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Mean-Squared-Error Equalization Using Manually Adjusted Equalizers

By Y.-S. CHO

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This paper describes an equalization procedure for systems using manually adjustable bump equalizers that is based on a mean-squared error criterion. We show that, in accordance with a Gauss-Seidel iteration process, gain adjustment always converges to the optimal value at which the minimum MSE of the equalized channel is obtained. Both zero forcing and MSE algorithms based on the Gauss-Seidel iteration method are derived, and hardware implementation of these algorithms is discussed. According to the error reduction analysis, an equalizer composed of orthogonal networks requires only one iteration to bring the equalizer to the optimum state. For the bump equalizers used in the latest L5 Coaxial Carrier Transmission System whose Bode networks are semi-orthogonal, two to three iterations are shown to be sufficient to achieve the optimum gain settings in the mean-squared error sense.

I. INTRODUCTION

In this paper, a manual adjustment of bump equalizers is described which uses a mean-squared error (MSE) criterion. In the existing L4 Coaxial Transmission System, the bump equalizers (realized with Bode equalizer networks[1]) are used for line equalization and adjusted according to a zero forcing (ZF) algorithm.[2] This method results in an optimum equalization in the MSE sense, but only under certain very restrictive conditions with respect to the transmission response of the channel. The latest L5 Coaxial Transmission System, which provides up to 10,800 toll-grade long-haul message channels on a pair of 0.375-inch coaxial cables over 4000 miles, also includes bump equalizers for equalization of the 65-MHz bandwidth channel. The equalization method used in the L5 Coaxial Carrier Transmission System, however, minimizes the MSE of the channel deviation. It is found in practice that the MSE algorithm has given a better equalization result than the ZF algorithm.

Since a coaxial cable system shows relatively stable channel characteristic, the bulk of the L5 line equalization is accomplished by manually adjustable equalizers. Normally, the time-varying channel deviation in a cable system is mostly the result of seasonal temperature variation and aging of the components in the system. In such case, a usage of complex automatic equalizers in the system is not economically desirable.

In Ref. 3, two MSE methods are discussed for the equalizer adjustment. Both methods are based on the steepest descent algorithm and could be easily implemented in an automatic equalizer, but not in a manual equalizer.

In this paper, an MSE algorithm based on the Gauss-Seidel iteration method is described for the gain adjustment of the manual bump equalizers. Under the specified assumptions, this method guarantees the convergence of the following iterative process. If a visual display of the gradient of the MSE with respect to each gain setting is available, adjust the first gain setting until the gradient becomes zero; next, adjust the second gain setting until the associated gradient becomes zero; similarly, adjust the third gain setting and all others up to the last one, thus completing one iteration. As the number of iterations increases, the residual error in the channel will be minimized in the MSE sense.

While the Gauss-Seidel iteration method may seem quite complicated, it has several distinct advantages. First of all, the Gauss-Seidel iteration method requires only one gradient at a time, which can simplify the hardware involvement, particularly for manual equalizers. As is shown in Section III, the number of iterations needed to bring the equalizers to the optimum state is not large. When the equalizer is composed of orthogonal networks, a single iteration is sufficient. Since most of the equalizer networks used in transmission systems are orthogonal or semi-orthogonal in nature, the number of iterations will usually be small. For the bump equalizers, which consist of semi-orthogonal terms, two to three iterations are satisfactory for the optimal equalization according to the error analysis given in Section III. This result has been verified experimentally in the field.

II. MSE ADJUSTING ALGORITHMS FOR BUMP EQUALIZERS

In this section, several assumptions are made before the ZF and MSE algorithms are presented.

2.1 Characterization of coaxial channel

The channel assumed in this section is represented by an infinite sin x/x series on the frequency domain. (Since the transfer function of the Bode network is symmetric on the log f plane, where f is the natural frequency in hertz, the frequency used throughout is defined by $w = \log f$.)

Let $M(w,\ t)$ represent the time-varying channel misalignment which is a real valued function of frequency in decibels. From the practical point of view, however, the channel can be assumed to be simply $M(w)$ since the time variation is negligible during the equalization interval. Further, assume that the Fourier transform of the channel is limited in the time domain by a certain positive constant. (It should be noted that the Fourier transform of the channel does not result in an impulse response of the channel because the channel is measured in dB. However, there is an implicit dual relationship between the channel studied in this paper and that involving a time-domain equalizer, e.g., a transversal equalizer, in which a frequency band limitation of the channel is implied.) Hence, the channel can be characterized on the frequency domain by the following series:

$$M(w) = \sum_{n=0}^{\infty} C_n \frac{\sin[2\pi p(w - w_n)]}{2\pi p(w - w_n)} \qquad \text{(dB)}, \tag{1}$$

where C_n, p, and w_n are real numbers and

$$w_{n+1} - w_n = \frac{1}{2p} \quad \text{for all } n = 0, 1, \cdots.$$

Note that w_n is equally spaced.

Equation (1) can also be written in the following way.

$$\begin{aligned} M(w) &= \int_0^1 \sum_{n=0}^{\infty} C_n \cos[2\pi p(w - w_n)x]dx \\ &= \int_0^1 \left\{ \sum_{n=0}^{\infty} C_n \cos(2\pi p w_n x) \cos(2\pi p w x) \right. \\ &\qquad \left. + \sum_{n=0}^{\infty} C_n \sin(2\pi p w_n x) \sin(2\pi p w x) \right\} dx \\ &= \int_0^1 \{F(x) \cos(2\pi p w x) + H(x) \sin(2\pi p w x)\} dx, \end{aligned} \tag{2}$$

where

$$F(x) = \sum_{n=0}^{\infty} C_n \cos(2\pi p w_n x) \quad \text{and} \quad H(x) = \sum_{n=0}^{\infty} C_n \sin(2\pi p w_n x).$$

Since $0 \leqq x \leqq 1$, eq. (2) implies that the shortest frequency domain ripple period found in the channel $M(w)$ is $1/p$.

