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Treatment of Microscopic Fluctuations in Noise Theory

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A new method is introduced and used to calculate the statistics of the microscopic fluctuations of charge carriers in devices. By expressing the fluctuations of the carriers in terms of elementary transfer fluctuations, we are able to separate the induced fluctuations from the spontaneous fluctuations experienced by such carriers. This enables us to treat correlation effects in the dynamical portion of the problem and reserve for the statistical portion only well-defined, uncorrelated random forces whose statistics are readily calculated. The method includes all important correlation effects as well as multiple-decay-time relaxation effects and, thus, it fills a gap in the Langevin method as well as in the impedance-field method of calculating noise in devices. The method is suitable for treating nonstationary as well as stationary noise, and in some cases can be used directly on macroscopic problems. We also present a derivation of a recently introduced expression for diffusion noise of carriers whose mobility is a nonlinear function of applied electric field. This microscopic approach may further illustrate the origin, nature, and treatment of fluctuations in devices.

I. INTRODUCTION

In this article we describe a very simple means of treating microscopic fluctuations in noise theory. The method is simple in the sense that it focuses attention directly on the heart of the matter, the elementary processes which give rise to device noise. It is also simple in the sense that no sophistication in probability theory is used beyond an understanding of simple shot noise. Nonetheless, the method is rigorous under the rather mild constraint that the fluctuations are sufficiently small that the equations governing the noise are linear. The method has the added advantage that it can be used nearly as easily for nonstationary noise as for stationary noise. We make no

claim that this method is an advance in the philosophy of noise; we do claim, however, that it is adequate for solving many noise problems of practical interest.

There are two equivalent[1,2] methods of calculating device noise, the Langevin[3] method (LM) and the impedance-field[4] method (IFM). Both methods are characterized by an inherent simplification: namely, the separation of the task of calculating the spontaneous fluctuations of the current carriers in each elemental region of the device, and the task of calculating the observable response to these fluctuations at the external contacts of the device. The former task, the treatment of the microscopic fluctuations, is simplified because in dealing with the source of the fluctuations one can focus attention on the statistics of the microscopic variations inherent to the local physical conditions, which in turn are determined by the (noiseless) state of the device during operation. As a result, both the LM and the IFM are primarily concerned with the latter task, the coupling of the microscopic fluctuations to the macroscopic, observable voltages and currents. This task is also well-defined because the influence of the carriers in one region of the device on the carriers in another region, and on the contacts, has often been studied in detail in attempts to understand the dc and ac operation of the device. Thus, it is important to complement the LM or the IFM with the microscopic method described below. When this is done, it can be claimed that in most cases, if one understands the device sufficiently to calculate its noiseless operation, one can calculate the device noise as well. This should be of assistance to those tackling the noise in new and/or unfamiliar devices from scratch.

For example, if a device is sufficiently well-understood to be characterizable by an equivalent circuit, one can often introduce equivalent, random voltage and current sources to simulate the noise in the device.[5–8] Using such sources, a circuit designer with little interest in noise theory can readily calculate the size of the noise in the circuit employing the devices of interest. In a similar way, a person working with individual devices may find it convenient to have a simple scheme to translate the physical processes with which he is familiar into noise sources and to quickly evaluate their effect on the performance of the device of interest.[9–10] It is hoped that the method presented here will be used in such situations.

It is important to realize that charge carriers in any device fluctuate in response to random forces exerted on them.[3] The response to a specific impulse continues in general long after the impulse causing it

has ceased, and, in the interim, subsequent impulses will further alter the induced fluctuation. In addition, a fluctuation in the distribution of one type of carrier can induce fluctuations in the distributions of other types of carriers. Thus, whereas the statistics of the spontaneous, random forces may be quite simple, those of the carriers can be somewhat complicated owing to the correlation between the various induced fluctuations. The key to the simplicity of the method presented here results from the separation of the correlation effects from the statistical problem. These correlation effects are *not* neglected. Rather, they are included in the dynamical portion instead of in the statistical portion of the treatment. As it turns out, this results in the primary simplification achieved with our method.

In what follows, we shall use several examples to introduce and elaborate our microscopic approach. The first example, the decay of charge stored on a leaky capacitor, will motivate the method and illustrate how this technique can be used to treat certain macroscopic problems as well as microscopic ones. The second example concerns transfers between a two-level system. Here correlations are of primary importance, and our method is seen to treat these adequately yet simply. The third example illustrates how velocity fluctuations can be decomposed into transfer fluctuations, which are much easier to treat. Complicated scattering mechanisms including multiple decay times, which are not normally covered in the usual Langevin[3] or impedance-field[4] methods, can be handled with relative ease. The fourth example considers recombination to illustrate how correlation effects can be treated efficiently and effectively. By working in the time domain, we can see how the method works for nonstationary[9] as well as for stationary statistics. The statistics are treated in detail in the appendices. In particular, a recently used expression[11] for the diffusion noise of carriers having a nonlinear mobility is derived.

It is hoped that our discussion will provide, for the nonspecialist, further insight into the physical nature and mathematical representation of noise in general.

II. SOME SIMPLE EXAMPLES

Since the primary purpose of this paper is to elucidate a general, practical approach to solving the microscopic portion of noise problems, it seems best at first to describe this approach in terms of simple examples. Although this method is best suited to treat noise at the microscopic level, for purposes of illustration we shall commence with a macroscopic example.

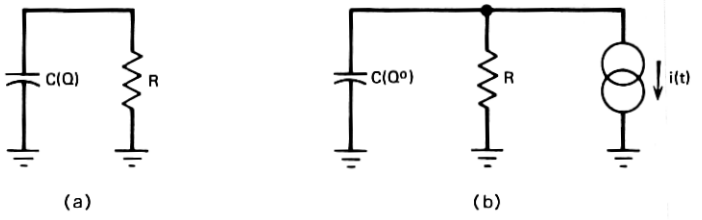

Fig. 1—(a) *RC* circuit in which charge initially placed on the capacitor decays to ground through the resistor. (b) Noise equivalent circuit of (a).

Consider the *RC* circuit shown in Fig. 1a in which we assume that capacitance C is a function $C(Q)$ of the charge Q, which it stores. Let us assume that initially ($t = 0$) a charge of size Q_o^o is stored on the capacitor. Subsequently ($t > 0$) the charge will decay away through the resistor R. In the absence of noise, the charge as a function of time $Q^o(t)$ satisfies the equation

$$\frac{dQ^o(t)}{dt} = -\frac{Q^o(t)}{RC[Q^o(t)]}, \tag{1}$$

subject to the boundary condition that $Q^o(0) = Q_o^o$.

We know, however, that the thermal (Brownian) motion of the electrons in the resistor gives rise to a noise current. Thus, rather than the uniform charge decay predicted by (1) for noiseless conditions, the charge decay is in fact somewhat random. Moreover, the noise is not solely governed by the resistor. If, during a time interval, the noise current in the resistor is such as to draw too many charges from the capacitor, then the voltage on the capacitor will decrease and the subsequent current will be reduced. This is just the response of the circuit to the noise generated by the resistor. In such a problem, we would ordinarily determine the statistics of the noise current $i(t)$ associated with the resistance itself, and then determine the fluctuations in the charge $Q(t)$ on the capacitor in terms of $i(t)$. We would be implicitly assuming, and rightly so, that only the resistor, and not the circuit exterior to the resistor, determines the statistics of $i(t)$, the thermal noise current generated by the resistor. We would then relate $Q(t)$ to $i(t)$ using an equation of the form

$$\frac{dQ}{dt} = -I(Q) - i(t), \qquad I(Q) \equiv \frac{Q}{RC(Q)}. \tag{2}$$

(The noise equivalent circuit is shown in Fig. 1b.) We would then write $Q(t)$ as a sum of its stochastic average $Q^o(t)$ and a fluctuation

$\delta Q(t) \equiv Q(t) - Q^o(t)$; assume that $\langle \delta Q(t)^2 \rangle^{\frac{1}{2}} \ll \Delta Q$, where

$$\Delta Q = (\partial I/\partial Q)/(\partial^2 I/\partial Q^2);$$

and expand eq. (2) to first order in $\delta Q(t)$, obtaining a nonlinear equation [eq. (1)] for the $Q^o(t)$ and a linear equation,

$$\frac{d\delta Q}{dt} = -\frac{\delta Q}{\tau(t)} - i(t), \tag{3}$$

for δQ in terms of $i(t)$. In eq. (3) $\tau(t)$ is defined by

$$\frac{1}{\tau(t)} \equiv \frac{d}{dQ}\left(\frac{Q}{RC(Q)}\right)\Bigg|_{Q=Q^o(t)}. \tag{4}$$

Solving (3) for $\delta Q(t)$ in terms of $i(t)$ permits the statistics of $\delta Q(t)$ to be obtained from those of $i(t)$, which for now we assume we know. In this way, we can determine the noise associated with the variable of interest $Q(t)$ from a knowledge of the noise associated with a more simply characterized noise variable $i(t)$.

The foregoing procedure has one drawback: one must be extremely careful, in general, to properly introduce such noise terms as $i(t)$ into otherwise noiseless dynamical relations. As we shall see in the examples considered below, there is a one-to-one correspondence between the transfer processes characterizing the problem of interest and the noise terms which one introduces. Thus, if one writes down several coupled-rate equations, each involving several transfer processes, and introduces but one noise term per equation, one finds, in general, that these noise terms are not simple, being correlated statistically to one another. Since, for simplicity, one would desire noise terms to be uncorrelated, some care must be used in including them in the rate equations. We now outline a procedure in the context of the above example that can be used in more complicated problems to insure that such noise terms are included properly.

The decay of charge from capacitor to ground is accomplished by transfer of individual electrons. Let t_i be the time at which the ith electron leaves the capacitor. Then we may write the following kinematic relation for the decay of the charge on the capacitor,

$$\frac{dQ(t)}{dt} = -q \sum_i \delta(t - t_i), \tag{5}$$

where q is the size of an elementary charge. If we consider an ensemble of RC circuits of the type shown in Fig. 1a, in which an initial charge of Q_o^o is decaying, each decay will be characterized by a different set of

times $\{t_i\}$ at which elementary charges leave the capacitors. Each t_i, therefore, is a random variable whose probability distribution is, in general, dependent upon all t_j for the preceding events ($t_j < t_i$). A completely rigorous derivation of the noise would, therefore, involve calculating the probability of each such sequence of times $\{t_i\}$, including all correlation effects, and then using these probabilities to ascertain the statistics of the noise. Were it not for these correlation effects, the t_i would be independent, and the statistical problem would be greatly simplified.

We shall now recast eq. (5) into a form that greatly simplifies the correlation problem by decomposing the current into a spontaneous portion $d(t)$ and an induced portion $R(t)$. The spontaneous portion is governed by the sources of the noise, and the induced portion is controlled by the instantaneous state of the device, in this case the stored charge $Q(t)$. Returning to the charge decay problem, we rewrite eq. (5) in the following form:

$$\frac{dQ}{dt} = -qR(t) - qd(t), \tag{6}$$

where

$$d(t) \equiv \sum_i \delta(t - t_i) - R(t), \tag{7}$$

and where $R(t)$ is the "dynamical" rate of charge loss. By "dynamical" rate we mean that $R(t)$ is, in general, a function of (that is, is determined by) the dynamical variables of the problem. In this case, we have

$$qR(t) = \frac{Q(t)}{RC[Q(t)]}. \tag{8}$$

If there is a fluctuation δQ in Q, then a fluctuation $\delta R(t)$ occurs in $R(t)$ also, which in this case is given to first order in $\delta Q(t)$:

$$q\delta R(t) = \delta Q(t)/\tau(t), \tag{9}$$

where τ is defined in eq. (4). Returning to eq. (6), if we write $Q = Q^o + \delta Q$ as we did in passing from eq. (2) to eq. (3), then we obtain an equation for the noise δQ [which corresponds to eq. (3) above],

$$\frac{d\delta Q}{dt} = -\frac{\delta Q}{\tau(t)} - qd(t). \tag{10}$$

In this case, $qd(t)$ corresponds to the noise current $i(t)$ generated by the resistor, which, as discussed above, is independent of the δQ associated with the capacitor. Thus, $d(t)$ serves as a statistical driving term. We

may calculate $\delta Q(t)$ in terms of $d(t)$ and from the statistics of $d(t)$ determine those of $\delta Q(t)$.

