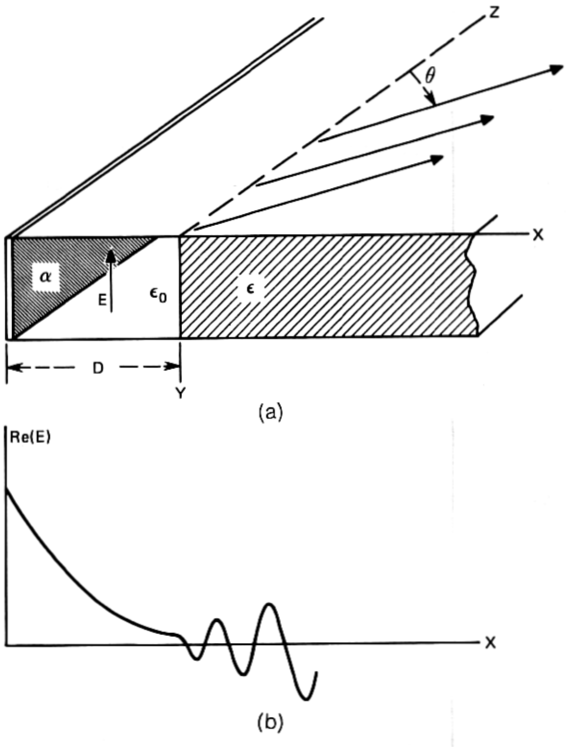

Fig. 1—(a) Reactive surface, with normalized susceptance α, coupled to a semi-infinite dielectric with permittivity $\epsilon \equiv n^2\epsilon_o$. For H modes, the structure is assumed terminated in the y direction by electric walls. Radiation takes place at the Cerenkov angle $\theta = \cos^{-1}[(k^2 + \alpha^2)^{\frac{1}{2}}/kn]$, $k \equiv 2\pi/\lambda$. (b) Variation of the field as a function of x.

independent of the y coordinate. Except for the changes $x \rightleftarrows y$ and $i \to -i$, we use Shevchenko's notation.[4]

For waves propagating along the z axis, the electric field has only a y component that we denote E. In a region with constant ϵ, E obeys the wave equation

$$d^2E/dx^2 + (\omega^2 \epsilon\mu_o - h^2)E = 0,$$
$$H_x = -h(\omega\mu_o)^{-1}E,$$
$$H_z = (i\omega\mu_o)^{-1}dE/dx. \tag{8}$$

If ϵ has a finite discontinuity, E and dE/dx remain continuous. The general solution of eq. (8) for $\epsilon = \epsilon_o$ and ϵ are, respectively,

$$E = A^+e^{i\chi x} + A^-e^{-i\chi x}, \qquad (\epsilon_o) \tag{9a}$$

$$E_s = A_s^+e^{igx} + A_s^-e^{-igx}, \qquad (\epsilon) \tag{9b}$$

$$\chi^2 \equiv \omega^2\epsilon_o\mu_o - h^2, \tag{10a}$$

$$g^2 \equiv \omega^2\epsilon\mu_o - h^2 = u^2 + \chi^2,$$
$$u^2 \equiv \omega^2(\epsilon - \epsilon_o)\mu_o. \tag{10b}$$

The loss can be evaluated by solving the boundary value problem. At the reactive surface ($x = -D$), we have the condition (see Ref. 4)

$$dE/dx + \alpha E = 0, \qquad x = -D, \tag{11}$$

where α is a positive real number proportional to the susceptance of the surface.* We assume that, in the dielectric, the wave propagates away from the structure, that is,

$$E_s = A_s^+e^{igx}. \tag{12}$$

Note that h is expected to have a small positive imaginary part expressing the radiation loss in the dielectric. Assuming that ϵ is real, that is, that the dielectric is free of dissipation losses, eq. (10b) shows that g has a small negative imaginary part. Thus, the wave amplitude grows exponentially as the distance to the structure increases. This solution of Maxwell's equations is called a "leaky wave."[4] It is not difficult to show that the curves of constant irradiance in the dielectric are straight lines making with the z axis an angle $\theta = \cos^{-1}(h_o/kn)$ (Cerenkov angle).

* A thin dielectric slab with permittivity ϵ and thickness d, supported by a magnetic wall, is equivalent to a reactive surface with normalized susceptance $\alpha = \omega^2(\epsilon - \epsilon_o)\mu_o d$. An equivalent configuration, obtained by symmetry with respect to the magnetic wall, is a thin slab of width $2d$ with dielectrics symmetrically located on both sides. Note that α has the dimension of a propagation constant.

From eq. (12), the boundary condition at $x = 0$ is

$$dE/dx - igE = 0, \qquad x = 0. \tag{13}$$

From eqs. (9a), (11), and (13), we obtain the equation defining χ, or h,

$$(\chi - i\alpha)(\chi + g) = (\chi + i\alpha)(\chi - g) \exp(2i\chi D). \tag{14}$$

If we let αD tend to infinity, the reactive surface is uncoupled to the dielectric and eq. (14) reduces to $\chi \equiv \chi_o = i\alpha$; that is,

$$\chi^2 \equiv \chi_o^2 \equiv \omega^2 \epsilon_o \mu_o - h_o^2 = -\alpha^2, \tag{15a}$$

$$g^2 \equiv g_o^2 = \omega^2 (\epsilon - \epsilon_o)\mu_o + \chi_o^2. \tag{15b}$$

Equation (15a) defines the propagation constant h_o of the uncoupled reactive surface.

Let us now consider

$$\exp(2i\chi_o D) \equiv \delta \tag{16}$$

as a small parameter and set

$$\begin{aligned} \chi &= \chi_o + \chi_1 \delta + \cdots, \\ g &= g_o + g_1 \delta + \cdots, \end{aligned} \tag{17}$$

in eqs. (14) and (10b). Collecting terms of first order in δ we get

$$\chi_1 = 2i\alpha(i\alpha - g_o)/(i\alpha + g_o). \tag{18}$$

From eqs. (10a) and (17) we have, to first order,

$$\mathrm{Im}(h) = -(\alpha\delta/h_o) \,\mathrm{Re}\,(\chi_1). \tag{19}$$

Thus the loss $\mathcal{L} \equiv \mathrm{Im}(h)$ is

$$\mathcal{L} = 4\alpha^3 u^{-2} h_o^{-1} g_o \exp(-2\alpha D), \tag{20a}$$

or, explicitly, in terms of k, n, D, and α,

$$\mathcal{L} = 4\alpha^3[k^2(n^2 - 1)]^{-1}(k^2 + \alpha^2)^{-\frac{1}{2}} \times [k^2(n^2 - 1) - \alpha^2]^{\frac{1}{2}} \exp(-2\alpha D). \tag{20b}$$

If the micron is used as the unit of length, the loss in dB/km is obtained by multiplying the r.h.s. of eq. (20b) by 8.7×10^9.