2.2 Representation of bump equalizers

The bump equalizer considered in this paper is a linear combination of adjustable-loss Bode networks. The input-output transfer function of an equalizer composed of N Bode networks can be represented by

$$\text{EQL}(w) = \sum_{k=1}^{N} g_k B_k(w) \qquad \text{(dB)}, \tag{3}$$

where N is the number of networks and g_k and B_k represent the gain and response respectively of the kth Bode network.

A typical Bode network is shown in Fig. 1a, where the loss is controlled by the resistor R. The transfer function, $B_k(w)$, can be analytically derived and, with a suitable flat gain amplifier, it can be expressed by the following equation:

$$B_k(w) = \frac{[E_k(1 + E_k) + D_k]^2 - D_k}{[(1 + E_k)^2 + D_k]^2} \qquad \text{(dB)}, \tag{4}$$

where

$$E_k = \frac{R_{0k}}{R_{1k}},$$

$$D_k = \frac{(w/w_k)^2 H_k}{[(w/w_k)^2 - 1]^2},$$

$$H_k = \frac{C_k}{L_k},$$

and

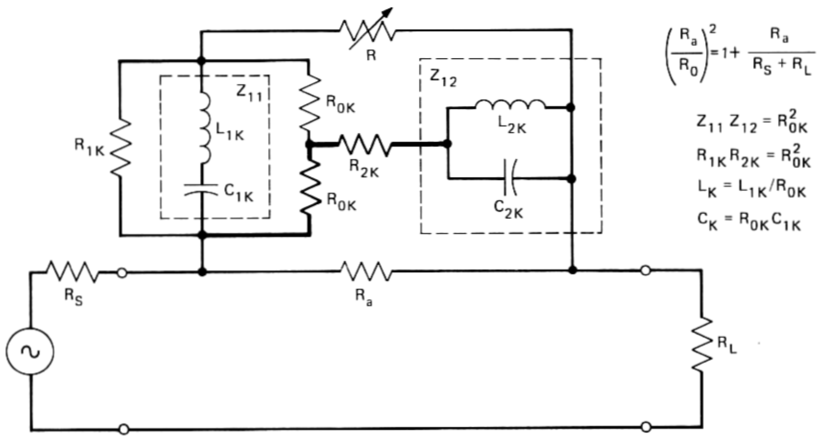
$$w_k = \log \frac{1}{2\pi\sqrt{L_k C_k}}.$$

Since eq. (4) shows $B_k(w)$ to be a quite complicated function of w, one of the following assumptions is used while analyzing the equalizer in detail.

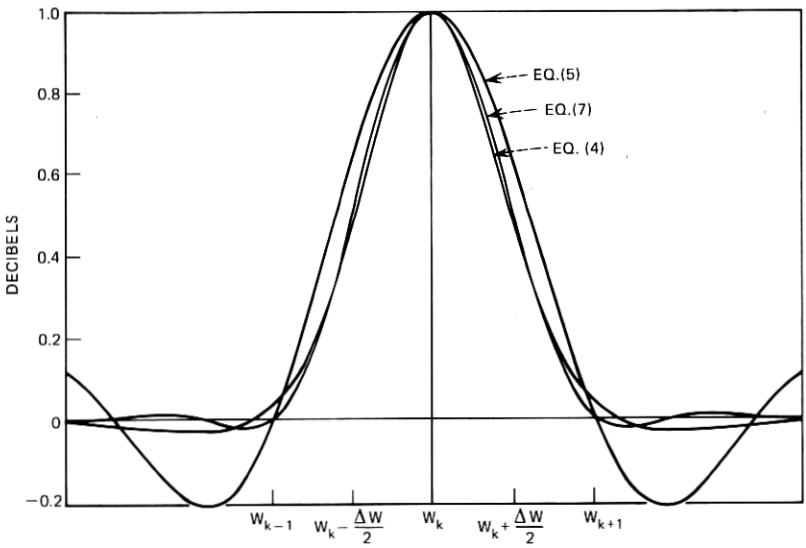
Assumption 1: Let $B_k(w)$ be approximated by

$$\text{sinc}\left[\frac{\pi}{\Delta w}(w - w_k)\right] = \frac{\sin[\pi(w - w_k)/\Delta w]}{\pi(w - w_k)/\Delta w} \qquad \text{(dB)}. \tag{5}$$

Since there are N Bode networks in the equalizer, which for the



(a) BODE NETWORK



(b) TRANSFER CHARACTERISTICS OF BODE NETWORK

Fig. 1—Adjustable Bode network.

present analysis are spaced equally on the w-axis with interval Δw (see Fig. 2), then the transfer function of equalizer can be expressed by

$$\mathrm{EQL}(w) = \sum_{k=1}^{N} g_k \,\mathrm{sinc}\left[\frac{\pi}{\Delta w}(w - w_k)\right] \qquad (\mathrm{dB}). \tag{6}$$