Let us return to eq. (7) for a moment to point out why $d(t)$ is to be regarded as the spontaneous portion of the current. The $-e\sum_i \delta(t - t_i)$ term is, of course, the entire current associated with the charge decay. The dynamical rate term $R(t)$, however, is a function only of the dynamical variables of the problem and does not contain noise sources. In our example, $R(t)$ involves only the charge $Q(t) = Q^o(t) + \delta Q(t)$, where $\delta Q(t)$ is the charge fluctuation induced by the noise sources acting on the device. $R(t)$ does not involve the noise sources themselves. Thus, if $d(t) = 0$, δQ, the response, vanishes. Hence, inasmuch as $d(t)$ is the difference between the total current and the noiseless-plus-induced portion of the current, it follows that $d(t)$ can contain only the spontaneous portion of the current. The advantage of starting with eq. (5) and proceeding as we did to eq. (10), rather than attempting to write eq. (2) [or eq. (3)] *a priori*, will become evident when more than one process is involved in the problem. By introducing a $d_i(t)$ and $R_i(t)$ for each process i, and noting that each $d_i(t)$ involves only spontaneous fluctuations and, hence, must be independent of all the other d_j, $j \neq i$, we can readily express the fluctuations of interest in terms of the independent statistical driving terms d_i.

If we solve eq. (10) for $\delta Q(t)$, we find that

$$\delta Q(t) = \int_{-\infty}^{t} dt' \exp\left[-\int_{t'}^{t} dt''/\tau(t'')\right][-qd(t')]. \tag{11}$$

Often one is most interested in the mean-square fluctuation $\langle \delta Q^2 \rangle$ for some time t. From (11) this is given by

$$\langle \delta Q^2(t) \rangle = q^2 \int_{-\infty}^{t} dt_1' \int_{-\infty}^{t} dt_2' \exp\left[-\int_{t_1'}^{t} dt''/\tau(t'')\right] \times \exp\left[-\int_{t_2'}^{t} dt''/\tau(t'')\right] \langle d(t_1')d(t_2') \rangle. \tag{12}$$

If we know the statistics of $d(t)$ (in this case, those of $i(t)/q$ associated with the resistor), we can calculate $\langle d(t_1)d(t_2) \rangle$, and hence $\langle \delta Q^2(t) \rangle$. Note that, in general, we must also know the noiseless solution $Q^o(t)$ [eq. (1), and also see eq. (4)]. For simplicity here, let us assume that $i(t)$ is pure thermal noise so that

$$q^2 \langle d(t_1)d(t_2) \rangle = 2kTG\delta(t_1 - t_2), \qquad G = 1/R.$$

It follows then that

$$\langle \delta Q^2(t) \rangle = kTC, \tag{13}$$

the usual result. [Note: $\langle \delta Q(t_1)\delta Q(t_2)\rangle = kTC \exp(-|t_1 - t_2|/\tau)$. This illustrates how eq. (10) maintains the correlation between fluctuations at two different times, while the fluctuations themselves are driven by a source without correlation, $\delta(t - t')$.] When we turn to purely microscopic processes, we shall find the statistics of $d(t)$ are governed by the rate function $R(t)$.

In what follows, the very important distinction between spontaneous and induced fluctuations will be used repeatedly. The advantage of this macroscopic example is that the separation between the two may be clearly visualized. From the point of view of the RC circuit, fluctuations in the current generator are spontaneous and induce fluctuations in the charge decay, δQ. In addition, the rate term $R(t)$ is a function only of Q and clearly includes only the induced and *not* the spontaneous fluctuations. The reader may find it helpful in subsequent examples to refer back to this simple model to clarify the somewhat more subtle distinctions between induced and spontaneous fluctuations at the microscopic level.

We now consider another simple example, this time a truly microscopic one. Let us consider two states a and b containing $n_a(t)$ and $n_b(t)$ charges, respectively. [These states, for example, might be two regions of phase space $(d\mathbf{x}d\mathbf{v})$:* one region for $(\mathbf{x}_a, \mathbf{v}_a)$ and one for $(\mathbf{x}_b, \mathbf{v}_b)$, or a might be trapped electrons and b free electrons.] Let us assume that charges are flowing to b from a at a rate $R_{ba} = R_{ba}(n_a, n_b)$ and to a from b at a rate $R_{ab} = R_{ab}(n_b, n_a)$. If charges leave a and enter b at times t_{bai} and leave b and enter a at times t_{abi}, then by analogy with (5) we write

$$\dot{n}_a = -\sum_i \delta(t - t_{bai}) + \sum_i \delta(t - t_{abi}) \tag{14a}$$

and

$$\dot{n}_b = -\sum_i \delta(t - t_{abi}) + \sum_i \delta(t - t_{bai}). \tag{14b}$$

Following the previous example, we rewrite (14) in the form

$$\dot{n}_a = -R_{ba}(t) + R_{ab}(t) - d_{ba}(t) + d_{ab}(t) \tag{15a}$$

and

$$\dot{n}_b = -R_{ab}(t) + R_{ba}(t) - d_{ab}(t) + d_{ba}(t), \tag{15b}$$

where, of course,

$$d_{ab}(t) \equiv \sum_i \delta(t - t_{abi}) - R_{ab}(t) \tag{16a}$$

* Note that boldface capital letters denote matrices; boldface lower-case letters denote vectors.

and

$$d_{ba}(t) \equiv \sum_i \delta(t - t_{bai}) - R_{ba}(t). \tag{16b}$$

If we write $n_a = n_a^o + \delta n_a$, $n_b = n_b^o + \delta n_b$, insert into (15), and expand, we obtain the following equations for the noiseless quantities n_a^o, n_b^o:

$$\dot{n}_a^o = -R_{ba}^o(t) + R_{ab}^o(t) \tag{17a}$$

and

$$\dot{n}_b^o = -R_{ab}^o(t) + R_{ba}^o(t), \tag{17b}$$

where

$$R_{ba}^o \equiv R_{ba}(n_a^o(t), n_b^o(t)), \qquad R_{ab}^o \equiv R_{ab}(n_b^o(t), n_a^o(t)).$$

For the noise δn_a, δn_b, we obtain the linear relations

$$\delta\dot{n}_a = -\left(\frac{\delta R_{ba}}{\delta n_a} - \frac{\delta R_{ab}}{\delta n_a}\right)\delta n_a - \left(\frac{\delta R_{ba}}{\delta n_b} - \frac{\delta R_{ab}}{\delta n_b}\right)\delta n_b - d_{ba}(t) + d_{ab}(t) \tag{18a}$$

and

$$\delta\dot{n}_b = -\left(\frac{\delta R_{ab}}{\delta n_a} - \frac{\delta R_{ba}}{\delta n_a}\right)\delta n_a - \left(\frac{\delta R_{ab}}{\delta n_b} - \frac{\delta R_{ba}}{\delta n_a}\right)\delta n_b - d_{ab}(t) + d_{ba}(t), \tag{18b}$$

from which δn_a and δn_b can be determined in terms of d_{ab} and d_{ba}. (The linear operators of the form $\delta R/\delta n$ are evaluated at their noiseless values.) Since there are only two states, and since $n_a + n_b =$ constant ($\dot{n}_a + \dot{n}_b = 0$), it comes as no surprise that $\delta n_a(t) = -\delta n_b(t)$; one state's loss is the other's gain. Nonetheless, the source or driving terms, d_{ab} and d_{ba}, are independent, and the correlations between δn_a and δn_b are included in (18) through the presence of the dynamical terms. Since this is an important point, we shall discuss it more fully below.

If we ignore for the moment the random nature of the flow of charges from a to b (and from b to a), then eq. (17) tells us that we have a "smooth" continuous flow of charges from a to b at a rate of $R_{ba}^o(t)$ and from b to a at a rate of $R_{ab}^o(t)$. Noise enters the problem when we note (as we have above) that charges are actually transferred at times $\{t_{bai}\}$ and $\{t_{abi}\}$, the (ensemble) average rate of occurrence of these times being $R_{ba}^o(t)$ and $R_{ab}^o(t)$. Since $R_{ba}^o(t)$ and $R_{ab}^o(t)$ depend only on the steady-state solution (n_a^o, n_b^o) to eq. (17), these quantities are not affected by the details of a particular set of fluctuations. This means that if R_{ba}^o and R_{ab}^o govern the statistics of the transfers from a to b and b to a, then the individual transfer times $\{t_{bai}^o\}$, $\{t_{abi}^o\}$ associated with

R^o_{ba} and R^o_{ab} must be statistically independent. [If they were correlated, then R^o_{ba} and R^o_{ab} would reflect this correlation much as do R_{ba} and R_{ab} in eq. (15).] In discussing the discharge of the capacitor, we pointed out how $d(t)$ acted as a noise source containing only the spontaneous fluctuations. Above, we have noted that these fluctuations are governed by the average rates R^o. This suggests that to calculate the statistics of d_{ab} and d_{ba}, which according to (18) are needed to calculate the statistics of δn_a and δn_b, we may write the $d(t)$'s in the following form:

$$d_{ab}(t) = \sum_i \delta(t - t^o_{abi}) - R^o_{ab}(t) \tag{19a}$$

and

$$d_{ba}(t) = \sum_i \delta(t - t^o_{bai}) - R^o_{ba}(t). \tag{19b}$$

We stress at this point that for a specific event ($\{t_{abi}, t_{bai}\}$) we need not demand that the right-hand sides of (16) and (19) be equal. This is because we eventually average the dependence of n_a and n_b on d_{ab} and d_{ba} over all events in calculating correlation functions and spectral densities. All that is necessary is that the statistical properties of the two forms of d_{ab} and d_{ba} be the same, at least to lowest order in the size of the fluctuations. We shall see below that this is indeed the case. We are able to calculate simply the statistics of d_{ab} and d_{ba} because in writing (19) we have cast these random variables into a form in which the distinction between spontaneous and induced fluctuation no longer enters. We can do this because d involves only spontaneous (hence independent) fluctuations and, therefore, can be written in terms of independent events.

With independent transfer times, it is straightforward to calculate the statistical distributions of d_{ab} and d_{ba} as given in (19). This calculation is carried out in Appendix A. The autocorrelation functions for d_{ab} and d_{ba}, which are the spontaneous-fluctuation, noise-source terms, are given by

$$\langle d_{ab}(t_1) d_{ab}(t_2) \rangle = R^o_{ab}(t_1)\delta(t_1 - t_2) \tag{20a}$$

and

$$\langle d_{ba}(t_1) d_{ba}(t_2) \rangle = R^o_{ba}(t_1)\delta(t_1 - t_2), \tag{20b}$$

as expected for pure shot noise associated with independent events. In addition, if R^o_{ab} and R^o_{ba} are independent of time (so that the noise is stationary), then the spectral densities of d_{ab} and d_{ba} are given by

$$S^d_{ab}(f) = 2R^o_{ab} \tag{20c}$$

and

$$S^d_{ba}(f) = 2R^o_{ba}. \tag{20d}$$

The form given d_{ab} and d_{ba} in eq. (19), or rather their statistical form, eq. (20), is the basic assumption that enters our approach. That it is valid so long as the fluctuations δR in the rates $R = R^o + \delta R$ satisfy

$$\langle \delta R^2 \rangle^{\frac{1}{2}} \ll R^o \tag{21}$$

may be motivated by the following. If a fluctuation δn in n leads to a fluctuation δR in R, then the statistics of the immediately following t_i will be governed by the altered rate. This suggests as a first-order iteration in the fluctuation that

$$\langle d(t_1)d(t_2) \rangle \approx R(t_1)\delta(t_1 - t_2). \tag{22}$$

However, $R = R^o + \delta R$; thus, the error in (20) is of the order of $\delta R/R^o$, which we assume to be small. A similar argument applies to residual correlation effects among the d's. Thus, so long as (21) is satisfied, we expect (20) to be valid to order $\delta R/R^o$. Since we are discussing a linear theory, this is good enough for our purposes.