This expression for the loss, applicable to small couplings, can be obtained alternatively from the equality

$$h - h_o = \omega \int (\epsilon - \epsilon_o)\mathbf{E}^+ \cdot \mathbf{E}_p dS \Big/ \int (\mathbf{E}^+ \times \mathbf{H}_p - \mathbf{E}_p \times \mathbf{H}^+) \cdot \mathbf{d}S, \tag{21}$$

where $(\mathbf{E}, \mathbf{H})$ and h_o denote the field and propagation constant of the wave guided by the reactive surface in the absence of the dielectric and $(\mathbf{E}^+, \mathbf{H}^+)$ denotes the field adjoint to $(\mathbf{E}, \mathbf{H})$ (see Part I). $(\mathbf{E}_p, \mathbf{H}_p)$ and h denote the field and propagation constant in the presence of the dielectric. The integral in the numerator extends to the dielectric cross section, and the integral in the denominator extends to the whole cross section. Equation (21) is exact and is readily obtained from Maxwell's equations.* The field $(\mathbf{E}_p, \mathbf{H}_p)$, unfortunately, is not known. It may differ considerably from the unperturbed field $(\mathbf{E}, \mathbf{H})$ when the dielectric supports modes almost synchronous with the waveguide mode. This is why this expression, eq. (21), is, in general, not practical to evaluate the coupling between waveguides, or waveguides and mode sinks. The configuration presently considered, however, is sufficiently simple to be handled on the basis of eq. (21).

For our case, eq. (21) becomes, with the approximation $h \approx h_o$,

$$h - h_o \approx -(\omega^2\mu_o/2h_o)\int_0^\infty (\epsilon - \epsilon_o)EE_p dx \Big/ \int_{-D}^\infty E^2 dx. \tag{22}$$

The unperturbed field, normalized to unity at $x = -D$, is

$$E = \exp(i\chi x)\exp(i\chi D). \tag{23}$$

The perturbed field is obtained by assuming as before an $\exp(igx)$ dependence in the dielectric, matching E and dE/dx at the vacuum-dielectric interface $(x = 0)$, and stating that $E_p \approx 1$ at $x = -D$. We obtain

$$E_p = 2(1 + g/\chi)^{-1}\exp(i\chi D)\exp(igx), \qquad x \geqq 0. \tag{24}$$

Substituting in eq. (22) and integrating, a result identical to eq. (20) is obtained.

Let us now apply to the same problem the method explained in Section II of this paper, which consists in adding the losses associated with each mode of the substrate. The coupling coefficient between two H modes, with fields E and E_s, was given in Part I. With our present notation we have

$$C^2 = \alpha^2 h_o^{-2}\left(E^2 \Big/ \int E^2 dx\right)\left(E_s^2 \Big/ \int E_s^2 dx\right), \tag{25}$$

where the integrals are over the whole cross section, and E, E_s are defined at some point located between the two waveguides.

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* The contribution at infinity is assumed to vanish. Thus, it is implicitly assumed that the rate of decay of the unperturbed field exceeds the rate of growth of the perturbed field. This condition is always satisfied for small couplings.

The field E of the reactive surface alone is, as we have seen,

$$E = \exp(-\alpha x). \tag{26}$$

Thus, at $x = 0$,

$$E^2 \Big/ \int_{-D}^{\infty} E^2 dx = 2\alpha \exp(-2\alpha D). \tag{27}$$

Let us consider next the dielectric alone and first assume that its thickness L_x is finite. By matching E and dE/dx at $x = 0$ and $x = L_x$, we obtain the field at the vacuum dielectric interface, and

$$E_s^2 \Big/ \int_{-\infty}^{+\infty} E_s^2 dx \approx E_s^2 \Big/ \int_0^{L_x} E_s^2 dx = 2g_o^2 u^{-2} L_x^{-1}. \tag{28}$$

Substituting eqs. (27) and (28) in eq. (25), we obtain the coupling coefficient

$$C^2 = 4\alpha^3 g_o^2 u^{-2} h_o^{-2} \exp(-2\alpha D) L_x^{-1}. \tag{29}$$

Let us now evaluate the number of modes ($N\ dh$) in the dielectric whose propagation constants lie between h and $h + dh$. Because we are far from cut-off, the boundary condition is almost the same as for a metallic waveguide, $E = 0$. Thus, the condition on g is

$$g_m = m\pi/L_x, \qquad m = 1, 2, \cdots. \tag{30}$$

Using the relation

$$g_m^2 = \omega^2 \epsilon \mu_o - h^2, \tag{31}$$

the mode number density is, from eq. (30),

$$N = hg^{-1}L_x/\pi. \tag{32}$$

The radiation loss is obtained from eqs. (29), (32), and (7), and $h = h_o$, $g = g_o$,

$$\mathfrak{L} = \pi C^2 N = 4\alpha^3 u^{-2} h_o^{-1} g_o \exp(-2\alpha D). \tag{33}$$

This result coincides with the result eq. (20) obtained by taking the limit of large D in the exact solution. The variation of the loss expressed in dB/km is given in Fig. 2 as a function of the normalized susceptance α of the surface, for $\lambda = 1\ \mu$m, $\epsilon/\epsilon_o = 2$, and $D = 1.5$, 1.75, and 2 μm.

For comparison, when the dielectric permittivity has the form $\epsilon = \epsilon_o + i\epsilon_i$ (the dielectric is perhaps a lossy foam) and the spacing D is chosen as large as consistent with a loss of 10 dB/km at $\alpha = 6.28$, the loss experienced is shown on the same figure as a dotted line. The comparison clearly shows the advantage of mode sinking over dissipation for mode selection.