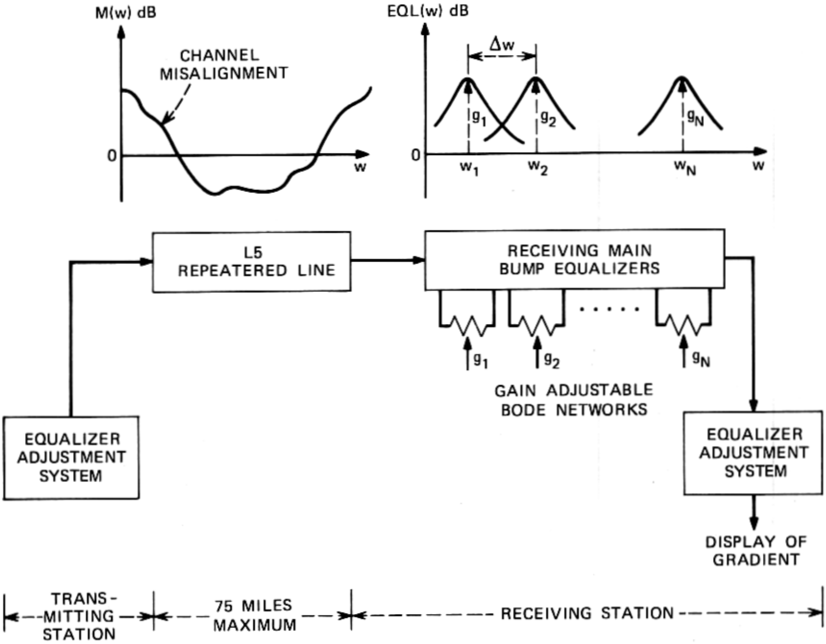


Fig. 2—Manual equalization of L5 coaxial system.

To permit a comparison between $B_k(w)$, as represented by eqs. (4) and (5), the two equations are plotted in Fig. 1b. The maximum differences between the two best matched curves are 0.165 and 0.183 dB when $|w - w_k| \leqq \Delta w$ and $|w - w_k| > \Delta w$, respectively.

Assumption 2: Let $B_k(w)$ be approximated by

$$\text{cosinc}\left[\frac{\pi}{\Delta w}(w - w_k)\right] = \frac{\sin[\pi(w - w_k)/\Delta w]}{\pi(w - w_k)/\Delta w} \cdot \frac{\cos[\pi(w - w_k)/\Delta w}{1 - 4[(w - w_k)/\Delta w]^2}. \quad (7)$$

Under the same conditions listed in Assumption 1, the transfer function of equalizer can be expressed by

$$\text{EQL}(w) = \sum_{k=1}^{N} g_k \,\text{cosinc}\left[\frac{\pi}{\Delta w}(w - w_k)\right] \quad \text{(dB)}. \quad (8)$$

Expression (7) is also plotted in Fig. 1b, and it can be seen that cosinc $[\pi(w - w_k)/\Delta w]$ approximates quite well the actual transfer function of the Bode network as expressed by eq. (4). The maximum difference between the two best-matched curves is 0.0327 dB when $|w - w_k| \leqq \Delta w$ and 0.0404 dB for $|w - w_k| > \Delta w$.

2.3 Mean-squared error algorithms (Gauss-Seidel iteration method)

The definition of optimal equalization used in this paper is the minimization of the MSE of the equalized channel as a result of adjusting the gain parameters, g_k.

On a decibel scale, the equalized error will be

$$E(w) = \sum_{k=1}^{N} g_k B_k(w) - M(w). \tag{9}$$

Then the MSE can be represented in the frequency domain by

$$\text{MSE} = \int_{-\infty}^{\infty} |E(w)|^2 dw. \tag{10}$$

Theorem 1: If the equalizer described by eq. (3) is composed of linearly independent networks, then there exists a unique set of g_k's which nulls all the gradients G_k; i.e.,

$$G_k = \frac{\partial MSE}{\partial g_k} = 0 \quad \textit{for all } k = 1, 2, \cdots, N, \tag{11}$$

where MSE is defined in eq. (10), and the corresponding set of g_k's results in minimum MSE.

The proof is given in Appendix A.

The bump equalizers considered in this paper belong to the class of the equalizers defined in Theorem 1.

As derived in eq. (25) of Appendix A, the gradient vector is given by

$$\mathbf{G} = \mathbf{B}\mathbf{g} - \mathbf{M}, \tag{12}$$

where $\mathbf{B}$, $\mathbf{g}$, and $\mathbf{M}$ are the system matrix, gain vector, and correlation vector, respectively, and are defined as follows. Defining an inner product

$$\langle A, B \rangle \equiv \int_{-\infty}^{\infty} A(w)B(w)dw,$$

$$\mathbf{G} = [G_1, G_2, \cdots, G_N]^T,$$

where T indicates the transpose,

$$\mathbf{g} = [g_1, g_2, \cdots, g_N]^T,$$
$$\mathbf{M} = 2[\langle B_1(w), M(w)\rangle, \langle B_2(w), M(w)\rangle, \cdots, \langle B_N(w), M(w)\rangle]^T,$$

and

$$\mathbf{B} = 2\begin{bmatrix} \langle B_1, B_1\rangle, \langle B_1, B_2\rangle, \cdots & \langle B_1, B_N\rangle \\ \langle B_2, B_1\rangle, \langle B_2, B_2\rangle, \cdots & \langle B_2, B_N\rangle \\ \vdots & \\ \langle B_N, B_1\rangle, \langle B_N, B_2\rangle, \cdots & \langle B_N, B_N\rangle \end{bmatrix}.$$

Now the gain vector is

$$\mathbf{g} = \mathbf{B}^{-1}(\mathbf{G} + \mathbf{M}), \tag{13}$$

provided that $\mathbf{B}^{-1}$ exists. The optimum gain, $\mathbf{g}^*$, which results in the minimum MSE is obtained by solving eq. (13) with $\mathbf{G} = \mathbf{0}$. The MSE algorithm given in Ref. 3 solves (13) with $\mathbf{G} = \mathbf{0}$ by the steepest descent method, which can be readily implemented in an automatic equalizer control circuit. For manually adjusted equalizers, however, the steepest descent algorithm cannot be easily implemented because the algorithm requires simultaneous adjustment of all the gain settings. A manual equalizer adjustment algorithm should have the following properties:

(*i*) The several gain settings can be adjusted one at a time, within a specified sequence.