Returning to (18), assuming stationary noise so that we may use Fourier-transform techniques, and representing that the linear operators are simple decay rates for purposes of illustration, we may write

$$i\omega \delta n_a = -\delta n_a/\tau_a + \delta n_b/\tau_b - d_{ba} + d_{ab} \tag{23a}$$

and

$$i\omega \delta n_b = +\delta n_a/\tau_a - \delta n_b/\tau_b - d_{ab} + d_{ba}, \tag{23b}$$

which become, upon solving for $\delta n_a(\omega)$ and $\delta n_b(\omega)$ in terms of $d_{ba}(\omega)$ and $d_{ab}(\omega)$,

$$\delta n_a = -\delta n_b = (d_{ab} - d_{ba})(i\omega + 1/\tau)^{-1}, \tag{23c}$$

where

$$1/\tau = 1/\tau_a + 1/\tau_b. \tag{23d}$$

Since the d_{ab} and d_{ba} are independent, for spectral densities we obtain

$$S^{\delta n}_a(f) = S^{\delta n}_b(f) = |i\omega + 1/\tau|^{-2}[S^d_{ab}(f) + S^d_{ba}(f)] \tag{24a}$$

$$= |i\omega + 1/\tau|^{-2}(2R^o_{ab} + 2R^o_{ba}) \tag{24b}$$

$$= -S^{\delta n}_{ab}(f) = -S^{\delta n}_{ba}(f), \tag{24c}$$

where the expressions in (24c) are cross-spectral densities ($f = \omega/2\pi$). In such a problem as this with transfers permitted only between two states, the correlation between δn_a and δn_b is maximal. (Introducing

more states will lessen the correlation between any two.) Nonetheless, the important features of the correlation are clearly obtained in the result (24b): the small-signal decay rate $(1/\tau)$ is the sum of the separate decay rates, and the spectral density is proportional to a full shot-noise term derived from the sum of the noiseless rates $(R^o_{ab} + R^o_{ba})$. This example is admittedly very trivial and could have been done more easily by other means. In the following sections, the advantages of the present method will become evident. Note, however, that there was no question as to how to include the driving terms when writing down the coupled noise equations (18) and no question regarding the independence of these driving terms when it came to calculating spectral densities in (24).

Let us pause before going on to summarize the logic we have used in arriving at our method. First we recognized that noise arises owing to the randomness in the times at which charge carriers change their state. Then for each transfer process, we separated the expression for the particle current flowing between any two states into a dynamical-rate term $R(t)$ that includes all induced fluctuations, and a driving term $d(t)$ that includes only spontaneous (and therefore independent) fluctuations. In this way, we are able to shift the statistical correlations, so to speak, out of the statistical term and into the dynamical term. To calculate the statistics of our driving term, we noted that since the fluctuations were characterized by a mean rate of $R^o(t)$, which depended only on the noiseless solution, we could rewrite $d(t)$ in a natural way in terms of independent events. [These latter events cannot be coupled since their statistics are governed only by $R^o(t)$ and not by preceding events.] Although this step represented an approximation, we argued that it should be all right for small fluctuations. With the new form for $d(t)$, its statistics were readily calculated. Finally, knowing the statistics of d and the linear relation between δn and d, the statistics of δn including all correlation effects of interest, which we desire, are straightforward to obtain. In the next two sections, we shall carry out the above procedure for two examples from start to finish.

In passing, we make reference to Lax's discussion[12] in which he showed the basic limitation of any source theory of noise. The heart of Lax's argument is that one cannot expect to model on the time scale of the duration of a spontaneous fluctuation (during which time the system appears to be reversible) by using the essentially irreversible dynamics embodied in eq. (18). As for time scales on the order of the response time of the carriers to spontaneous fluctuations, however, all is well. Thus, as Langevin probably recognized long ago, the price paid

for using source terms is small. The whole trick is to include them unambiguously.

III. NOISE IN CHARGE TRANSFER

The purpose of this section is to stress the insight one can gain by starting with the most elementary transfer processes taking place in a device. In this case, we avoid making assumptions about the statistics of the velocity fluctuations of carriers by calculating them from occupancy fluctuations, which can be understood much more simply using our method. For simplicity, we shall ignore correlation effects, which in fact are shown to be negligible for the problem we discuss. In the next section, which is on recombination, we shall stress correlation effects and how they can be dealt with using the microscopic method.

One of the most basic approaches to the problem of the storage and transport of carriers in a device is first to assign a density function to each carrier state. For example, $n_a(\mathbf{x}, \mathbf{v}, t)d\mathbf{x}d\mathbf{v}$ represents the number of carriers of type a at $\mathbf{x}$ with velocity $\mathbf{v}$ in the element $d\mathbf{x}d\mathbf{v}$ of phase space. Similarly, $n_b(\mathbf{x}, t)d\mathbf{x}$ represents the number of carriers of type b trapped at $\mathbf{x}$ in volume $d\mathbf{x}$. In this example, let us ignore trapping and recombination effects and focus attention on the transport via scattering of a single type of carrier. This will lead to a general expression for diffusion noise which we can compare with the results obtained by Langevin and by Shockley et al. using the Langevin method and the impedance-field methods, respectively.

We proceed as follows. If at times t_{ijl} a particle is scattered from state j to state i, then we have (by analogy with our treatments above) that

$$\frac{dn_i}{dt} = -\sum_{jl} \delta(t - t_{jil}) + \sum_{jl} \delta(t - t_{ijl}) \tag{25}$$

and

$$\frac{dn_i}{dt} = -\sum_{j} R_{ji}(t) + \sum_{j} R_{ij}(t) - \sum_{j} d_{ji}(t) + \sum_{j} d_{ij}(t). \tag{26}$$

The subscripts i, j each designate a particular region of phase space $d\mathbf{x}d\mathbf{v}$ at $(\mathbf{x}, \mathbf{v})$. (We do this to simplify the notation.) In (26), $d_{ij}(t)$ is defined by

$$d_{ij}(t) = \sum_{l} \delta(t - t_{ijl}) - R_{ij}(t). \tag{27}$$

It is important to note that we introduce a source term for *each* elementary process, that is, for each transfer process indexed by the ordered pair (i, j).

As before, we let $n_i \equiv n_i^o + \delta n_i$, insert into (26), and expand to lowest order in δn_i. The noiseless equation that results in

$$\frac{dn_i^o}{dt} = -\sum_j R_{ji}^o(t) + \sum_j R_{ij}^o(t) \tag{28}$$

is just the Boltzmann equation, R_{ij}^o being a function of n_i^o and n_j^o. Using $\mathbf{x}$, $\mathbf{v}$ notation, (28) becomes

$$\begin{aligned}\frac{dn(\mathbf{x}, \mathbf{v}, t)}{dt} &= \frac{\partial n(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \frac{\partial n(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{x}} + \frac{\mathbf{F}}{m} \cdot \frac{\partial n(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{v}} \\ &= -\int d\mathbf{x}'\, d\mathbf{v}' (R(\mathbf{x}, \mathbf{v}; \mathbf{x}', \mathbf{v}') - R(\mathbf{x}', \mathbf{v}'; \mathbf{x}, \mathbf{v})),\end{aligned}$$

the more usual form of the Boltzmann equation, where $R(1, 2)$ is the average rate at which particles are scattered from 2 to 1. In general, $R(1, 2)$ is a function of n_1 and n_2. Usually $R(1, 2)$ is taken to be proportional to n_2, however. To determine such noiseless quantities as mobility, etc., (28) must be solved for the noiseless $n^o(\mathbf{x}, \mathbf{v}, t)$.

The equation for the noise $\delta n(\mathbf{x}, \mathbf{v}, t) = \delta n_i$ obtained from (26) is

$$\begin{aligned}\delta \dot{n}_i = &-\sum_{j,k} \frac{\delta R_{ji}}{\delta n_k} \delta n_k + \sum_{j,k} \frac{\delta R_{ij}}{\delta n_k} \delta n_k \\ &- \sum_j d_{ji}(t) + \sum_j d_{ij}(t).\end{aligned} \tag{29}$$

In general, this is a coupled (linear, integral-differential) equation among the δn_i. [As before, the linear operator $\delta R(n)/\delta n$ is evaluated at its noiseless value by inserting the noiseless solutions n_i^o of (28).] For our purposes here, we simplify (29) by assuming (*i*) that R_{ji} is a function of (and not an operator on) the n_k, and (*ii*) that in (29) we can ignore δn_k where $k \neq i$. Assumption (*i*) is in fact the usual situation one has with the Boltzmann equation. Assumption (*ii*) can be made plausible in the following manner. By ignoring in (29) δn_k, $k \neq i$, we are ignoring correlations among the fluctuations δn_i and δn_k. Let us suppose that the scattering rates between i and k are equal for all $N(N + 1)$ (i, k) pairs. Then a fluctuation δn_i will lead to a δn_k on the order of $\delta n_i/N$. The effect of δn_k back on state i will be of the order $\delta n_k/N$. Thus, summing over the N states $k \neq i$, the correlated contribution to δn_i which we lose by ignoring the δn_k in (29) is of the order of $\delta n_i/N$ $[= N \cdot (\delta n_i/N)/N]$, which for large N is entirely negligible. While we have assumed that the transfer rates between all i and k states are equal, a similar argument applies so long as the scattering to a few states is not favored. In this case, the δn_k dependence of these

few states must be included in (29). (We have already considered the two-state problem where correlation is largest. In our next example, we shall consider recombination in which correlation among several states is important.) Finally, we should note that our approximation is *not* equivalent to a relaxation-time approximation, which might be used to simplify (28).

Returning to our example, and based upon the reasoning given above, we approximate (29) by

$$\begin{aligned}\delta\dot{n}_i = &- \sum_j \frac{\delta R_{ji}}{\delta n_i}\,\delta n_i + \sum_j \frac{\delta R_{ij}}{\delta n_i}\,\delta n_i \\ &- \sum_j d_{ji}(t) + \sum_j d_{ij}(t)\end{aligned} \tag{30a}$$

and

$$\delta\dot{n}_i = -\delta n_i/\tau_i(t) - d_i^o(t) + d_i^e(t), \tag{30b}$$

where

$$1/\tau_i(t) \equiv \sum_j \left(\frac{\delta R_{ji}}{\delta n_i} - \frac{\delta R_{ij}}{\delta n_i}\right), \tag{30c}$$

$$d_i^o(t) \equiv \sum_{j,l} \delta(t - t_{jil}) - \sum_j R_{ji}(t), \tag{31a}$$

and

$$d_i^e(t) \equiv \sum_{j,l} \delta(t - t_{ijl}) - \sum_j R_{ij}(t). \tag{31b}$$

The superscripts on d_i, "o" and "e," designate "out" and "in" scattering, respectively. We are able to lump the driving terms together in this way because (30a, b) involve only δn_i and because the residual correlation between d_i^o and d_i^e is of order $1/N$.