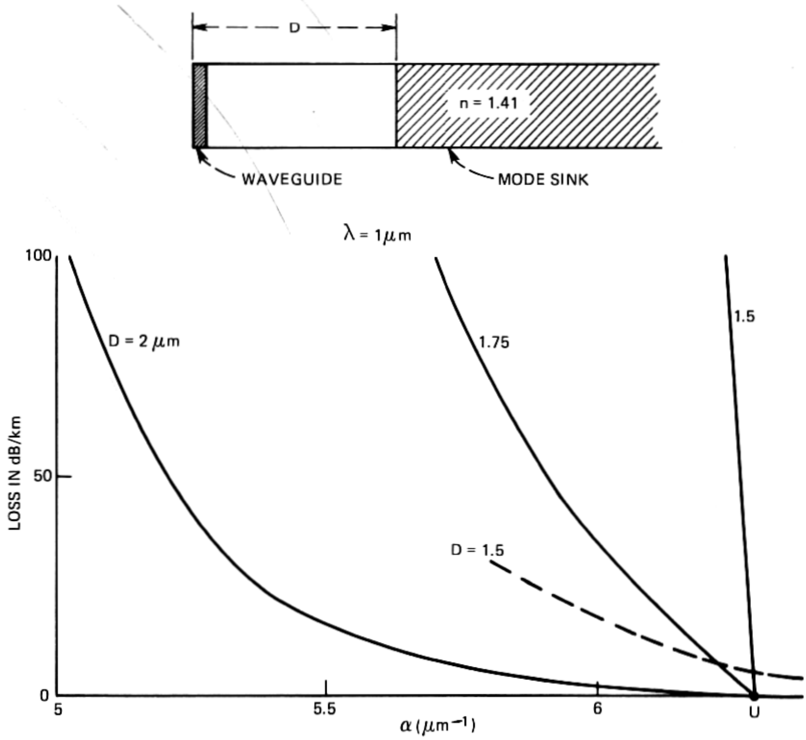

Fig. 2—Radiation loss in dB/km as a function of the normalized surface susceptance α of the waveguide for a wavelength $\lambda = 1\ \mu m$, $n^2 = 2$, and $D = 1.5$, 1.75, and $2\mu m$. The dotted line is applicable to a dissipative dielectric.

IV. COUPLING TO PLANAR SUBSTRATES

Let us now consider a waveguide with propagation constant h_1 coupled to a substrate that extends to infinity in the y direction, but has a finite thickness in the x direction. This substrate is perhaps a reactive plane (e.g., a corrugated conductor) or a dielectric slab, as illustrated in Fig. 3. In any case, homogeneity of the substrate in the y, z plane is assumed.

Because of the assumed homogeneity of the substrate, plane wave solutions

$$E_s(x, y, z) = E_s(x) \exp (ih_{sy}y + ih_{sz}z), \tag{34}$$

where

$$h_{sz} = f(h_{sy}, \omega), \tag{35}$$

exist at some angular frequency ω (ω is now considered a fixed parameter and is omitted).

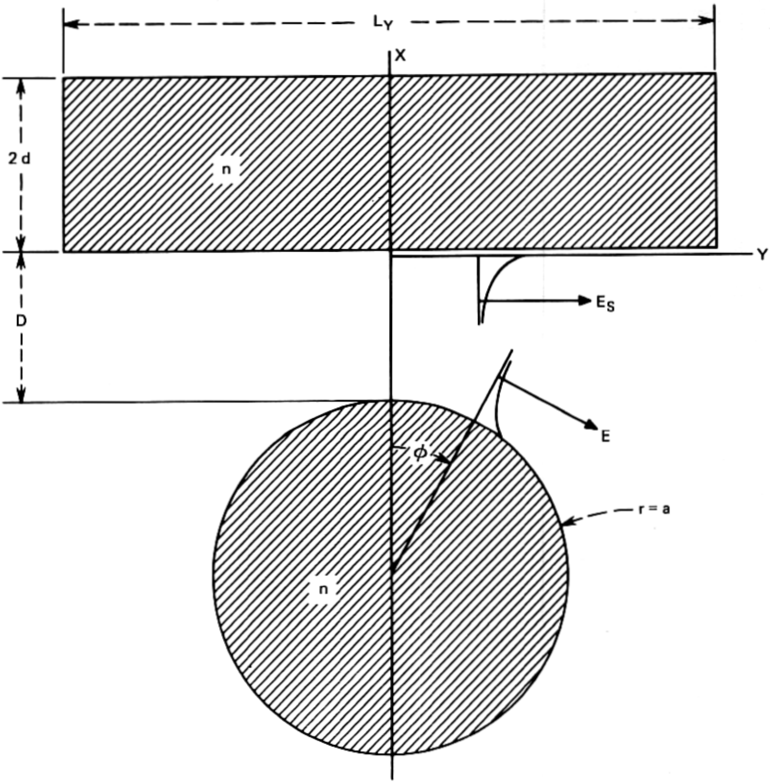

FIG. 3—Dielectric rod coupled to a dielectric slab. The rod field E is shown for the spurious H_{01} mode, and the slab mode is $H_{1\nu}$ (ν is a continuous index in the limit $L_y \to \infty$). Coupling takes place at $\phi \approx 0$.

In the discussion that follows, we consider only waveguide and substrate modes that are even in y. Assuming that f is even in h_{sy} and that the slab is terminated by electric walls, even modes satisfy the relation

$$h_{sy}L_y = 2n\pi, \qquad n = 0, 1, 2 \cdots, \tag{36}$$

where L_y denotes the width of the substrate. L_y will be later assumed to tend to infinity. The density N of even modes is from eqs. (35) and (36)

$$N = (df/dh_{sy})^{-1}(L_y/2\pi). \tag{37}$$

If the substrate is isotropic, with wave vector h_s, eq. (35) is

$$h_{sz} \equiv f(h_{sy}) = (h_s^2 - h_{sy}^2)^{\frac{1}{2}}, \tag{38}$$

and the mode density is, from eq. (37),

$$N = (h_{sz}/h_{sy})L_y/2\pi. \tag{39}$$

The loss is then obtained from eqs. (7) and (39).

$$\mathfrak{L} = \tfrac{1}{2}[h_1/f^{-1}(h_1)]C^2L_y, \tag{40}$$

the coupling coefficient C^2 being evaluated from eq. (6) in Part I.