(*ii*) Repeating step (*i*), $\mathbf{g}$ approaches $\mathbf{g}^*$.

The converging rate of the initial $\mathbf{g}$ to the final $\mathbf{g}^*$ depends on the type of algorithm used and the system matrix $\mathbf{B}$. For the bump equalizer, this question is discussed in Section 3.2.

Theorem 2 (Gauss-Seidel iteration algorithm): If the system matrix $\mathbf{B}$ *in (12) has dominant diagonal elements such that*

$$|\langle B_k, B_k\rangle| > \sum_{j=1}^{N}{}' \, |\langle B_k, B_j\rangle| \tag{14}$$

$$\textit{for all } k = 1, 2, \cdots, N,$$

where $\sum'$ *indicates the summation of all terms excluding the case* $j = k$, *then every* $\mathbf{g}$ *converges to the optimum gain* $\mathbf{g}^* = \mathbf{B}^{-1}\mathbf{M}$ *by the following iteration process:*

Iteration 1: Let $g_{k(i)}$ *indicate the kth gain at the ith iteration; thus, the initial gain settings are* $g_{1(0)}, g_{2(0)}, \cdots, g_{N(0)}$. *Adjust* $g_{1(0)}$ *until its corresponding gradient* $G_1 = 0$ *and designate the resultant gain* $g_{1(1)}$. *The gain settings are then* $g_{1(1)}, g_{2(0)}, \cdots, g_{N(0)}$. *Adjust* $g_{2(0)}$ *until the gradient* $G_2 = 0$, *resulting in the gain settings* $g_{1(1)}, g_{2(1)}, g_{3(0)}, \cdots, g_{N(0)}$. *Repeating the operation for each setting results in* $g_{1(1)}, g_{2(1)}, g_{3(1)}, \cdots, g_{N(1)}$, *and completes the first iteration.*

Iteration 2: Adjust $g_{1(1)}$ until $G_1 = 0$, resulting in the gain settings $g_{1(2)}$, $g_{2(1)}$, $g_{3(1)}$, $\cdots$, $g_{N(1)}$. Obtain $g_{2(2)}$, $g_{3(2)}$, $\cdots$, $g_{N(2)}$ by similar operation, completing the second iteration. Similarly, iterations 3, 4, $\cdots$, n can be carried out as required.

The proof is given in Appendix B.

If equalizer networks satisfy the inequality (14), they are called, in this paper, semi-orthogonal terms. The bump equalizers defined in this paper satisfy the inequality (14), and hence Theorem 2 can be used as a manual-equalizer adjusting algorithm. To implement this algorithm, a visual display of each gradient is required before the corresponding gain is adjusted. The following two theorems provide simple ways of determining the gradient.

Theorem 3 (ZF algorithm): Let the channel be represented by eq. (1) and the equalizer satisfy assumption 1. If the interval Δw between two adjacent Bode networks is no greater than half the shortest ripple period ($= 1/p$) found in the channel, i.e.,

$$\Delta w \leqq \frac{1}{2p}, \tag{15}$$

then the optimum gain setting is obtained by repeating the Gauss-Seidel iteration process defined in Theorem 2 with the gradient given by

$$G_k = 2E(w_k), \tag{16}$$

where $k = 1, 2, \cdots, N$ and $E(w_k)$ is the frequency domain error value measured at frequency w_k, the center frequency of the kth Bode network.

The proof is given in Appendix C.

Thus, if signals are transmitted at a set of frequencies equal to the center frequencies, w_k, of the Bode networks, and if the errors are measured at these frequencies at the receiving station, the gradients, G_k, can be obtained directly. Then each gain, g_k, would be adjusted until its gradient, G_k, reduced to zero. This is the well-known "zero-forcing" technique used in Ref. 2; it also achieves the optimum equalized channel in the MSE sense, if the stipulated assumptions apply.

Theorem 4 (MSE algorithm): Let the bump equalizer in this case satisfy assumption 2 and assume that the interval, Δw, between adjacent Bode networks is no greater than the shortest frequency domain ripple period in the channel, i.e.,

$$\Delta w \leqq \frac{1}{p}. \tag{17}$$

Then the optimum gain setting is obtained by repeating the Gauss-Seidel *iteration process with the gradient in this instance given by*

$$G_k = \tfrac{1}{2}E\left(w_k - \frac{\Delta w}{2} \right) + E(w_k) + \tfrac{1}{2}E\left(w_k + \frac{\Delta w}{2} \right) \qquad (18)$$

$$k = 1, 2, \cdots, N,$$

where $E(w_k)$ is the frequency domain error at the center frequency of the kth Bode network, and $E[w_k - (\Delta w/2)]$ and $E[w_k + (\Delta/2)]$ are the frequency domain errors measured at lower and upper frequencies midway between adjacent Bode networks. Equation (18) is derived in Ref. 3.