To proceed further, we approximate (31a, b) in the manner discussed above [see (19)] and determine the statistics of δn_i. Using the general results derived in Appendix A for d and then in Appendix B for $2n$ as a function of d, we find for the correlation function of δn_i, δn_j the expression

$$\langle \delta n_i(t_1)\delta n_j(t_2)\rangle = \delta_{ij}[R_i^o(\text{in}) + R_i^o(\text{out})](\tau_i/2)e^{-|t_1-t_2|/\tau_i}, \tag{32}$$

where we have assumed that the noiseless rates $R_i^o(\text{in}) = \sum_j R_{ij}^o$ and $R_j^o(\text{out}) = \sum_j R_{ji}^o$ [and therefore τ_i defined in (30)] are independent of time. One further simplification is often justified. Under the stationary conditions taken here to arrive at (32), $R_i^o(\text{in}) = R_i^o(\text{out})$. Furthermore, if R_{ji} depends only on n_i and is directly proportional thereto, it follows

that

$$R_i^o(\text{out})\tau_i = n_i^o. \tag{33}$$

Thus, we find the expected result that

$$\langle \delta n_i(t_1)\delta n_j(t_2)\rangle = \delta_{ij} n_i^o e^{-|t_1-t_2|/\tau_i}. \tag{34}$$

We shall now use this result to determine the statistics of the velocity fluctuations of the individual carriers. It is velocity fluctuations rather than occupancy fluctuations that are usually taken as basic. We shall see, however, that occupancy fluctuations are the more fundamental of the two.

Diffusion noise arises from the velocity fluctuations that a charge carrier undergoes owing to its interaction with the material in which it is confined as well as with the other carriers. Since the current in a device is made up of the linear superposition of the currents carried by each carrier, and since each contribution is proportional to the velocity of the individual carrier, a calculation of the current correlation function (to determine device noise) requires knowledge of the velocity autocorrelation function for each carrier. This latter quantity we can calculate at once from (34). We proceed in the following manner.

By definition, the velocity of a carrier is given by

$$\mathbf{v}(\mathbf{x}, t) = \frac{\int d^3\mathbf{v}'\mathbf{v}' n(\mathbf{x}, \mathbf{v}', t)}{\int d^3\mathbf{v}' n(\mathbf{x}, \mathbf{v}', t)}. \tag{35}$$

If we recall that $n = n^o + \delta n$, we can write (35) in the form $\mathbf{v} = \mathbf{v}^o + \delta\mathbf{v}$ to determine that

$$\mathbf{v}^o(x_{n,t}) = \frac{\int d^3\mathbf{v}'\mathbf{v}' n^o(\mathbf{x}, \mathbf{v}', t)}{\int d^3\mathbf{v}' n^o(\mathbf{x}, \mathbf{v}', t)} \tag{36}$$

and that

$$\delta\mathbf{v}(x, t) = \frac{\int d^3\mathbf{v}'\mathbf{v}'\delta n(\mathbf{x}, \mathbf{v}', t)}{\int d^3\mathbf{v}' n^o(\mathbf{x}, \mathbf{v}', t)}, \tag{37}$$

where we have assumed that the total number of carriers within a volume element $d\mathbf{x}$ remains at the noiseless value. We make this assumption only because we are interested here in diffusion noise only

and not noise from fluctuations in particle density. It follows at once from (34) (assuming stationarity) and (37) that the velocity correlation function of interest is given by

$$\langle \delta v_i(\mathbf{x}_1, t)\delta v_j(\mathbf{x}_2, t_2)\rangle = \delta(\mathbf{x}_1 - \mathbf{x}_2)\frac{\int d^3\mathbf{v}' v_i' v_j' n^o(\mathbf{x}_1, \mathbf{v}')}{n^o(\mathbf{x}_1)\int d^3\mathbf{v}' n^o(\mathbf{x}_1, \mathbf{v}')} e^{-|t_1-t_2|/\tau(\mathbf{x}_1,\mathbf{v}')}. \quad (38)$$

In (38), i and j now correspond to the components of the velocity $\mathbf{v} = (v_x, v_y, v_z)$.

When one is dealing with stationary noise, it is more convenient to work with spectral densities than autocorrelation functions. Using the standard definitions of the spectral density function,[13] one finds using (34) that

$$\begin{aligned} S_{\delta n}(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, f) &= \delta(\mathbf{x}_1 - \mathbf{x}_2)\delta(\mathbf{v}_1 - \mathbf{v}_2)4\int_0^\infty e^{-t/\tau(\mathbf{x}_1,\mathbf{v}_1)} n^o(\mathbf{x}_1, \mathbf{v}_1)\cos \omega t dt \\ &= \delta(\mathbf{x}_1 - \mathbf{x}_2)\delta(\mathbf{v}_1 - \mathbf{v}_2)\frac{4n^o(\mathbf{x}_1, \mathbf{v}_1)\tau(\mathbf{x}_1, \mathbf{v}_1)}{1 + \omega^2\tau^2(\mathbf{x}_1, \mathbf{v}_1)}. \end{aligned} \quad (39)$$

It follows that the spectral density of the velocity fluctuations is given by

$$S_{\delta\mathbf{v},ij}(\mathbf{x}_1, \mathbf{x}_2, f) = \delta(\mathbf{x}_1 - \mathbf{x}_2)\int d\mathbf{v}\frac{4n^o(\mathbf{x}_1, \mathbf{v})\tau(\mathbf{x}_1, \mathbf{v})}{1 + \omega^2\tau^2(\mathbf{x}_1, \mathbf{v})}\frac{v_i v_j}{n^o(\mathbf{x}_1)^2}, \quad (40)$$

an expression which could have been obtained directly from (38).

The spectral density given in (40) has several interesting features that we shall touch on briefly. Ordinarily for the frequencies of interest in devices, $\omega^2\tau^2 \ll 1$, owing to the very rapid carrier-scattering rates. In this case (40) reduces to[4,14]

$$S_{\delta\mathbf{v},ij} = \delta(\mathbf{x}_1 - \mathbf{x}_2)4D_{ij}/n^o(\mathbf{x}_1), \quad (41)$$

where D_{ij} is the ij component of the diffusion tensor defined by

$$D_{ij} \equiv \frac{1}{2}\frac{d}{dt}\langle \delta\mathbf{x}_i(t_1)\delta\mathbf{x}_j(t_1 + t)\rangle. \quad (42)$$

This expression (42) can be derived from (38) by noting that

$$\delta x(t) = \int_{-\infty}^{t} dt'\delta v(t').$$

The diagonal components of D, $D_{ij} \equiv D_i$ satisfy

$$D_i = \frac{\int d\mathbf{v} v_i^2 n^o(\mathbf{x}, \mathbf{v}) \tau(\mathbf{x}, \mathbf{v})}{\int d\mathbf{v} v_i^2 n^o(\mathbf{x}, \mathbf{v})} \int d\mathbf{v} v_i^2 n^o(\mathbf{x}, \mathbf{v}) / n^o(\mathbf{x}). \tag{43}$$

In this form, we recognize that the first factor is $m\mu_i(\mathbf{x})/q$, while for a thermal distribution, the second factor is kT/m. Here, $\mu_i(\mathbf{x})$ is the carrier mobility in the ith direction at $\mathbf{x}$ derived using a standard Boltzmann equation approach. Inserting into (41), we find the usual result for thermal noise, namely that

$$S_{\delta v}(\mathbf{x}_1, \mathbf{x}_2, f) = \delta(\mathbf{x}_1 - \mathbf{x}_2) 4kT\mu(\mathbf{x}) / (qn^o(\mathbf{x}_1)). \tag{44a}$$

In Appendix C we generalize (44) to the case of field-dependent mobilities $\mu(E)$. The result when an effective temperature T can be defined is that

$$S_{\delta v}(\mathbf{x}_1, \mathbf{x}_2, f) = \delta(\mathbf{x}_1 - \mathbf{x}_2) 4 \left(\frac{kT}{q} \right) \frac{d[\mu(E)E]}{dE} \Big/ n^o(\mathbf{x}_1). \tag{44b}$$

We are now in a position to compare the results obtained here with those obtained by Langevin[3] and by Shockley et al.[4] using additional assumptions. For simplicity, we shall work in one dimension and ignore spatial variations. In the Langevin[3] method, one begins by decomposing the force acting on each carrier into two parts, a damping force proportional to the velocity and a stochastic force of zero mean. The purpose of the latter is, of course, to produce the random fluctuations in the velocity. Thus, one writes

$$\delta\dot{v} = -\delta v/\tau + h(t), \tag{45}$$

where $\tau = m\mu/q$. One then calculates the spectral density of δv in terms of that of h, obtaining

$$S_{\delta v}(f) = \frac{S_h(f)\tau^2}{1 + \omega^2\tau^2}. \tag{46}$$

If one assumes that $S_h(\omega) = S_h(0)$ [white noise corresponding to totally uncorrelated $h(t)$], then since

$$\frac{kT}{m} = \langle \delta v^2 \rangle = \int_0^\infty df S_{\delta v}(f) = S_h(0)\tau/4, \tag{47}$$

one finds that $S_h(\omega) = 4kT/m\tau$. It follows then from (46) that

$$S_{\delta v}(f) = \frac{4kT\tau/m}{1 + \omega^2\tau^2} = \frac{4kT\mu/e}{1 + \omega^2\tau^2} \tag{48a}$$

$$= \frac{4\int dv n^o(v)v^2\tau}{1 + \omega^2\tau^2}. \tag{48b}$$

This result agrees with (44) for $\omega\tau \ll 1$. If, however, we are interested in ω, for which $\omega\tau \approx 1$, then this result differs markedly from (40), which we write as

$$S_{\delta v}(f) = 4\int dv \frac{n^o(v)v^2\tau(v)}{1 + \omega^2\tau^2(v)}, \tag{49}$$

unless $\tau(v) = \tau$, a constant. This, however, is seldom the case in realistic scattering problems. We may rescue the Langevin approach if, in place of (45), we pass to the frequency domain to write

$$i\omega\delta v = -\delta v/\tau_\omega + h_\omega, \tag{50}$$

in which the effective damping $1/\tau_\omega$ is frequency dependent; and now

$$S_{\delta v}(f) = \frac{S_h(f)\tau_\omega^2}{1 + \omega^2\tau_\omega^2}. \tag{51}$$

Again, we choose $S_h(f) = S_h(0)$ and note that as $f \to 0$ we must obtain (48). Thus, $S_h(f) = 4kT/m\tau$, where $\tau = \tau(\omega = 0)$. Inserting into (51) we obtain

$$S_{\delta v}(f) = \frac{4kT/m\tau}{\omega^2 + 1/\tau_\omega^2} = \frac{4\int dv n^o(v)v^2/\tau}{\omega^2 + 1/\tau_\omega^2}. \tag{52}$$

If (52) is equated to (49), one can solve for $1/\tau_\omega$, that is, for the appropriate damping term to be used in the Langevin equation. This illustrates a major defect of the Langevin[3] approach, quite apart from inserting h_ω in a rather arbitrary manner. If the appropriate τ_ω is not known *a priori*, one must return to a more fundamental approach such as that given above. This is important, for if $\dot{x}_\omega = \mu_\omega E_\omega$, then

$$\mu_\omega = (e/m)(i\omega + 1/\tau_\omega')^{-1} = (e/kT)\int dv v^2 n(v)[i\omega + 1/\tau(v)]^{-1}.$$

Setting $\tau_\omega = \tau_\omega'$ does *not* make (52) equal (49). Thus, the original Langevin idea is not internally consistent in general.

In developing the impedance-field[4] method, Shockley et al. focus attention on the trajectory of a given carrier and on the carrier's deviations from its noiseless trajectory. Nonetheless, when it comes to writing an expression for the autocorrelation function of the velocity, they must postulate an expression equivalent to (38). [See eq. (48) of Ref. (4).] In more complicated problems, it seems best to have a fundamental approach from which such microscopic correlation functions can be derived. For example, were it necessary to include correlation between certain pairs of states, several characteristic decay times would be present in the correlation expression. These can be included in a natural way if we derive $\delta \mathbf{v}$ from eq. (37), as we have done here. Using less fundamental approaches, one must often rely on analogy and intuition.