It should be noted that, when the propagation constant of the waveguide mode (h_1) is just equal to the propagation constant (h_s) of the 2-dimensional substrate, h_{sy} is equal to zero and the loss, according to eq. (40), is infinite if C^2L_y remains finite. (This was not the case for the 3-dimensional mode sinks considered in Section III because, as $L_x \rightarrow \infty$, the field at the surface of the dielectric tends sufficiently rapidly to zero to make C^2L_x vanish in the limit.) This infinity at $h_1 = h_s$ would be removed if some finite dissipation loss in the substrate were present. Even in the absence of dissipation losses, the radiation loss remains finite at $h_1 = h_s$, because the perturbation method on which eq. (40) is based is no longer applicable. The peak in the loss curve predicted by eq. (40) (analogous to a sound barrier) is pronounced only for small couplings.

Our general result, eq. (40), is now applied to a dielectric rod coupled to a dielectric slab. The thickness and permittivity of the slab can always be chosen in such a way that only the fundamental mode of the rod propagates without radiation loss. The calculation of the loss of higher-order modes is carried out for the case where the rod diameter and the slab thickness are very large compared with the wavelength; that is, when the rod is highly multimoded in the absence of coupling.

Approximate expressions for the modes and propagation constant in the slab and the rod are given in the next subsections.

4.1 Modes of the slab

Let us consider first the modes in the dielectric slab. If the thickness $2d$ of the slab is very large (more precisely, if $\omega^2(\epsilon - \epsilon_o)\mu_o d^2 \gg 1$), the propagation constant of the fundamental H_1 mode is approximately given by the condition that the field E vanishes at the boundary

$$E(x, z') \approx E_{so} \cos (g_s x) \exp (ih_s z').$$

Thus, we have*

$$g_s^2 \equiv \omega^2 \epsilon \mu_o - h_s^2 = (\pi/2d)^2. \tag{41}$$

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* A more accurate and general expression is (see Ref. 5) $g_s d = m(\pi/2)(1 - V^{-1})$ for H modes and $g_s d = m(\pi/2)(1 - n^{-2}V^{-1})$ for E modes, where $m = 1, 2 \cdots$ is the mode number and $V \equiv ud$. These expressions show that the H_1 mode that we are considering in this section is the fundamental mode; that is, the mode that has the largest propagation constant. The difference Δh in propagation constants is, for $m = 1$, equal to $d^{-1}(\pi/2knd)^2(1 - 1/n^2)^{\frac{1}{2}}$.

(The axial coordinate is denoted z' instead of z to avoid changing our notation when waves propagating at some angle to the z axis are considered. The origin of the x axis is, in this subsection, at the center of the slab.) The axial (z') and transverse (x) components of the magnetic field are, within the slab, as we have seen before

$$H_x = -h_s(\omega\mu_o)^{-1}E, \tag{42}$$

$$H_{z'} = (i\omega\mu_o)^{-1}dE/dx, \tag{43}$$

and the power per unit width is approximately

$$P \approx -\int_{-d}^{+d} EH_x dx = dh_s(\omega\mu_o)^{-1}E_{so}^2. \tag{44}$$

The field at the boundary is in fact not exactly equal to zero. To obtain its value, we use the fact that the dependence of E on x in vacuum is exp $(-p_s x)$, where $p_s^2 = h_s^2 - \omega^2\epsilon_o\mu_o$, and the continuity of dE/dx. We obtain

$$E(d) = (\pi/2d)p_s^{-1}E_{so}. \tag{45}$$

Now let the slab have a finite width L_y with electric walls at $y = \pm L_y/2$. The modes even in y can be described as a superposition of two infinite slab waves whose propagation constants are such that

$$h_{sy} = \pm 2\pi n/L_y, \qquad n = 0, 1, 2, \cdots. \tag{46}$$

We have, by definition,

$$h_{sy}^2 + h_{sz}^2 = h_s^2, \tag{47}$$

h_s being given in eq. (41).

The field has all its components different from zero with the exception of E_x, which vanishes. The components E_y and H_z are obtained by adding the field of the two waves. We obtain

$$E_{sy} = 2h_{sz}h_s^{-1} \cos (h_{sy}y) \cos (\pi x/2d)E_{so}, \tag{48}$$

$$H_{sz} = -2(i\omega\mu_o)^{-1}h_{sz}h_s^{-1}(\pi/2d) \cos (h_{sy}y) \sin (\pi x/2d)E_{so}. \tag{49}$$

The energy flowing through the slab is obtained by multiplying P, given in eq. (44), by $2h_{sz}h_s^{-1}L_y$

$$P_s = 2h_{sz}(\omega\mu_o)^{-1}dL_yE_{so}^2. \tag{50}$$

The y component of the field at the boundary ($x = d$) is obtained from eq. (45) or directly from $H_{sz} = (i\omega\mu_o)^{-1}\partial E/\partial x$:

$$E_{sy}(d) = -p_s^{-1}i\omega\mu_o H_{sz}. \tag{51}$$

4.2 Modes of the rod

Let us now turn our attention to the modes of the dielectric rod. We assume that the radius a of the rod is much larger than the wavelength ($a \gg \lambda$).

In the limit of large radii, the propagation constant of the fundamental HE_{11} mode is given (see the appendix) by the first root of $J_0(g_o a)$, namely,

$$g_o a \equiv (\omega^2 \epsilon \mu_o - h_o^2)^{\frac{1}{2}} a = 2.4 \cdots, \qquad a \to \infty. \tag{52}$$

The next higher order mode of the dielectric rod is the H_{01} mode.* In the limit of large radii, the boundary condition at $r = a$ is $E_\phi = 0$, as for a round metallic pipe. The propagation constant h_1 is therefore given by

$$J_1(g_1 a) = 0, \tag{53}$$

whose first root is

$$g_1 a \equiv (\omega^2 \epsilon \mu_o - h_1^2)^{\frac{1}{2}} a = 3.8 \cdots, \qquad a \to \infty. \tag{54}$$

Within our approximation, the field of the H_{01} mode in the rod ($r < a$) has components

$$\begin{aligned} E_\phi &= J_1(g_1 r), \\ H_r &= -h_1(\omega\mu_o)^{-1} J_1(g_1 r), \\ H_z &= (i\omega\mu_o)^{-1} g_1 J_0(g_1 r), \end{aligned} \tag{55}$$

and the energy flow is

$$P = -\int_0^a E_\phi H_r 2\pi r dr = \pi h_1 (\omega\mu_o)^{-1} a^2 J_0^2(g_1 a). \tag{56}$$

To obtain the field E_ϕ at the boundary ($r = a$), we use the fact that dE/dr is continuous and that the r dependence of E_ϕ in vacuum is approximately† $\exp(-p_1 r)$ where $p_1^2 \equiv h_1^2 - \omega^2 \epsilon_o \mu_o$. We obtain

$$E_\phi(a) = p_1^{-1} i\omega\mu_o H_z. \tag{57}$$

4.3 Synchronization conditions

For simplicity and because this is a case of practical significance, we assume that the rod and the slab have the same permittivity ϵ.