Proof: In this case,

$$\langle B_k, B_k \rangle = 0.75$$

and

$$\sum_{j=1}^{N}{}' \, |\langle B_k, B_j \rangle| = 0.25$$

for all k, thus satisfying the inequality (14). Hence, the Gauss-Seidel iteration process converges to the optimum gain settings.

To implement the MSE algorithm, a measure of the error at $2N - 1$ points in the frequency domain is required (see Fig. 2). In practice, the MSE technique results in better equalization than that obtained by the ZF method. Note that assumption 2 for the MSE algorithm approximates the actual equalizer more precisely than assumption 1 does. Moreover, inequality (15) for the ZF algorithm derived in this section is a conservative assumption. The channel ripple period allowed by the MSE algorithm can be half the period assumed by the ZF algorithm.

III. CONTROL OF MANUAL BUMP EQUALIZERS

In this section, the Gauss-Seidel iteration process derived in the previous section is applied to the manual equalizer for optimum gain control. The number of iterations required to obtain acceptable gain settings is reflected in inequality (14). The larger the diagonal components [left-hand side of (14)] compared to the off-diagonal components [right-hand side of (14)], the fewer the iterations needed. The rate of convergence of the iteration is described in Section 3.2 based on an error-reduction analysis. It is shown that one iteration is sufficient to obtain the optimum gain settings for the ZF Gauss-Seidel iteration algorithm derived in Theorem 3. When, in the more general case, the channel is initially equalized by the ZF algorithm, one or two more iterations are usually sufficient to achieve a practically optimum equalized channel for the MSE algorithm.

3.1 Hardware realization of Gauss-Seidel iteration process

For the L5 line equalization, an equalizer adjustment system has been developed for the adjustment of bump equalizers by the Gauss-Seidel iteration process. It is composed of a precision transmission measuring set, 90G oscillator—90H detector—digital control unit (Ref. 4), and a hardwired, special-purpose computer which contains a programmed memory and an arithmetic unit called an equalizer adjustment unit (EAU) (see Fig. 3). Referring to Fig. 2, we assume that the equalizers in the receiving station are to be adjusted by the MSE algorithm. By selecting the particular Bode network to be adjusted, the EAU in the transmitting station causes the 90G oscillator to generate sequentially an appropriate set of three frequencies (f_{k1}, f_{k2}, and f_{k3}). The EAU in the receiving station measures the channel error at the same three frequencies and computes the gradient, G_k,

Fig. 3—Equalizer adjustment unit.

by the following relationship:

$$G_k = B_{k1}E(f_{k1}) + B_{k2}E(f_{k2}) + B_{k3}E(f_{k3}).$$

Normally, $B_{k1} = B_{k3} = \frac{1}{2}$ and $B_{k2} = 1$. The resultant is displayed in a digital readout of the EAU and the operator subsequently adjusts g_k until $G_k = 0$. Then the next Bode network is selected and the transmitting EAU causes the generation of the required set of three frequencies, and the receiving EAU processes the received error signals and displays the calculated gradient. Again, the corresponding gain adjustment is made. When the ZF algorithm is selected, the gradient displayed is simply the error at the center frequency of the Bode network. Hence, as far as the operator is concerned, the adjustment procedure for the ZF and MSE algorithms is identical.

In practice, it is found that the equalizer should be initially adjusted by the ZF algorithm to bring the system near the optimum state. In this way, any large initial gain deviations from the optimum value are quickly reduced to within about ±0.5 dB. Then the MSE algorithm is used for the "fine tuning" of the gain adjustments. Usually, one or two iterations with the MSE algorithm will be sufficient for the equalizer to reach the optimum MSE state when starting from the ZF state. According to inequality (23) derived in the following section, and given initial gain settings within ±0.5 dB of the optimum value, the gain settings after two iterations are within ±0.036 dB of the ideal values in the worst case. The actual deviation from optimum in most cases will be smaller than 0.036 dB and in any case will be less than the inherent accuracy limitations of the 90-type transmission measuring sets to be used.

3.2 Error analysis

The Gauss-Seidel iteration provides fast convergence of initial gain settings to the optimum values for the bump equalizers. As derived in Appendix B, the Gauss-Seidel iteration can be expressed by

$$\mathbf{g}_{(i+1)} = -\mathbf{L}^{-1}\mathbf{U}\mathbf{g}_{(i)} + \mathbf{L}^{-1}\mathbf{M}, \tag{19}$$

where $\mathbf{L}$ is the lower triangular portion of the system matrix $\mathbf{B}$ including the diagonal and $\mathbf{U}$ is the upper triangular portion of $\mathbf{B}$ not including the diagonal.

Defining an error vector after the ith iteration to be

$$\mathbf{e}_{(i)} = \mathbf{g}^* - \mathbf{g}_{(i)}, \tag{20}$$

where $\mathbf{g}^*$ is the optimum value, then

$$\begin{aligned}\mathbf{e}_{(i+1)} &= \mathbf{g}^* - \mathbf{g}_{(i+1)} \\ &= \mathbf{g}^* + \mathbf{L}^{-1}\mathbf{U}\mathbf{g}_{(i)} - \mathbf{L}^{-1}\mathrm{M} \\ &= -\mathbf{L}^{-1}\mathbf{U}\mathbf{e}_{(i)}. \end{aligned} \tag{21}$$

To derive (21), the equality, $\mathbf{L}^{-1}\mathbf{M} = \mathbf{g}^* + \mathbf{L}^{-1}\mathbf{U}\mathbf{g}^*$, was used. Hence, the magnitude of the eigenvalues of $\mathbf{L}^{-1}\mathbf{U}$ determines the speed of convergence.