IV. RECOMBINATION-GENERATION NOISE

Up to this point, we have been concerned with deriving the statistical distributions of microscopic variables. From a knowledge of the dependence of the macroscopic quantities of interest on these microscopic variables, the statistical distributions of the former can be calculated from those of the latter. In some cases, e.g., thermal noise in resistors, the macroscopic current or voltage spectral densities can be obtained very simply using thermodynamic arguments without recourse to microscopic methods. We now turn to an example, that of recombination noise, for which the power of treating correlations among fluctuations using the microscopic method developed in this paper becomes apparent.

Let us consider the following model of recombination. Let n be the number of mobile electrons, p the number of holes, n_{to} the number of unoccupied traps, n_{tp} the number of traps containing a trapped hole, and n_{tn} the number of traps containing trapped electron, all within a unit volume. Recombination occurs when a trap containing an electron captures a hole or when a trap containing a hole captures an electron. In Fig. 2, we present schematically the eight different processes which enter into this recombination-generation model. In Fig. 2, as in the equations for the dynamics of this system, the r's represent trapping rates, e.g., r_{ton} the rate at which electrons (n) are trapped by empty traps (t_o) per electron per trap, and the g's represent release rates from the traps per trap. As we shall see, the key to determining the noise correctly within this model is to introduce source terms for each of these eight processes.

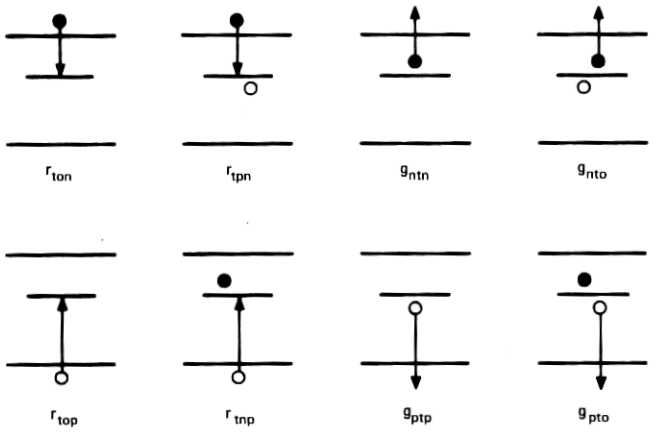

Fig. 2—The eight trapping processes contributing to recombination.

Even before we consider this problem quantitatively, we realize that correlations among fluctuations in the various particle densities will be complicated. For example, a positive fluctuation δn in the number of mobile electrons will lead to positive fluctuations in the recombination directly via increased trapping in the t_p traps and indirectly via increased trapping in the t_o traps, and the increased number of t_o traps. Using our microscopic approach we can deal with these correlations in a routine manner.

The step-by-step application of our method to this problem is carried out in some detail in Appendix D. If we make use of the definitions $\delta n \equiv \delta n_1$, $\delta p \equiv \delta n_2$, $\delta n_{tn} = \delta n_3$, $\delta n_{tp} = \delta n_4$, $\delta \mathbf{n} = (\delta n_1, \delta n_2, \delta n_3, \delta n_4)$, then the cross-spectral density of δn_i, δn_j is given by

$$S_{ij}^{\delta \mathbf{n}}(f) = \sum_{kl} M_{ik}(\omega) S_{kl}^{\mathbf{d}}(f) M_{lj}^{*}(\omega), \tag{53}$$

according to (124a). In (53), $\mathbf{S}^{\mathbf{d}}$ is the (4×4) cross-spectral density of the driving sources given by (124b) and $\mathbf{M}$ is the (4×4) matrix expressing the induced fluctuations $\delta \mathbf{n}$ in terms of the spontaneous ones $\mathbf{d}$:

$$\delta \mathbf{n}(\omega) = \mathbf{M}(\omega) \mathbf{d}(\omega), \tag{54}$$

according to (123). [Note from Appendix D that the vector $\mathbf{d} \equiv (d_1, d_2, d_3, d_4)$ consists of four linear combinations of the eight fundamental noise driving terms that arise using this method. That these terms are mutually correlated can be seen from (124b).] Thus, from (53) we can calculate the spectral densities of interest. These include S_{11} for the

spectral density of δn, S_{22} for that of δp, and S_{12} and S_{21} for the cross-spectral densities between δn and δp, all of which contribute to the current fluctuations observed at the contacts of the device. Even more components of $\mathbf{S}^{\delta n}$ are needed to calculate fluctuations in recombination radiation, if such results are desired.

We should stress that (53) contains all correlation effects (through $\mathbf{S}^d$) and all relaxation effects (through $\mathbf{M}$ and $\mathbf{M}^*$) contained in this problem. If certain trapping or release rates are small relative to others, the contribution of such processes to $\mathbf{S}^d$ can be neglected; if $\mathbf{M}$ can be characterized by a single time constant τ, $\mathbf{MM}^*$ becomes proportional to $(\omega^2 + 1/\tau^2)^{-1}$, as is usually assumed.[8] The point to be made is that with *no* additional effort of a statistical nature we can solve device-noise problems which contain rather complicated statistical correlations. The key is to put the correlations in the dynamics of the problem of interest, thereby keeping the statistics simple.

Finally, the reader is cautioned against making a normal mode analysis of the linearized noise equations and then introducing a noise-source term for each normal mode, since such terms will not be statistically independent in general. We note in concluding that, in obtaining (53), nowhere did we have to make use of normal-mode analysis.

V. CONCLUSION

In this paper we have discussed a straightforward, microscopic approach which can be used to calculate the statistical fluctuations accompanying any transfer process. Since nearly all charge-carrier velocity fluctuations can be characterized in terms of transfer processes, we, therefore, are able to calculate the statistics of the velocity fluctuations from those of the transfer fluctuations. Knowledge of the velocity fluctuations is often all the microscopic information that is needed to insert into the Langevin method or the impedance-field method to calculate device noise. (These methods explain in great detail how to convert velocity fluctuations into observed current and voltage fluctuations.) In so doing, we are able to insure that all important correlation effects and relaxation effects are included in the results. We further simplify the statistical portion of the calculation by separating the spontaneous from the induced fluctuations, and expressing the statistics of the latter in terms of the former. In this way, we have to calculate the statistics of only the uncorrelated, spontaneous fluctuations from probability theory, while the more complicated statistics of the induced fluctuations can be obtained directly from those of the spontaneous

type. The correlation effects are thus shifted from the statistical portion of the problem to the dynamical portion. We have included (in Appendix C) a derivation of an important result for the diffusion noise of "hot" charge carriers, that is, for charge carriers whose mobilities are field dependent. As the method developed here is readily generalized to nonstationary noise, our results will be used extensively in treating noise in charge-transfer devices.

VI. ACKNOWLEDGMENT

It is a pleasure to thank J. R. Brews for helpful discussions.

APPENDIX A

In this appendix, we consider in detail a single elementary random process and calculate its statistical distribution. As we have shown in the text, many complicated random processes can be decomposed into simpler processes of the type considered here. Once the statistical distribution of these simpler processes is understood, one can determine the distribution of the more complicated process of interest. In the text, we have focused attention on autocorrelation functions, and on how to distinguish the autocorrelation function of a complicated process from those of the simpler processes of which it is composed. In Appendix B, we show how to obtain distribution functionals of complicated processes from those of simpler ones.

The random variable whose statistics we seek is

$$d(t) = \sum_i \delta(t - t_i) - R^o(t), \tag{55}$$

where $R^o(t)$ (a specific function of time) is the mean rate of occurrence of the t_i at time t. In other words, the t_i are independent random variables, each specifying the time at which an independent random event occurs. To be a meaningful construct, it is necessary that $R^o(t)$ satisfy

$$R^o(t) \gg \frac{dR^o(t)}{dt} \frac{1}{R^o(t)}, \tag{56}$$

that is, that the characteristic time in which $R^o(t)$ changes is much longer than the average time necessary for an event to occur. If (56) is satisfied, many events will almost surely occur in time intervals during which $R^o(t)$ changes only by a small amount. During these intervals, the statistics of the t_i will be Poisson with mean rate $R^o(t)$.

In probability theory, one often works with distribution functions. Thus, if a random variable x has a probability of $p_x(x_1)dx_1$ at x_1, that

x is in dx_1, then the distribution function $q_p(k)$ of $p_x(x)$ is defined by

$$q_p(k) = \int_{-\infty}^{\infty} dx e^{ikx} p_x(x). \tag{57}$$

Since

$$p_x(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} q_p(x), \tag{58}$$

it is clear that $q_p(k)$ contains as much information about x as does $p_x(x)$. What is important is that it contains this information in a more convenient form, since, for example, all moments of x can be obtained at once from $q_p(k)$ by suitable differentiations with respect to k.

Here, we shall calculate the distribution functional[15] $Q_d(k(t))$ of $d(t)$ defined by

$$Q_d[k(t)] = \left\langle \exp\left(i \int_{t_1}^{t_2} dt k(t) d(t) \right) \right\rangle, \tag{59}$$

where the brackets denote averaging over the probability distribution of $d(t)$. Rather than use the form of $d(t)$ given in (55), let us use instead the form

$$d(t) = \sum_i f(t - t_i) - R^o(t), \tag{60}$$

where $f(t)$ is a function of t satisfying

$$\int_{-\infty}^{\infty} dt f(t) = 1. \tag{61}$$

Equation (59) then becomes

$$Q_d[k(t)] = \left\langle \exp\left[i \int_{t_1}^{t_2} dt k(t) \left(\sum_i f(t - t_i) - R^o(t) \right) \right] \right\rangle. \tag{62}$$

Let us now evaluate this average.

To determine $Q_d[k(t)]$ we note the following.

(*i*) $R^o(t)$ is a specific function of time and, hence,

$$\exp\left(-i \int_{t_1}^{t_2} dt k(t) R^o(t) \right)$$

can be factored out of the brackets.

(*ii*) The t_i are independent and, hence, the average of the exponential in $\sum_i f(t - t_i)$ can be factored into products of averages of an exponential in each $f(t - t_i)$, each of the n averages being equal to all the others.

(*iii*) The probability that n events occur in the interval $t_1 < t < t_2$ is Poisson.

From these three considerations, it follows that

$$Q_d[k(t)] = \exp\left(-i\int_{t_1}^{t_2} dtk(t)R^o(t)\right) \times \sum_{n=0}^{\infty} P_n\left[\left\langle\exp\left(i\int_{t_1}^{t_2} dtk(t)f(t-t_i)\right)\right\rangle\right]^n, \quad (63)$$

where P_n is given by

$$P_n = e^{-\Lambda}\Lambda^n/n! \quad (64)$$

and Λ is given by

$$\Lambda = \int_{t_1}^{t_2} R^o(t)dt. \quad (65)$$

To evaluate the average in (63), we note that the probability that t_i occurs in dt' at t' is $R^o(t')dt'/\Lambda$. Thus,

$$\left\langle\exp\left(i\int_{t_1}^{t_2} dtk(t)f(t-t_i)\right)\right\rangle = \frac{1}{\Lambda}\int_{t_1}^{t_2} dt_i \exp\left(i\int_{t_1}^{t_2} dtk(t)f(t-t_i)\right) R^o(t_i). \quad (66)$$

Inserting this into (63) and carrying out the sum on n, one obtains finally for $Q_d[k(t)]$

$$Q_d[k(t)] = \exp\left\{\int_{t_1}^{t_2} dt'R^o(t')\left[\exp\left(i\int_{t_1}^{t_2} dtk(t)f(t-t')\right)-1\right]\right\} \times \exp\left(-i\int_{t_1}^{t_2} k(t)R^o(t)dt\right). \quad (67)$$

We note from (59) that if $k(t) = 0$, then Q_d should equal 1. Since (67) satisfies this condition, we are assured that $Q_d[k(t)]$ is properly normalized. As we shall see in Appendix B, from (67) we can calculate the distribution functional of nearly any function of $d(t)$.