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* The E_{01} and HE_{21} modes have almost the same propagation constant as the H_{01} mode for large rod radii. For small radiation losses, they can be considered independently of the H_{01} mode (see appendix).

† The exact dependence of E_ϕ on r is $K_0(p_1 r)$, where K_0 denotes the modified Bessel function of the second kind. For large arguments, $K_0(x) \approx (2/\pi x)^{\frac{1}{2}} \exp(-x)$ and $K_0'(x) \approx -(2/\pi x)^{\frac{1}{2}} \exp(-x) \approx -K_0(x)$.

The fundamental HE_{11} mode of the rod is free of radiation loss if its propagation constant h_o given in eq. (52) is slightly larger than the propagation constant h_s in the slab given in eq. (41). For simplicity, we set $h_s = h_o$ or, equivalently, $g_s = g_o$; that is,

$$\pi/2d = 2.4/a, \tag{58a}$$

or

$$d = 0.65a. \tag{58b}$$

Thus the ratio of the slab thickness to rod diameter is 0.65. (In practice, the slab has finite dissipation losses and a finite width. Furthermore, it is difficult to control accurately the thickness of the slab. For these reasons, it might be preferable to choose the value of h_s midway between the propagation constants of the HE_{11} and H_{01} modes rather than equal to the propagation constant of the HE_{11} mode. If the former condition were to hold, we would find that the slab thickness should be equal to half the rod diameter.) Figure 4 gives the propagation constants of the rod and the slab for $n = 1.41$ and a rod radius of 10 μm ($\lambda = 1$ μm).

Let us now consider one of the next higher order modes of the rod, the H_{01} mode. This mode radiates into the substrate modes that have the same propagation constant along the z axis ($h_{sz} = h_1$). Using eq. (54), we obtain

$$\omega^2 \epsilon \mu_o - h_{sz}^2 = (3.8/a)^2. \tag{59}$$

Since

$$h_{sz}^2 + h_{sy}^2 = h_s^2, \tag{60}$$

and h_s has the value h_o given in eq. (52), we have

$$h_{sy}^2 = (3.8/a)^2 - (2.4/a)^2, \tag{61}$$

or

$$h_{sy} = 3.0/a. \tag{62}$$

In the next subsection we evaluate the coupling coefficient between the H_{01} mode of the rod and the substrate mode defined by eq. (62).

4.4 Coupling coefficient

The contour of integration for the evaluation of the coupling coefficient being arbitrary, it is convenient to choose this contour as the rod boundary, $r = a$. Along that contour, the H_{01} mode field is a constant.

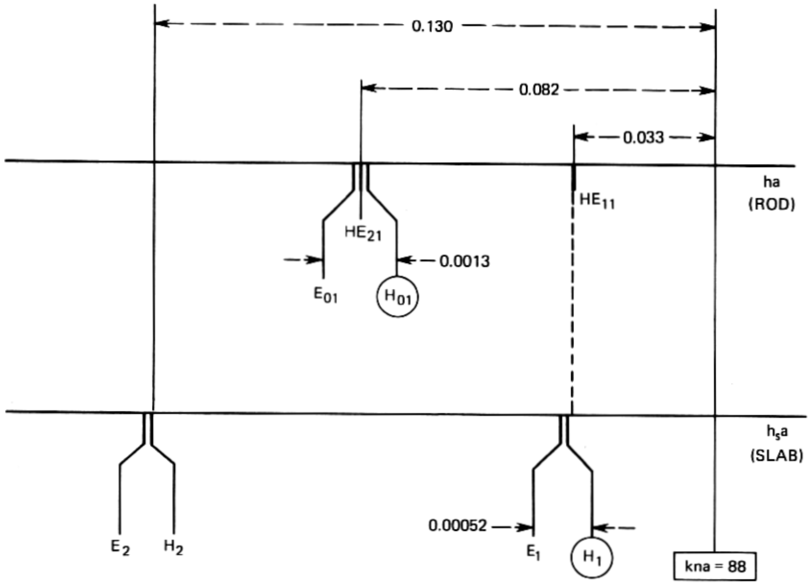

Fig. 4—Propagation constants (h) of the trapped modes of the rod and maximum value (h_s) of the propagation constants of the radiation modes in the slab. It is assumed that $n = 1.41$, $\lambda = 1\ \mu$m, and $a = 10\ \mu$m. The modes circled are those whose coupling is discussed in this paper.

Let ϕ denote the angle from the x axis shown in Fig. 3 and D the spacing between the rod and the slab. We have

$$\begin{aligned} x &= -D - a(1 - \cos \phi), \\ y &= a \sin \phi. \end{aligned} \tag{63}$$

Because $a \gg \lambda$, the coupling takes place near the point of closest approach of the rod to the slab; that is, $\phi \approx 0$. We can therefore write

$$\begin{aligned} x &\approx -D - a\phi^2/2, \\ y &\approx a\phi. \end{aligned} \tag{64}$$

The y dependence of the field slab is $\cos (h_{sy}y) = \cos (h_{sy}a\phi)$. However, since, according to eq. (62), h_{sy} is of the order of a^{-1}, the argument of the cosine function is small compared with unity in the range where the coupling is significant. Thus, we can neglect the dependence of the field of the slab on y. This approximation could be relaxed with little additional complication.