If the equalizer belongs to the class defined in assumption 1, $\mathbf{B}$ is a positive definite diagonal matrix and $\mathbf{L}^{-1}\mathbf{U}$ is a null matrix. Hence, $\mathbf{e}_{(i+1)} = \mathbf{0}$ for all $i = 0, 1, 2, \cdots$. In other words, we obtain the optimum gain settings by the ZF Gauss-Seidel iteration algorithm in one iteration. The same result can be obtained from eq. (19), since $\mathbf{L}^{-1}\mathbf{U}$ is a null matrix and $\mathbf{L}^{-1} = \mathbf{B}^{-1}$.

If the equalizer satisfies assumption 2, the MSE Gauss-Seidel iteration algorithm can be used. Now the system matrix is

$$\mathbf{B} = 2\begin{bmatrix} b_{11}, & b_{12}, & b_{13}, & \cdots, & b_{1N} \\ b_{21}, & b_{22}, & b_{23}, & \cdots, & b_{2N} \\ \vdots & & & & \\ b_{N1}, & b_{N2}, & b_{N3}, & \cdots, & b_{NN} \end{bmatrix},$$

where

$$\begin{aligned} b_{ij} &= 0.75 \quad \text{if} \quad i = j \\ b_{ij} &= 0.125 \quad \text{if} \quad |i - j| = 1 \end{aligned}$$

and

$$b_{ij} = 0 \qquad \text{if} \quad |i - j| \geqq 2$$

for all $i, j = 1, 2, \cdots, N$. Splitting $\mathbf{B}$ into two parts, $\mathbf{L}$ and $\mathbf{U}$, which are defined above, and performing some algebra,

$$\mathbf{L}^{-1}\mathbf{U} = \tfrac{4}{3}\begin{bmatrix} 0, & 6^{-1}, & 0, & 0, & \cdots, & 0 \\ 0, & -6^{-2}, & 6^{-1}, & 0, & \cdots, & 0 \\ 0, & 6^{-3}, & -6^{-2}, & 6^{-1}, & \cdots, & 0 \\ \vdots & & & & & \\ 0, & (-1)^{N+1}6^{-N}, & (-1)^{N}6^{-(N-1)}, & \cdots & 6^{-3}, & -6^{-2} \end{bmatrix}.$$

Hence, one can calculate the new error vector by eq. (21). After one iteration, the upper bound on the maximum residual error becomes

$$\begin{aligned} |e_{k(1)}|_{\max} &\leqq \tfrac{4}{3}(6^{-1} + 6^{-2} + 6^{-3} + \cdots)|e_{j(0)}|_{\max} \\ &= 0.26667|e_{j(0)}|_{\max} \end{aligned} \tag{22}$$

for all $j, k = 1, 2, \cdots, N$. Similarly, after the second iteration,

$$|e_{k(2)}|_{\max} \leqq 0.26667|e_{j(1)}|_{\max} \leqq 0.07111|e_{i(0)}|_{\max} \tag{23}$$

for all i, j, $k = 1, 2, \cdots, N$. The equality in eqs. (22) and (23) will be obtained if and only if

$$e_{1(0)} = -e_{2(0)} = e_{3(0)} = -e_{4(0)} = \cdots = |e_{k(0)}|_{\max}.$$

This in general will not be the case, and the maximum setting error after the second iteration usually will be less than 0.07111 times the maximum setting error prior to the first iteration. Consequently, if the gain settings are within 0.5 dB of optimum at the start (which a single ZF iteration will establish), the deviation from optimum settings in the MSE sense is within a few hundredths of a decibel after two additional iterations.

IV. CONCLUSION

This paper shows that a manual equalization process which can be described by a Gauss-Seidel iteration method provides optimal control for bump equalizers in the MSE sense. Compared to the steepest descent method discussed in Ref. 3, the Gauss-Seidel iteration method can be more economically implemented for manual equalization such as in the L5 system. The Gauss-Seidel iteration process requires knowledge of the gradient of the MSE with respect to the gain setting for each Bode network to be adjusted. The ZF algorithm derived in this paper requires just one Gauss-Seidel iteration, but in practice the ZF equalized channel is not optimum and can be further improved by the MSE algorithm. This is because the gradient obtained by the MSE algorithm is more accurate than the one obtained by the ZF algorithm for realizable equalizer shapes. It should be noted that three tones are required to determine the gradient of the MSE with respect to the gain setting of each Bode network for the MSE algorithm, while only one tone is used to obtain the gradient for the ZF algorithm. The number of iterations that are necessary to bring the equalizer to the optimum state depends on how close the initial settings are to optimum. When the channel is initially ZF-equalized, only one or two more iterations are needed to optimize the channel with the MSE algorithm. This result agrees completely with experiments conducted in the L5 field trial.