It often happens that the particular process of interest involves numerous events, each of which contributes only a very small portion to the total fluctuation. In this case, we can expand the exponential in $f(t-t')$ to quadratic order, obtaining for $Q_d[k(t)]$

$$Q_d[k(t)] = \exp\left[-\frac{1}{2}\int_{t_1}^{t_2} dt'R^o(t')\left(\int_{t_1}^{t_2} dtf(t-t')k(t)\right)^2\right], \quad (68)$$

where we have assumed that $f(t)$ is of sufficiently short duration that

$$\int_{t_1}^{t_2} dt'R^o(t')f(t-t') = R^o(t). \quad (69)$$

Recalling finally that $f(t - t') = \delta(t - t')$ for the $d(t)$ of (55), it follows that

$$Q_d[k(t)] = \exp\left(-\frac{1}{2}\int_{t_1}^{t_2} dtk(t)k(t)R^o(t)\right). \tag{70}$$

In the text, we made repeated use of the autocorrelation function of d, $\langle d(t_1)d(t_2)\rangle$. This we may calculate from (67) with $f(t) = \delta(t)$, or from (70), the results being the same. To do this, we note that from (59)

$$\langle d(t_1)d(t_2)\rangle = -\frac{\partial}{\partial\gamma_1}\bigg|_{\gamma_1=0}\frac{\partial}{\partial\gamma_2}\bigg|_{\gamma_2=0} Q_d[\gamma_1\delta(t - t_1) + \gamma_2\delta(t - t_2)]. \tag{71}$$

Thus, from (70) we readily find that

$$\langle d(t_1)d(t_2)\rangle = R^o(t_1)\delta(t_1 - t_2). \tag{72}$$

Although (67) is our most general expression for the statistics of $d(t)$, (72) is usually all that is needed in noise calculations for devices. In simplest terms, it is the autocorrelation of the (shot) noise associated with the process of which $d(t)$ represents the source term and $R^o(t)$ the average rate at time t. It is the building block from which most device noise can be constructed. For stationary processes, $R^o(t_1) = R^o$, a constant depending on the noiseless solution. The spectral density of d under such conditions is a useful concept and is given by

$$S_d(f) = 2R^o. \tag{73}$$

APPENDIX B

In this appendix, we show how to obtain the distribution functional of a statistical process composed of a number of simple independent processes of the type discussed in Appendix A. Suppose that the process of interest, $\delta n(t) \equiv n(t) - n^o(t)$, is a linear functional of a number of $d(t)$, $\{d_i(t), 1 \leqq i \leqq m\}$. Then

$$\delta n(t) = \mathfrak{F}_n[d_1(t), \cdots, d_m(t), t]. \tag{74}$$

Since δn is linear in the d_i, we can write $\mathfrak{F}_n$ in the form

$$\mathfrak{F}_n = \sum_{i=1}^{m} \mathfrak{F}_{ni}[d_i(t), t]. \tag{75}$$

The distribution functional $Q_n[k(t)]$ of $\delta n(t)$ is then given by

$$Q_n[k(t)] = \left\langle \exp\left(i \int_{t_1}^{t_2} k(t)\mathfrak{F}_n dt \right) \right\rangle \tag{76}$$

$$= \prod_{i=1}^{m} \left\langle \exp\left(i \int_{t_1}^{t_2} k(t)\mathfrak{F}_{ni}[d_i(t), t]dt \right) \right\rangle \tag{77}$$

$$\equiv \prod_{i=1}^{m} Q_{ni}[k(t)], \tag{78}$$

where (77) follows from the independence of the d_i.

A typical functional dependence of $\delta n(t)$ on $d_i(t)$ is

$$\delta n(t) = \sum_{i=1}^{m} a_i \int_{t_1}^{t} dt' \exp\left(- \int_{t'}^{t} dt''/\tau_i(t'') \right) d_i(t'). \tag{79}$$

It follows from (78) that $Q_{ni}[k(t)]$ is given by

$$Q_{ni}[k(t)]$$

$$= \left\langle \exp\left[i \int_{t_1}^{t_2} dt k(t) \int_{t_1}^{t} dt' a_i d_i(t') \exp\left(- \int_{t'}^{t} dt''/\tau_i(t'') \right) \right] \right\rangle \tag{80}$$

$$= \left\langle \exp\left[i \int_{t_1}^{t_2} dt a_i d_i(t) \int_{t}^{t_2} dt' k(t') \exp\left(- \int_{t}^{t'} dt''/\tau_i(t'') \right) \right] \right\rangle \tag{81}$$

$$= Q_{di}\left[a_i \int_{t}^{t_2} dt' k(t') \exp\left(- \int_{t}^{t'} dt''/\tau_i(t'') \right] \right). \tag{82}$$

If we use (70) for Q_{di}, then we find that

$$Q_{ni}[k(t)] = \exp\left[- \frac{1}{2} \int_{t_1}^{t_2} dt \int_{t_1}^{t_2} dt' k(t)k(t')F_i(t, t') \right], \tag{83}$$

where

$$F_i(t, t') = a_i^2 \exp\left[- \int_{\tau_{\min}}^{\tau_{\max}} dt''/\tau_i(t'') \right] \times \int_{t_1}^{\tau_{\min}} d\tau R_i^o(\tau) \exp\left[-2 \int_{\tau}^{\tau_{\min}} dt''/\tau_i(t'') \right] \tag{84}$$

and where $\tau_{\max} = \max(t, t')$, $\tau_{\min} = \min(t, t')$. Inserting (83) into (78) determines Q_n as desired.

From Q_n we can determine the autocorrelation function of $n(t)$. First, we note from (74) and (76) that

$$\langle \delta n(t_1)\delta n(t_2) \rangle = - \frac{\partial}{\partial \gamma_1}\bigg|_{\gamma_1=0} \frac{\partial}{\partial \gamma_2}\bigg|_{\gamma_2=0} Q[\gamma_1\delta(t - t_1) + \gamma_2\delta(t - t_2)]. \tag{85}$$

Using (78) and (83) this becomes

$$\langle \delta n(t_1)\delta n(t_2)\rangle = \sum_i F_i(t_1, t_2), \tag{86}$$

where F_i is given in (84).

If τ_i and R_i^o are independent of t, an interesting result is obtained. By performing the integrations in (84), it follows at once that

$$F_i(t_1, t_2)/a_i^2 = (R_i^o\tau_i/2) \exp(-|t_2 - t_1|/\tau_i), \tag{87}$$

where the correlation of the fluctuations contributing to δn is readily apparent. Expression (87) occurs often in the theory of stationary processes. [Note in (87) that in the limit $\tau_i \to 0$, $F_i''(t_1, t_2)/a_i^2$ approach $R_i^o\delta(t_2 - t_1)$ as expected from (72).]

APPENDIX C

Fluctuation-Dissipation Theorem

Relations between different physical phenomena have always attracted considerable interest, and justifiably so. No exception is the Einstein relation between the mobility μ and the diffusion coefficient D:

$$\mu = \frac{qD}{kT}. \tag{88}$$

In this expression, μ relates the carrier's velocity v to the electric field E acting on the carrier according to

$$v = \mu E, \tag{89}$$

and D relates the mean-square distance which the carrier diffuses in equilibrium to the time t in which it has been diffusing according to

$$\langle (x_t - x_o)^2\rangle = 2Dt. \tag{90}$$

Although usually not emphasized, all derivations of (88) are based on small fluctuations from equilibrium (in the absence of driving forces). In the presence of driving forces, one must be careful to use the appropriate small-signal quantities when relating diffusion to transport. One of the purposes of this appendix is to explain how this can be done in some cases.

It is very important at the outset to understand the physical origin of the Einstein relation, as well as that of the more general fluctuation-dissipation theorem[16–18] (FD theorem). This is especially important for noise theory since it often enables one to express fluctuation properties in terms of transport properties. This is a valuable aid since transport properties have already been carefully studied in attempts to

understand the noiseless behavior of the device. In addition, an understanding of how the microscopic noiseless motion of a single carrier contributes to the noiseless device current can be carried over at once to calculating how the microscopic fluctuations contribute to the device noise. Thus, we ask the question, Why are fluctuations and dissipation so closely related?

What is the physical origin of the fluctuations undergone by charge carriers? The answer clearly lies in the scattering of all sorts that such carriers experience within the material. What is the physical origin of the damping force experienced by charge carriers? The answer clearly is the same. This connection may be phrased in several ways. A fluctuation corresponds to the response of the carrier to a random force. The velocity-field dependence is a similar relation of response to applied force. Alternatively, under steady-state conditions, the average gain in the energy of a carrier from the material due to fluctuations must be dynamically balanced by the loss of energy due to damping. The gain and loss are, therefore, closely linked. Still another way of seeing the connection is to note that a fluctuation is a departure from equilibrium, such as when a small, disturbing probe force is applied to the system. However it is viewed, a close connection between fluctuation and dissipation must clearly exist, which we shall derive below.

In the impedance-field[4] method, we arrive at the following equation for the voltage spectral density of the device in terms of the elementary thermal-velocity fluctuations of the carriers:

$$S(\delta V_N, f) = \sum_{i=1}^{3} \int d\mathbf{x} \, |\nabla_i Z_{N\mathbf{x}}|^2 4q^2 n(\mathbf{x}) D_{\mathbf{x}}(\delta v_i, f), \tag{91}$$

where q is the elementary carrier charge, $n(\mathbf{x})$ is the carrier density, $Z_{N\mathbf{x}}$ is the impedance field between the contact at N and the field point at $\mathbf{x}$, and

$$S_v(\mathbf{x}, f) = 4D_{\mathbf{x}}(\delta v_i, f) = 4\mathrm{Re} \int_0^\infty e^{i\omega t} \langle \delta v_i(t) \delta v_i(0) \rangle dt \tag{92}$$

is the velocity spectral density in the ith ($=\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$) direction. [See eqs. (54), (56), (27), (28), and (35) of Ref. 4.] The quantity $D_{\mathbf{x}}(\delta v_i, f)$ is also referred to as the diffusion of δv_i at frequency f in the region of $\mathbf{x}$ for a single carrier. In discussing the impedance-field method for the case of $v = \mu E$, where μ is a constant, it is found that

$$D_{\mathbf{x}}(\delta v_x, f) = \left(\frac{kT}{q}\right) \mu \tag{93}$$

for f much less than the scattering frequency. In what follows, we use the FD theorem to prove that if $\mu = \mu(E) = v(E)/E$ and if an effective temperature T can be defined, then[11]

$$D_x(\delta v_x, f) = \left(\frac{kT}{q}\right)\frac{d[\mu(E)E]}{dE}. \tag{94}$$

[Intuitively, (94) follows from (93) if we recall that fluctuations in velocity result from spontaneous fluctuations in the electric field, and if $v = v(E)$, then $\delta v = [dv(E)/dE]\cdot\delta E = (d[\mu(E)E]/dE)\cdot\delta E.$] In the Langevin[3] method, diffusion noise enters through a spectral density of the form

$$S_h(\mathbf{x}, \mathbf{x}', f) = 4q^2 n(\mathbf{x}) D_n \delta(\mathbf{x} - \mathbf{x}'), \tag{95}$$

where again D_n is the diffusion constant, as defined in (42) for frequencies much less than the scattering frequency. Noting how (42) is related to $\langle \delta v(t_1)\delta v(t_2)\rangle$, we note that, once we have shown that (94) is true, it will follow at once that

$$S_h(\mathbf{x}, \mathbf{x}', f) = 4q^2 n(\mathbf{x})\frac{kT}{q}\frac{d(\mu E)}{dE}\delta(\mathbf{x} - \mathbf{x}'). \tag{96}$$

The first quantum-mechanical derivation of the FD theorem is usually attributed to Callen and Welton.[16] This subject has subsequently been treated in greater detail in Refs. 19 and 20. For the sake of completeness, we shall rederive the FD theorem here primarily for the purpose of calling attention to its application to cases of steady-state but nonequilibrium conditions. We conclude this appendix with the derivation of more general relations for fluctuations and dissipation valid for any energy distribution (especially nonthermal) of the states of the system. These relations show how closely the two are related even under nonthermal conditions.