Using the above approximation, we obtain for the field of the H_{01} mode (rod) at $r = a$, from eqs. (55), (56), and (57),

$$H_z = (i\omega\mu_o)^{-1} g_1 J_0(g_1 a), \tag{65a}$$

$$E_\phi = p_1^{-1} i\omega\mu_o H_z, \tag{65b}$$

$$P = \pi h_1 (\omega\mu_o)^{-1} a^2 J_0^2(g_1 a), \tag{65c}$$

where

$$\begin{aligned} g_1^2 &\equiv \omega^2 \epsilon\mu_o - h_1^2 = (3.8/a)^2, \\ p_1^2 &\equiv h_1^2 - \omega^2 \epsilon_o\mu_o \approx \omega^2(\epsilon - \epsilon_o)\mu_o \equiv u^2. \end{aligned} \tag{66}$$

For the slab we have, at $r = a$, from eqs. (49), (50), and (51), setting the arbitrary constant E_{so} equal to unity, $h_{sz} \approx h_s$ and taking into account the exp $(p_s x)$ dependence of the field below the slab

$$H_{sz} = -2(i\omega\mu_o)^{-1}(\pi/2d) \exp\left[-p_s(D + a\phi^2/2)\right], \tag{67a}$$

$$E_{sy} = -p_s^{-1} i\omega\mu_o H_{sz}, \tag{67b}$$

$$P_s = 2h_s(\omega\mu_o)^{-1} d L_y, \tag{67c}$$

with

$$\begin{aligned} h_s &\approx h_1 \approx kn, \\ p_s &\approx p_1 \approx u = k(n^2 - 1)^{\frac{1}{2}}, \\ \pi/2d &= 2.4/a. \end{aligned} \tag{68}$$

The coupling coefficient C^2 is c^2/PP_s, where

$$c = a \int_{-\pi}^{+\pi} \left[E_\phi H_{sz} - E_{sy} \cos(\phi) H_z\right] d\phi. \tag{69}$$

From eqs. (65) and (67), it is apparent that the two terms in the integrand in eq. (69) are equal and add up if we make the approximation $\cos\phi \approx 1$. Thus,

$$c \approx 2aE_\phi \int_{-\infty}^{+\infty} H_{sz} d\phi. \tag{70}$$

Using eq. (67a) for H_{sz}, we have

$$\int_{-\infty}^{+\infty} H_{sz} d\phi = -2(i\omega\mu_o)^{-1}(\pi/2d) \exp(-p_s D)(2\pi/p_s a)^{\frac{1}{2}}, \tag{71}$$

if we make use of the identity

$$\int_{-\infty}^{+\infty} e^{-bx^2} dx = (\pi/b)^{\frac{1}{2}}. \tag{72}$$

Thus,

$$c = 4ap_1^{-1} g_1 (i\omega\mu_o)^{-1}(\pi/2d) J_0(g_1 a) \exp(-p_s D)(2\pi/p_s a)^{\frac{1}{2}}, \tag{73}$$

and

$$C^2 = (32/\pi) g_1^2 (\pi/2d)^3 (u^3 h^2 a L_y)^{-1} \exp(-2uD). \tag{74}$$

Since the mode number density is given by eq. (40), the loss

$$\mathfrak{L} = \tfrac{1}{2}(h_{sz}/h_{sy}) C^2 L_y \tag{75}$$

is finally obtained from eqs. (74), (62), (66), and (68),

$$\mathfrak{L} = 340 n^{-1} (n^2 - 1)^{-\frac{3}{2}} (k^4 a^5)^{-1} \exp\left[-2(n^2 - 1)^{\frac{1}{2}} kD\right]. \tag{76}$$

The loss in dB/km is obtained by multiplying the r.h.s. of eq. (76) by 8.7×10^9, the μm being used as the unit of length. Thus, for $n = 1.41$ and $n = 1.01$ we have, respectively,

$$\mathfrak{L}_{\mathrm{dB/km}} = 1.35 \times 10^9 \lambda_{\mu m}^{-1} (a/\lambda)^{-5} \exp(-12.5 D/\lambda), \qquad n = 1.41, \tag{77}$$

$$\mathfrak{L}_{\mathrm{dB/km}} = 675 \times 10^9 \lambda_{\mu m}^{-1} (a/\lambda)^{-5} \exp(-1.76 D/\lambda), \qquad n = 1.01. \tag{78}$$

For example, if $D = 0.15$ μm, $n = 1.41$, $\lambda = 1$ μm and $a = 40$ μm, we find that the radiation loss of the H_{01} mode is $\mathfrak{L} = 2$ dB/km. If $D = 1$ μm, $n = 1.01$, $\lambda = 1$ μm, and $a = 40$ μm, the loss is as high as 1140 dB/km. The radiation loss is shown as a function of a/λ and D/λ in Figs. 5 and 6 for a wavelength of 1 μm, and for $n = 1.41$ and 1.01, respectively. The amount of loss required to prevent the power

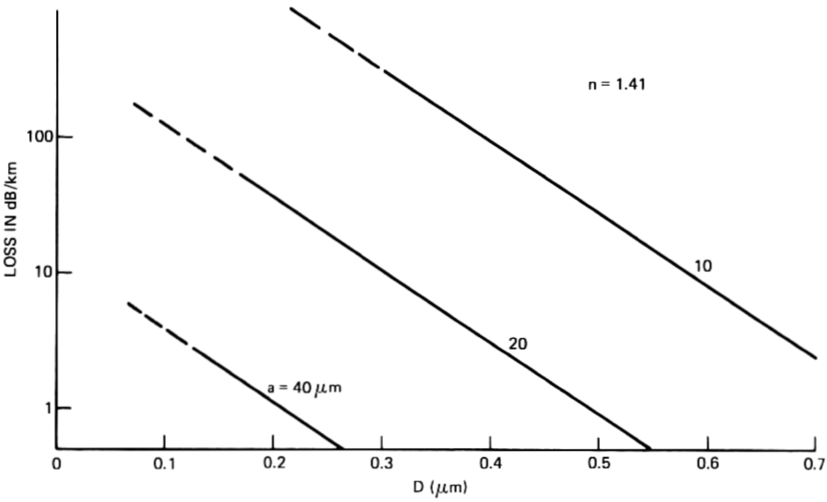

Fig. 5—Radiation loss in dB/km of the rod H_{01} mode in the slab as a function of spacing D with the rod radius a as a parameter, for $n_{\mathrm{rod}} = n_{\mathrm{slab}} = 1.41$. These curves are valid for large values of D.