APPENDIX A

Proof of Theorem 1

Substituting eqs. (9) and (10) into eq. (11), we obtain

$$G_k = 2 \int_{-\infty}^{\infty} B_k(w) \left\{ \sum_{j=1}^{N} g_j B_j(w) - M(w) \right\} dw. \tag{24}$$

Defining an inner product

$$\langle A, B\rangle = \int_{-\infty}^{\infty} A(w)B(w)dw,$$

$$\mathbf{G} = [G_1, G_2, \cdots, G_N]^T,$$

where T indicates the transpose,

$$\mathbf{g} = [g_1, g_2, \cdots, g_N]^T$$

$$\mathbf{M} = 2[\langle B_1(w), M(w)\rangle, \langle B_2(w), M(w)\rangle, \cdots, \langle B_N(w), M(w)\rangle]^T$$

and

$$\mathbf{B} = 2\begin{bmatrix} \langle B_1, B_1\rangle, & \langle B_1, B_2\rangle, & \cdots, & \langle B_1, B_N\rangle \\ \langle B_2, B_1\rangle, & \langle B_2, B_2\rangle, & \cdots, & \langle B_2, B_N\rangle \\ \vdots & & & \\ \langle B_N, B_1\rangle, & \langle B_N, B_2\rangle, & \cdots, & \langle B_N, B_N\rangle \end{bmatrix},$$

a simultaneous equation of the type of eq. (24) for all k from 1 to N can be written as

$$\mathbf{G} = \mathbf{Bg} - \mathbf{M}$$

or

$$\mathbf{Bg} = \mathbf{G} + \mathbf{M}. \tag{25}$$

Since eq. (25) is a nonhomogeneous system of N equations and $\mathbf{G} + \mathbf{M}$ is a vector with N real-numbered components in the case considered, for the given $\mathbf{G}$, the unknown $\mathbf{g}$ is uniquely obtained by

$$\mathbf{g} = \mathbf{B}^{-1}(\mathbf{G} + \mathbf{M}), \tag{26}$$

provided that $\mathbf{B}$ is a nonsingular matrix.

However, if $\mathbf{B}$ were a singular matrix, then a linear combination of the columns could be made zero, i.e.,

$$\sum_{k=1}^{N} h_k\langle B_k, B_j\rangle = 0$$

for each $j = 1, 2, \cdots, N$ where h_k are real nonzero numbers.

In addition, the following relationship could also hold:

$$\sum_{k=1}^{N} h_1h_k\langle B_k, B_1\rangle + \sum_{k=1}^{N} h_2h_k\langle B_k, B_2\rangle + \cdots + \sum_{k=1}^{N} h_Nh_k\langle B_k, B_N\rangle = 0. \tag{27}$$

But (27) can also be written as

$$\int_{-\infty}^{\infty} \left\{ \sum_{k=1}^{N} h_kB_k(w)\right\}^2 dw = 0. \tag{28}$$

Equation (28) contradicts the assumption that $B_k(w)$'s are linearly independent. Hence, **B** is indeed a nonsingular matrix and there exists but one set of g_k's for which $\mathbf{G} = \mathbf{0}$ in eq. (26). It is yet necessary to prove that this stationary point is the global minimum of the MSE defined in eq. (10), which is established if the following relationship can be proved:

$$\int_{-\infty}^{\infty} \left\{ M(w) - \alpha \sum_{k=1}^{N} g_k^* B_k(w) - \beta \sum_{k=1}^{N} g_k^{**} B_k(w) \right\}^2 dw$$
$$< \alpha \int_{-\infty}^{\infty} \left\{ M(w) - \sum_{k=1}^{N} g_k^* B_k(w) \right\}^2 dw$$
$$+ \beta \int_{-\infty}^{\infty} \left\{ M(w) - \sum_{k=1}^{N} g_k^{**} B_k(w) \right\}^2 dw, \quad (29)$$

where

$$\sum_{k=1}^{N} g_k^* B_k(w) \quad \text{and} \quad \sum_{k=1}^{N} g_k^{**} B_k(w)$$

indicate distinct equalizer settings, $\alpha + \beta = 1$ and $\alpha, \beta > 0$, then MSE is a strict convex function of gain settings g_k's and has a global minimum.

Subtracting the left-hand side from the right-hand side of inequality (29), we obtain the following:

$$-2\alpha\beta \int_{-\infty}^{\infty} \left\{ \left[\sum_{k=1}^{N} g_k^* B_k(w) \right]^2 + \left[\sum_{k=1}^{N} g_k^{**} B_k(w) \right]^2 \right\} dw. \quad (30)$$

Since $B_k(w)$'s are linearly independent and at least one equalizer setting,

$$\sum_{k=1}^{N} g_k^* B_k(w) \quad \text{or} \quad \sum_{k=1}^{N} g_k^{**} B_k(w),$$

is not zero, (30) is negative. Hence, inequality (29) is correct, and the proof of Theorem 1 is complete.

APPENDIX B

Proof of Theorem 2

The gradient of MSE with respect to the gain settings is represented by the following equation:

$$\mathbf{G} = \mathbf{Bg} - \mathbf{M}, \quad (31)$$

where **B**, **g**, and **M** are defined in (24). Splitting the **B** matrix as follows

$$\mathbf{B} = \mathbf{L} + \mathbf{U}, \quad (32)$$

where $\mathbf{L}$ is the lower triangular portion of $\mathbf{B}$ including the diagonal and $\mathbf{U}$ is the remainder of $\mathbf{B}$,

$$\mathbf{G} = \mathbf{Lg} + \mathbf{Ug} - \mathbf{M}. \tag{33}$$

According to the iterative procedure described in the theorem, $\mathbf{g}_{(i+1)}$ is obtained from $\mathbf{g}_{(i)}$ by setting $\mathbf{G} = \mathbf{0}$. With the aid of eq. (33), this procedure can be expressed by

$$\mathbf{0} = \mathbf{L}\mathbf{g}_{(i+1)} + \mathbf{U}\mathbf{g}_{(i)} - \mathbf{M}$$

or

$$\mathbf{g}_{(i+1)} = -\mathbf{L}^{-1}\mathbf{U}\mathbf{g}_{(i)} + \mathbf{L}^{-1}\mathbf{M}. \tag{34}$$

By successive calculation, eq. (34) can be modified by

$$\mathbf{g}_{(i+1)} = [-\mathbf{L}^{-1}\mathbf{U}]^{i+1}\mathbf{g}_{(0)} + \sum_{k=0}^{i} [-\mathbf{L}^{-1}\mathbf{U}]^{k}\mathbf{L}^{-1}\mathbf{M}, \tag{35}$$

where $\mathbf{g}_{(0)}$ is the initial value.