Let us suppose that we have some system which can be described by a Hamiltonian H^s, which includes applied electric and magnetic fields giving rise to currents, etc. Let $\hat{r}_i$ be an operator whose expectation value we seek as a function of time in response to a unit impulse in a probe force f_j, which enters the total Hamiltonian H according to

$$H = H^s - f_j\hat{r}_j, \tag{97}$$

where $\hat{r}_j$ is another operator. Then the FD theorem states that

$$\text{Re}\,[K_{ij}(\omega)] = \text{Im}\,[X_{ij}(\omega)]\frac{\hbar}{2}\frac{1 + \exp(-\beta\hbar\omega)}{1 - \exp(-\beta\hbar\omega)}, \tag{98a}$$

where $\beta = 1/kT$, T is the effective temperature of the system, if such can be defined,

$$K_{ij}(\omega) \equiv \int_0^\infty dt k_{ij}(t) e^{i\omega t}, \tag{98b}$$

$$k_{ij}(t) \equiv \tfrac{1}{2}\langle[\hat{r}_i(t), \hat{r}_j(0)]_+\rangle = \tfrac{1}{2}\langle \hat{r}_i(t)\hat{r}_j(0) + \hat{r}_j(0)\hat{r}_i(t)\rangle, \tag{98c}$$

$$X_{ij}(\omega) \equiv \int_{-\infty}^\infty dt x_{ij}(t) e^{i\omega t}, \tag{98d}$$

and $x_{ij}(t)$ is the impulse response of $\hat{r}_i$ to the unit impulse f_j in (97). From (98b, c) it is clear that 4 Re $[K_{ij}(\omega)] = S_{ij}(f)$, the cross-spectral density between r_i and r_j, whereas from (98d) it follows that Im $[X_{ij}(\omega)]$ is the absorption through r_i of the energy put into the system via the coupling (interaction energy) $-f_j r_j$.

Returning for the moment to (92), we note that if we let $\hat{r}_i = \hat{r}_j = \delta v_i = \delta\dot{x}$ (for $i = x$), in the classical limit ($\hbar \to 0$), we obtain from (98a)

$$S_v(\mathbf{x}, f) = 4 \text{ Im } [X_{\dot{x}\dot{x}}(\omega)]kT/\omega. \tag{99}$$

After we derive (98a), we shall show that for ω much less than the scattering frequency

$$X_{\dot{x}\dot{x}}(\omega) = i\omega\mu_{ac}/q, \tag{100}$$

where $\mu_{ac} = d[\mu(E)E]/dE$. The important result (94) then follows at once.

We shall derive (98a) by evaluating $K_{ij}(\omega)$ and $X_{ij}(\omega)$ and comparing the results. By definition

$$x_{ij}(t) = \frac{i}{\hbar}\langle \hat{r}_i(t)\hat{r}_j(0) - \hat{r}_j(0)\hat{r}_i(t)\rangle, \tag{101}$$

where we assume the impulse occurs at $t = 0$. Clearly $x_{ij}(t) = 0, t < 0$. Thus,

$$X_{ij}(\omega) = \frac{i}{\hbar}\int_0^\infty dt e^{i\omega t} \sum_{lm} \langle l|\hat{r}_i|m\rangle\langle m|\hat{r}_j|l\rangle(e^{-\beta E_l} - e^{-\beta E_m}) \times e^{-i/\hbar(E_m - E_l)t}/\sum_l e^{-\beta E_l}, \tag{102}$$

which follows if we assume that the system described by H^s can be characterized by a temperature T. This insures that the probability that the system is in eigenstate ψ_n with eigenenergy E_n is exp $(-\beta E_n)/N$, where N is the normalization factor used in (102).

Performing the integration on t, we obtain

$$X_{ij}(\omega) = \sum_{lm} (e^{-\beta E_l} - e^{-\beta E_m})\langle l|\hat{r}_i|m\rangle\langle m|\hat{r}_j|l\rangle \times (-1)[\hbar\omega - (E_m - E_l) + i\epsilon]^{-1}/N, \quad (103)$$

and, finally, taking the imaginary part we find that

$$\text{Im}\,[X_{ij}(\omega)] = \pi \sum_{lm} (e^{-\beta E_l} - e^{-\beta E_m})\langle l|\hat{r}_i|m\rangle\langle m|\hat{r}_j|l\rangle\delta[\hbar\omega - (E_m - E_l)]/N. \quad (104)$$

If now we compare the definition of $k_{ij}(t)$ (98c) with (101), it follows at once from making the appropriate changes in (103) that

$$K_{ij}(\omega) = \frac{\hbar}{2i} \sum_{lm} (e^{-\beta E_l} + e^{-\beta E_m})\langle l|\hat{r}_i|m\rangle\langle m|\hat{r}_j|l\rangle \times (-1)[\hbar\omega - (E_m - E_l) + i\epsilon]^{-1}/N. \quad (105)$$

And upon taking the real part of (105),

$$\text{Re}\,[K_{ij}(\omega)] = \pi \frac{\hbar}{2} \sum_{lm} (e^{-\beta E_l} + e^{-\beta E_m}) \times \langle l|\hat{r}_i|m\rangle\langle m|\hat{r}_j|l\rangle\delta[\hbar\omega - (E_m - E_l)]/N \quad (106)$$

is obtained. In (104) and (106), we may replace E_m in the exponent by $(E_l + \hbar\omega)$ owing to the presence of the delta function. This then permits the factor $[1 - \exp(-\beta\hbar\omega)]$ to be pulled out of the sum in (104) and $[1 + \exp(-\beta\hbar\omega)]$ to be pulled out in (106). The FD theorem (98a) follows at once.

To derive (100), we must look more closely at the exact response x_{ij}^e of $\hat{r}_i$ to a general time dependent f_j in $H = H_s - f_j\hat{r}_j$. If H is the Hamiltonian, then

$$x_{ij}^e(t) = \langle\hat{r}_i\rangle_t = \text{Tr}\,[\hat{r}_i\rho(t)]/\text{Tr}\,[\rho(t)], \quad (107)$$

where $\rho(t)$ is the density matrix of the system at time t. The density matrix $\rho(t)$ at t can be obtained from that at t_1 according to

$$\rho(t) = P(t, t_1)\rho(t_1)P^\dagger(t, t_1), \quad (108)$$

where the propagator $P(t, t_1)$ is defined by

$$P(t, t_1) = \exp\left(-\frac{i}{\hbar}\int_{t_1}^{t} d\tau H_\tau\right), \quad (109)$$

and where we are using the Feynman[21] ordered-operator notation. [If

t_1 is prior to the turning on of f and the system can be described by an effective temperature T, then $\rho(t_1) = \exp(-\beta H^s)$.]

The specific problem in which we are most interested is the case where $\hat{r}_i$, $\hat{r}_j$ represent velocity fluctuations from the expectation value of the velocity $\mathbf{v}_o$ of a carrier under the influence of an electric field of arbitrary strength contained in H^s. Let us place ourselves in a reference frame drifting with velocity $\mathbf{v}_o$. In such a frame, we may calculate either the velocity fluctuations $k_{\dot{x}\dot{x}}(t)$ from the noiseless motion or the response $\dot{x}_{ij}(t)$ to a probe force f_j. If we were applying a small, ac electric field $e(t)$ to the carrier, then $H = H^s - qe(t)\hat{x}$, where $\hat{x}$ is the position operator corresponding to the x coordinate (in the drifting frame). Also, we know that the velocity response to such a field is given by

$$\delta v(t) = [dv(E)/dE]e(t) \tag{110}$$

or, taking Fourier transforms,

$$\delta v(\omega) = \{d[\mu(E)E]/dE\}e(\omega), \tag{111}$$

since $v(E) = \mu(E)E$, μ being the mobility. However, $X_{\dot{x}\dot{x}}(\omega)$ arises from the velocity response to $H = H^s - f(t)\hat{\dot{x}}$. We can relate $f(t)$ to $e(t)$ if we note that from (109), $f(t)$ enters in the form

$$\exp\left(-\frac{i}{\hbar}\int_{t_1}^{t} d\tau f(\tau)\dot{x}\right) = \exp\left(-\frac{i}{\hbar}\int_{t_1}^{t} d\tau(-)\dot{f}(\tau)x\right), \tag{112}$$

where we can take $f(t) = 0$ for simplicity. Thus, $qe(t) = -\dot{f}(t)$, or $qe(\omega) = i\omega f(\omega)$. Also, using (111), we obtain

$$\begin{aligned} X_{\dot{x}\dot{x}}(\omega) &\equiv \frac{\delta v(\omega)}{f(\omega)} = \frac{\delta v(\omega)}{e(\omega)}\frac{i\omega}{q} = \frac{d[\mu(E)E]}{dE}\frac{i\omega}{q} \\ &= i\omega\mu_{ac}/q, \end{aligned} \tag{113}$$

which proves (86). Our principal result (94) is therefore demonstrated.

We promised that we would conclude with more general relations for fluctuations and dissipation valid for any probability distribution of the eigenstates of a system according to eigenenergies. We proceed as follows.

Let the (normalized) probability that the system be in a state of eigenenergy E be $f(E)$. Choose the scale of energy such that the lowest eigenenergy is zero (0). Take the Laplace transform of $f(E)$ to obtain

$$F(s) = \int_0^\infty dE f(E)e^{-sE} \tag{114a}$$

and, of course,

$$f(E) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds e^{sE} F(s). \tag{114b}$$

[For a thermal distribution, $f(E) = \exp(-\beta E)/N$ and $F(s) = (s + \beta)^{-1}/N$.] It follows from (104) and (106) that

$$\frac{2}{\hbar} \operatorname{Re}[K_{ij}(\omega)] = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds G(s, \omega)(1 + e^{s\hbar\omega}) \tag{115a}$$

and

$$\operatorname{Im}[X_{ij}(\omega)] = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} ds G(s, \omega)(1 - e^{s\hbar\omega}), \tag{115b}$$

where

$$G(s, \omega) \equiv \pi F(s) \sum_{lm} e^{sE_l} \langle l | \hat{r}_i | m \rangle \langle m | \hat{r}_j | l \rangle \delta[\hbar\omega - (E_m - E_l)]. \tag{115c}$$

The difference between (115a) and (115b) consists of only one sign: plus in (115a), minus in (115b). For the special case of a thermal distribution, this sign difference yields the ratio factor that appears in (98a). In general, while such a simple relation between K_{ij} and X_{ij} is no longer valid, it is clear from (115) that fluctuations and dissipation have a common origin. These are the more general relations we promised.

APPENDIX D

The purpose of this appendix is merely to carry out the routine mathematical steps necessary to arrive at the spectral densities of recombination-generation noise. These results are referred to in Section IV.