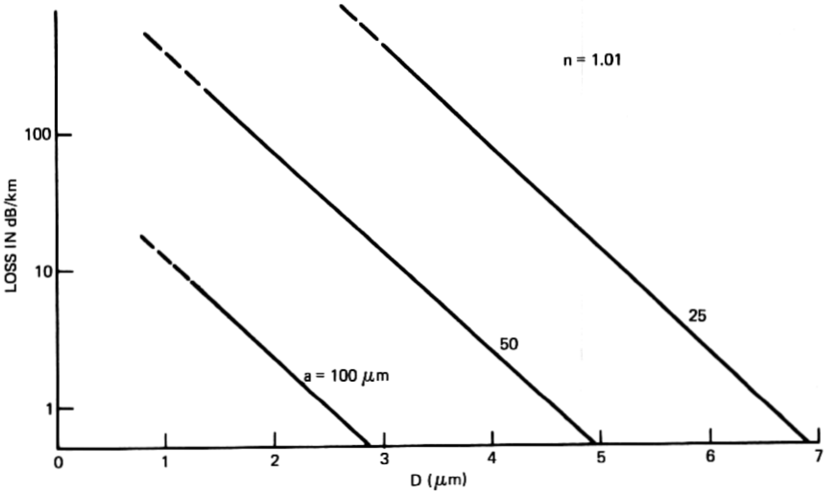

Fig. 6—Continuation of Fig. 5 for $n = 1.01$.

transferred to the H_{01} mode to be transferred back to the HE_{11} mode and to cause pulse spreading depends on the fiber irregularities and is not accurately known.

The above results are approximate and, to some extent, incomplete. In particular, the perturbation method that we used is not accurate when D is small. Also it would be useful to ascertain that the radiation losses of the other higher-order modes are at least equal to the loss calculated for the H_{01} mode. For some of these higher-order modes of the rod, it is necessary to take into account the higher-order modes of the substrate, both E and H, and this involves some complication.[9] In spite of these limitations, our result, eq. (76), should provide preliminary information concerning the mode-selection mechanism afforded by 2-dimensional mode sinks. In particular, the very fast dependence of the loss on the rod radius (a^{-5}) indicates that very large rods cannot be used if single-mode operation is to be achieved in air. However, if the gap between the rod and the slab is filled up with a material whose permittivity is only slightly smaller than the rod and slab permittivities, the rod radius a and the spacing D can be large, as Fig. 6 suggests.

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APPENDIX

Limit forms of the propagation constants in optical fibers

Two approximations can be made, applicable to low-order modes in highly multimoded fibers and to fibers with small transverse variation of permittivity. A simplified presentation is given in this appendix.

Low-order modes propagating in highly multimoded fibers correspond to waves propagating almost along the axial direction, z. The propagation constant h is therefore close to kn if n denotes the refractive index on axis. If the fiber refractive index is a constant within some contour and assumes a lower value outside that contour, the wave near a section of the contour can be assumed plane. Because it is incident at grazing angles, the electric and magnetic fields tend to zero compared with their values in the bulk. Thus, the electric and magnetic fields at the boundary of a dielectric rod vanish, compared to their values on axis, as the transverse dimensions of the rod tend to infinity for a given mode number.

For a round fiber with refractive index n and radius a, the exact equation defining h is, using the notation of the main text,[6]

$$\left[\frac{n^2 J_\nu'(u_1)}{u_1 J_\nu(u_1)} + \frac{K_\nu'(u_2)}{u_2 K_\nu(u_2)}\right]\left[\frac{J_\nu'(u_1)}{u_1 J_\nu(u_1)} + \frac{K_\nu'(u_2)}{u_2 K_\nu(u_2)}\right] = \left[\frac{\nu h}{k}\frac{(u_1^2 + u_2^2)}{u_1^2 u_2^2}\right]^2, \quad (79)$$

the axial and azimuthal variations of the field being denoted $\exp\,(ihz + i\nu\phi)$ and

$$\begin{aligned} u_1 &\equiv ga \equiv (k^2 n^2 - h^2)^{\frac{1}{2}} a, \\ u_2 &\equiv pa \equiv (h^2 - k^2)^{\frac{1}{2}} a, \\ k &\equiv \omega(\epsilon_o \mu_o)^{\frac{1}{2}}. \end{aligned} \quad (80)$$

In the limit $a \to \infty$, u_2 tends to infinity and the second terms in the brackets on the l.h.s. of eq. (79) vanish $[K_\nu'(x)/K_\nu(x) \to -1$ if $x \to \infty]$. On the r.h.s. of eq. (79), h can be replaced by kn. Thus, it is apparent that eq. (79) becomes

$$J_\nu'(u_1)/J_\nu(u_1) = \pm\, \nu/u_1, \quad (81)$$

or, equivalently, using well-known formulas involving Bessel's functions and their derivatives:*

$$J_{\nu\pm 1}(u_1) = 0. \quad (82)$$

* We have $\nu J_\nu \mp x J_\nu' = x J_{\nu\pm 1}$ and (for later use) $\nu K_\nu \pm x K_\nu' = \mp x K_{\nu\pm 1}$.

Table I

	ν	-2	-1	0	1	2		
$\nu < 0$	EH:	$J_{-3} = 0$	$J_{-2} = 0$	$J_{-1} = 0$	$J_0 = 0$	$J_1 = 0$:	HE	$\nu > 0$
		$(\mu = -2)$	$(\mu = -1)$	$(\mu = 0)$	$(\mu = 1)$	$(\mu = 2)$		
	HE:	$J_{-1} = 0$	$J_0 = 0$	$J_1 = 0$	$J_2 = 0$	$J_3 = 0$:	EH	

[Note: $J_{-\nu} = 0 \Longleftrightarrow J_\nu = 0$]

For symmetry reasons, modes with opposite values of ν have the same propagation constants. For $\nu = 2$, for instance, the propagation constants of the two sets of modes are given by the roots J_1 and J_3. For $\nu = -2$, they are given by the roots J_{-3} and J_{-1}. However, these are the same because $J_{-\nu} = (-)^\nu J_\nu$. Equation (82) was given by Snitzer.[7] For the HE_{11} ($\nu = 1$) and H_{01} ($\nu = 0$) modes, the relevant solutions of eq. (82) are the first roots of

$$J_0(u_1) = 0, \qquad u_{10} = 2.4\cdots \tag{83a}$$

and

$$J_1(u_1) = 0, \qquad u_{10} = 3.8\cdots. \tag{83b}$$

These are the results used in the main text.