If

$$[-\mathbf{L}^{-1}\mathbf{U}]^{i} \to [\mathbf{0}] \quad \text{as} \quad i \to \infty,$$

$$\sum_{k=0}^{i} [-\mathbf{L}^{-1}\mathbf{U}]^{k}\mathbf{L}^{-1} \to [\mathbf{L} + \mathbf{U}]^{-1} = \mathbf{B}^{-1}.$$

Hence, eq. (35) becomes

$$\mathbf{g}_{(i+1)} = \mathbf{0} + \mathbf{B}^{-1}\mathbf{M},$$

which is the desired result.

Hence, the theorem is proved if $\mathbf{L}^{-1}\mathbf{U}$ is a convergent matrix, i.e., eigenvalues of the matrix $\mathbf{L}^{-1}\mathbf{U}$ are all less than one in absolute value. However, if condition (14) is satisfied, $\mathbf{L}^{-1}\mathbf{U}$ is a convergent matrix and $[-\mathbf{L}^{-1}\mathbf{U}]^{i} \to [\mathbf{0}]$ as $i \to \infty$ (see Theorems 3.3 and 3.4 in Ref. 5).

Note: The iteration process defined by (34) is known as the Gauss-Seidel or, simply, the Seidel iteration.

APPENDIX C

Proof of Theorem 3

When assumption 1 is satisfied,

$$\langle B_k, B_j \rangle = 1, \quad k = j$$
$$= 0, \quad k \neq j$$

for all j and k.

Hence, Theorem 2 is satisfied and we have to prove now

$$G_k = 2E(w_k). \tag{36}$$

From eq. (24),

$$\begin{aligned} G_k &= 2\int_{-\infty}^{\infty} B_k(w)\left\{\sum_{j=1}^{N} g_j B_j(w) - M(w)\right\}dw \\ &= 2\int_{-\infty}^{\infty} \mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_k)\right]\sum_{j=1}^{N} g_j B_j(w)dw \\ &\qquad - 2\int_{-\infty}^{\infty} \mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_k)\right]M(w)dw. \end{aligned} \tag{37}$$

Since

$$\int_{-\infty}^{\infty} \mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_k)\right]\mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_j)\right]dw = 0 \quad \text{if} \quad k \neq j$$

and

$$= 1 \quad \text{if} \quad k = j,$$

the first integration in (37) is simply $2g_k$.

Substituting $M(w)$ of (2) into (37), the second integration of (37) becomes

$$\begin{aligned} 2\int_{-\infty}^{\infty} \mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_k)\right]\int_0^1 \{F(x)\cos(2\pi pwx) \\ + H(x)\sin(2\pi pwx)\}dxdw \\ = \int_{-\infty}^{\infty} \mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_k)\right]\int_0^1 \{f(x)\cos[2\pi p(w-w_k)x] \\ + h(x)\sin[2\pi p(w-w_k)x]\}dxdw, \end{aligned} \tag{38}$$

where

$$f(x) = F(x)\cos(2\pi pw_kx) + H(x)\sin(2\pi pw_kx)$$

and

$$h(x) = H(x)\cos(2\pi pw_kx) - F(x)\sin(2\pi pw_kx).$$

Since

$$\int_0^1\int_{-\infty}^{\infty} h(x)\,\mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_k)\right]\sin[2\pi p(w-w_k)x]dwdx = 0,$$

Eq. (38) becomes

$$2\int_{-\infty}^{\infty} \mathrm{sinc}\left[\frac{\pi}{\Delta w}(w-w_k)\right]\int_0^1 \{f(x)\cos[2\pi p(w-w_k)x]\}dxdw. \tag{39}$$

Replacing $w = u + w_k$ and changing the "cos" into "exponential" form, (39) becomes

$$\int_{-\infty}^{\infty} \text{sinc}\left(\frac{\pi}{\Delta w}u\right) \int_0^1 \{f(x)[\exp(i2\pi pux) + \exp(-i2\pi pux)]\} dxdu$$

$$= \int_0^1 f(x) \int_{-\infty}^{\infty} \text{sinc}\left(\frac{\pi}{\Delta w}u\right) \{\exp(i2\pi pux) + \exp(-i2\pi pux)\} dudx, \quad (40)$$

where $i^2 = -1$. Since $0 \leqq x \leqq 1$ and $2p \leqq 1/\Delta w$ by assumption, integration of (40) is simply

$$2 \int_0^1 f(x)dx.$$

Note that the inner integration in (40) is the Fourier transformation of the sinc function.

Combining the results, G_k in (37) becomes

$$G_k = 2g_k - 2 \int_0^1 f(x)dx. \quad (41)$$

However,

$$M(w_k) = \int_0^1 f(x)dx \quad \text{and} \quad \text{EQL}(w_k) = g_k.$$

Hence, (41) becomes

$$\begin{aligned} G_k &= 2[\text{EQL}(w_k) - M(w_k)] \\ &= 2E(w_k). \end{aligned}$$

This proves the theorem.

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