We proceed as follows. We can abbreviate our exposition since we have been through the necessary steps several times in the previous sections. The steps are as follows:

(*i*) Write the microscopic kinematic equations:

$$\frac{dn}{dt} = -\sum_m \delta[t - t_{ton}(m)] - \sum_m \delta[t - t_{tpn}(m)] + \sum_m \delta[t - t_{ntn}(m)] + \sum_m \delta[t - t_{nto}(m)], \tag{116a}$$

$$\frac{dp}{dt} = -\sum_m \delta[t - t_{top}(m)] - \sum_m \delta[t - t_{tnp}(m)] + \sum_m \delta[t - t_{ptp}(m)] + \sum_m \delta[t - t_{pto}(m)], \tag{116b}$$

$$\frac{dn_{tn}}{dt} = + \sum_m \delta[t - t_{ton}(m)] - \sum_m \delta[t - t_{ntn}(m)] - \sum_m \delta[t - t_{tnp}(m)] + \sum_m \delta[t - t_{pto}(m)], \quad (116c)$$

$$\frac{dn_{tp}}{dt} = - \sum_m \delta[t - t_{tpn}(m)] + \sum_m \delta[t - t_{nto}(m)] + \sum_m \delta[t - t_{top}(m)] - \sum_m \delta[t - t_{ptp}(m)], \quad (116d)$$

$$\frac{dn_{to}}{dt} = - \frac{dn_{tn}}{dt} - \frac{dn_{tp}}{dt}. \quad (116e)$$

(*ii*) Rewrite in terms of sources and responses:

$$\frac{dn}{dt} = -r_{ton}n_{to}n - r_{tpn}n_{tp}n + g_{ntn}n_{tn} + g_{nto}n_{to} - d_{ton} - d_{tpn} + d_{ntn} + d_{nto}, \quad (117a)$$

$$\frac{dp}{dt} = -r_{top}n_{to}p - r_{tnp}n_{tn}p + g_{ptp}n_{tp} + g_{pto}n_{to} - d_{top} - d_{tnp} + d_{ptp} + d_{pto}, \quad (117b)$$

$$\frac{dn_{tn}}{dt} = r_{ton}n_{to}n - g_{ntn}n_{tn} - r_{tnp}n_{tn}p + g_{pto}n_{to} + d_{ton} - d_{ntn} - d_{tnp} + d_{pto}, \quad (117c)$$

$$\frac{dn_{tp}}{dt} = - r_{tpn}n_{tp}n + g_{nto}n_{to} + r_{top}n_{to}p - g_{ptp}n_{tp} - d_{tpn} + d_{nto} + d_{top} - d_{ptp}, \quad (117d)$$

$$\frac{dn_{to}}{dt} = - \frac{dn_{tn}}{dt} - \frac{dn_{tp}}{dt}, \quad (117e)$$

where a typical d function is defined as

$$d_{ton} \equiv \sum_m \delta(t - t_{ton}) - r_{ton}n_{to}n \quad (118a)$$

and is equivalent for small fluctuations to

$$d_{ton} = \sum_m \delta(t - t^o_{ton}) - r^o_{ton}n^o_{to}n^o. \quad (118b)$$

(*iii*) Write each variable as the sum of a noiseless contribution and a noise contribution, and linearize (117) to obtain the following nonlinear equations for the noiseless solution:

$$\dot{n}^o = -r_{ton}n^o_{to}n^o - r_{tpn}n^o_{tp}n^o + g_{ntn}n^o_{tn} + g_{nto}n^o_{to}, \quad (119a)$$

$$\dot{p}^o = -r_{top}n^o_{to}p^o - r_{tnp}n^o_{tn}p^o + g_{ptp}n^o_{tp} + g_{pto}n^o_{to}, \quad (119b)$$

$$\dot{n}^o_{tn} = r_{ton}n^o_{to}n^o - g_{ntn}n^o_{tn} - r_{tnp}n^o_{tn}p^o + g_{pto}n^o_{to}, \tag{119c}$$

$$\dot{n}^o_{tp} = -r_{tpn}n^o_{tp}n^o + g_{nto}n^o_{to} + r_{top}n^o_{to}p^o - g_{ptp}n^o_{tp}, \tag{119d}$$

$$\dot{n}^o_{to} = -\dot{n}^o_{tn} - \dot{n}^o_{tp}, \tag{119e}$$

and to obtain the following linear equations for the noise solution:

$$\begin{aligned}\delta\dot{n} = {} & -r_{ton}(n^o_{to}\delta n + \delta n_{to}n^o) - r_{tpn}(n^o_{tp}\delta n + \delta n_{tp}n^o) \\ & + g_{ntn}\delta n_{tn} + g_{nto}\delta n_{to} - d_{ton} - d_{tpn} + d_{ntn} + d_{nto},\end{aligned} \tag{120a}$$

$$\begin{aligned}\delta\dot{p} = {} & -r_{top}(n^o_{to}\delta p + \delta n_{to}p^o) - r_{tnp}(n^o_{tn}\delta p + \delta n_{tn}p^o) \\ & + g_{ptp}\delta n_{tp} + g_{pto}\delta n_{to} - d_{top} - d_{tnp} + d_{ptp} + d_{pto},\end{aligned} \tag{120b}$$

$$\begin{aligned}\delta\dot{n}_{tn} = {} & r_{ton}(n^o_{to}\delta n + \delta n_{to}n^o) - g_{ntn}\delta n_{tn} - r_{tnp}(n^o_{tn}\delta p + \delta n_{tn}p^o) \\ & + g_{pto}\delta n_{to} + d_{ton} - d_{ntn} - d_{tnp} + d_{pto},\end{aligned} \tag{120c}$$

$$\begin{aligned}\delta\dot{n}_{tp} = {} & -r_{tpn}(n^o_{tp}\delta n + \delta n_{tp}n^o) + g_{nto}\delta n_{to} + r_{top}(n^o_{to}\delta p + \delta n_{to}p^o) \\ & - g_{ptp}\delta n_{tp} - d_{tpn} + d_{nto} + d_{top} - d_{ptp},\end{aligned} \tag{120d}$$

$$\delta n_{to} = -\delta n_{tn} - \delta n_{tp}. \tag{120e}$$

(*iv*) Finally, solve (120) for δn, δp, δn_{tn}, δn_{tp}, and δn_{to} in terms of d_{ton}, $\cdots$. This latter step involves solving five equations for five unknowns. If we are interested in the stationary solution to (119) ($\dot{n}^o = \dot{p}^o = \dot{n}^o_{tn} = \dot{n}^o_{tp} = 0$), then the coefficients of the δn, δp, $\cdots$ (120) are independent of time and a Fourier analysis of (120) is most expedient. Using (120e) to eliminate δn_{to}, one obtains the following set of equations:

$$i\omega\delta n = -R_n\delta n + R_{ntn}\delta n_{tn} - R_{ntp}\delta n_{tp} + d_1, \tag{121a}$$

$$i\omega\delta p = -R_p\delta p - R_{ptn}\delta n_{tn} + R_{ptp}\delta n_{tp} + d_2, \tag{121b}$$

$$i\omega\delta n_{tn} = -R_{tn}\delta n_{tn} + R_{tnn}\delta n - R_{tnp}\delta p - R_{tntp}\delta n_{tp} + d_3, \tag{121c}$$

$$i\omega\delta n_{tp} = -R_{tp}\delta n_{tp} - R_{tpn}\delta n + R_{tpp}\delta p - R_{tptn}\delta n_{tn} + d_4, \tag{121d}$$

where

$$d_1 \equiv -d_{ton} - d_{tpn} + d_{ntn} + d_{nto}, \tag{122a}$$

$$d_2 \equiv -d_{top} - d_{tnp} + d_{ptp} + d_{pto}, \tag{122b}$$

$$d_3 \equiv d_{ton} - d_{ntn} - d_{tnp} + d_{pto}, \tag{122c}$$

and

$$d_4 \equiv -d_{tnp} + d_{nto} + d_{top} - d_{ptp}. \tag{122d}$$

By combining the coefficients of the δn, δp, $\cdots$ in (120), the reader can determine the R_n, R_p, $\cdots$ rates in (121). For example, $R_n = r_{ton}n^o_{to} + r_{tpn}n^o_{tp}$, etc. The important point to be made here is that the d_1, d_2, d_3, d_4 are correlated, that is, they are not mutually independent. Thus,

one cannot *a priori* introduce independent d_1, d_2, d_3, d_4 into equations of the form of (121) at this stage of the calculation. Using our microscopic approach, however, these source terms and their correlations arise in a well-defined manner.

If we let $\delta n \equiv \delta n_1$, $\delta p \equiv \delta n_2$, $\delta n_{tn} \equiv \delta n_3$, $\delta n_{tp} = \delta n_4$, $\mathbf{\delta n} \equiv (\delta n_1, \delta n_2, \delta n_3, \delta n_4)$, $\mathbf{d} \equiv (d_1, d_2, d_3, d_4)$, then (121) can be written in the form

$$i\omega \mathbf{I} \delta \mathbf{n} = \mathbf{R} \delta \mathbf{n} + \mathbf{d} \tag{123a}$$

and solved for $\delta \mathbf{n}$ at once:

$$\delta \mathbf{n}(\omega) = \mathbf{M}(\omega)\mathbf{d}(\omega), \tag{123b}$$

where

$$\mathbf{M}(\omega) \equiv (i\omega \mathbf{I} - \mathbf{R})^{-1}. \tag{123c}$$

The cross-spectral density of δn_i, δn_j is then given by ($\omega = 2\pi f$):

$$S_{ij}^{\delta \mathbf{n}}(f) = \sum_{kl} M_{ik}(\omega) S_{kl}^{\mathbf{d}}(f) M_{lj}^{*}(\omega), \tag{124a}$$

where $S_{kl}^{\mathbf{d}}(f)$ is the cross-spectral density of d_k, d_l:

$$\mathbf{S}^{d} = \begin{bmatrix} S_{ton}^{d}+S_{tpn}^{d}+S_{ntn}^{d}+S_{nto}^{d} & 0 & -S_{ton}^{d}-S_{ntn}^{d} & S_{tpn}^{d}+S_{nto}^{d} \\ 0 & S_{top}^{d}+S_{tnp}^{d}+S_{ptp}^{d}+S_{pto}^{d} & S_{tnp}^{d}+S_{pto}^{d} & -S_{top}^{d}-S_{ptp}^{d} \\ -S_{ton}^{d}-S_{ntn}^{d} & S_{tnp}^{d}+S_{pto}^{d} & S_{ton}^{d}+S_{ntn}^{d}+S_{tnp}^{d}+S_{pto}^{d} & 0 \\ S_{tpn}^{d}+S_{nto}^{d} & -S_{top}^{d}-S_{ptp}^{d} & 0 & S_{tpr}^{d}+S_{nto}^{d}+S_{top}^{d}+S_{ptp}^{d} \end{bmatrix}, \tag{124b}$$

and

$$\begin{aligned} &S_{ton}^{d} = 2r_{ton}n^{o},\ S_{top}^{d} = 2r_{top}p^{o},\ S_{tpn}^{d} = 2r_{tpn}n^{o}, \\ &S_{tnp}^{d} = 2r_{tnp}p^{o},\ S_{ntn}^{d} = 2g_{ntn},\ S_{ptp}^{d} = 2g_{ptp},\ S_{nto}^{d} = 2g_{nto}, \\ &S_{pto}^{d} = 2g_{pto} \end{aligned}$$

from (73). Using (123b, c) and (124a, b), the cross-spectral density of δn_i, δn_j including *all* correlation effects can be readily calculated. Of greatest interest are the spectral densities of δn and δp given by $S_{21}^{\delta \mathbf{n}}(f)$ $S_{22}^{\delta \mathbf{n}}(f)$, *and* of δn, δp given by $S_{12}^{\delta \mathbf{n}}(f) = S_{21}^{\delta \mathbf{n}}(f)$. These are the terms involving mobile carriers, which contribute to the current fluctuations seen at the contacts of the device. It should be clear that trapping can be treated by using an approach similar to the above.

There is little point in carrying out the algebra implicit in (124a). Evaluating the inverse of the 4×4 matrix in (123c) and carrying out the sum in (124a) can be done readily, although it is somewhat tedious.

What we have accomplished is to express the cross-spectral densities of the electrons, holes, and various trapping states in terms of known, easily calculated, cross-spectral densities of the source-driving terms. And in so doing we have included all important correlation effects in a very natural way.

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