Because the modes H_{01}, E_{01}, and HE_{21} have almost the same propagation constants (see Table I), the validity of the calculations given in the main text can be questioned where the mode H_{01} was considered independently of the two other modes. It is therefore important to evaluate the actual splitting between these three modes. For simplicity, we consider only the H_{01} and E_{01} modes. The expressions giving the exact propagation constants of the H_{01} and E_{01} modes are, setting $\nu = 0$ in eq. (79),

$$J_1(u_1)/u_1J_0(u_1) = -K_1(u_2)/u_2K_0(u_2), \qquad (H_0), \tag{84}$$

and

$$J_1(u_1)/u_1J_0(u_1) = -n^2K_1(u_2)/u_2K_0(u_2), \qquad (E_0). \tag{85}$$

Setting

$$u_1 = u_o + \delta, \qquad \delta \ll 1, \tag{86}$$

where

$$J_1(u_o) = 0, \qquad u_o = 3.8\cdots \tag{87}$$

on the l.h.s. of eqs. (84) and (85) and

$$u_2 = k(n^2 - 1)^{\frac{1}{2}}a \tag{88}$$

on the r.h.s., we obtain for the difference Δh in propagation constant between the H_{01} and E_{01} modes

$$\Delta ha = (3.8/kna)^2(1 - 1/n^2)^{\frac{1}{2}}. \tag{89}$$

Except for a numerical factor, this result is the same as for a slab (see Section 4.1). If $a = 10\ \mu$m, $n = 1.41$, and $\lambda = 1\ \mu$m, the beat wavelength $2\pi/\Delta h$ is, from eq. (89), equal to 5 cm.

The individuality of the H_{01} mode is preserved and the calculations given in the main text are valid if the loss $\mathcal{L}$ is small over that length (e.g., $\mathcal{L} \ll 1$ dB/cm for $a = 10\ \mu$m). In fact, this restriction on $\mathcal{L}$ may be even less stringent than that calculated above because the degeneracy between the three modes may be lifted further by the presence of the slab when the coupling is increased.

The second approximation referred to at the beginning of this appendix is the scalar approximation widely used in optics. If the transverse variations of the medium permittivity are small, the x and y components of the field satisfy approximately the scalar Helmholtz equation

$$(\partial^2/\partial x^2 + \partial^2/\partial y^2)E_x + [k^2n^2(x, y) - h^2]E_x = 0. \tag{90}$$

A similar equation holds for E_y, which need not be written down.

Because all quantities are bounded in eq. (90), E_x and its first derivatives are continuous functions of x and y.

For the rod considered earlier, eq. (90) becomes, assuming an $\exp(i\mu\phi)$ dependence of E_x on ϕ,

$$\begin{aligned} d^2E_x/dr^2 + r^{-1}dE_x/dr + (k^2n^2 - h^2 - \mu^2/r^2)E_x &= 0, \qquad r < a, \\ d^2E_x/dr^2 + r^{-1}dE_x/dr + (k^2 - h^2 - \mu^2/r^2)E_x &= 0, \qquad r > a. \end{aligned} \tag{91}$$

These are differential equations for Bessel functions. The bounded solutions of eq. (91) are

$$\begin{aligned} E_x &= J_\mu(gr), \qquad g^2 \equiv k^2n^2 - h^2, \qquad r < a \\ E_x &= AK_\mu(pr), \qquad p^2 \equiv h^2 - k^2, \qquad r > a. \end{aligned} \tag{92}$$

Continuity of E_x and dE_x/dr imposes

$$J_\mu(u_1)/K_\mu(u_2) = (u_1/u_2)J_\mu'(u_1)/K_\mu'(u_2), \tag{93}$$

or, using the transformation formulas given before,

$$u_1J_{\mu+1}(u_1)/J_\mu(u_1) = u_2K_{\mu+1}(u_2)/K_\mu(u_2), \tag{94}$$

a result previously derived by Snyder[5] from the exact equation, eq.

(79). In the limit $a \to \infty$, eq. (94) reduces to

$$J_\mu(u_1) = 0, \tag{95}$$

in agreement with eq. (82). To each value of μ we must associate modes corresponding to the two states of polarization of the electromagnetic field. This is illustrated in Table I.

The physical significance of the scalar approximation is that if, for instance, a linearly polarized field, solution of eq. (90), is launched into a fiber, this field configuration is approximately maintained over a certain length. Eventually, however, the polarization is transformed because the two electromagnetic modes have slightly different real propagation constants as we have seen (for a report of experimental observations, see Ref. 8 in which the mode $\mu = \pm 1$ is illustrated in Figs. 3 and 4d) and/or different losses. The scalar approximation is useful to obtain approximate expressions for the propagation constants. This approximation is not applicable to the evaluation of radiation losses if these losses are polarization dependent. This is the case, for instance, if the propagation constant of the rod mode lies between the propagation constants of the slab E and H modes. Because the split between these two modes is very small, this is unlikely to happen unless the optical waveguide has been specially designed for that purpose. In that sense, the scalar approximation may be applied to problems of radiation losses.

REFERENCES

1. J. A. Arnaud, "Transverse Coupling in Fiber Optics—Part I: Coupling Between Trapped Modes," B.S.T.J., *53*, No. 2 (February 1974), pp. 217–224.
2. E. A. J. Marcatili, "Slab-Coupled Waveguides," B.S.T.J., this issue, pp. 645–674. Experimental results for the case $D = 0$ were reported by P. Kaiser, E. A. J. Marcatili, and S. E. Miller, "A New Optical Fiber," B.S.T.J., *52*, No. 2 (Feburary 1973), pp. 265–469.
3. J. A. Arnaud, "Note on the Use of Whispering Gallery Modes in Communication," September 1971, unpublished work.
4. V. V. Shevchenko, *Continuous Transitions in Open Waveguides*, Boulder, Colorado: The Golem Press, 1971, Chapter 2.
5. A. W. Snyder, "Asymptotic Expressions for Eigenfunctions and Eigenvalues of a Dielectric or Optical Waveguide," IEEE Trans. on Microwave Theory and Techniques, *MTT-17*, No. 12 (December 1969), pp. 1130–1138.
6. R. E. Collin, *Field Theory of Guided Waves*, New York: McGraw-Hill, 1960, p. 482.
7. E. Snitzer, "Cylindrical Dielectric Waveguide Modes," J. Opt. Soc. Amer., *51*, No. 5 (May 1961), pp. 491–498.
8. E. Snitzer and H. Osterberg, "Observed Dielectric Waveguide Modes in the Visible Spectrum," J. Opt. Soc. Amer., *51*, No. 5 (May 1961), pp. 499–505.
9. J. A. Arnaud, "Transverse Coupling in Fiber Optics—Part III: Bending Losses," to appear in B.S.T.J., *53*, No. 7 (September 1974).