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Theory of Noise in Charge-Transfer Devices

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The noise introduced into charge packets transferred through and stored in charge-transfer devices is calculated in a manner that includes all important relaxation, suppression, and correlation effects. First, the noise induced into each packet during each transfer phase from thermal, trapping, emission-current, and leakage-current fluctuations, whose statistics are nonstationary, and from clock-voltage fluctuations, whose statistics are stationary, is determined. Relaxation of the transferring charge to these fluctuations is found to suppress their size. Second, the accumulation (collecting) of the noise as each packet is transferred through the device is calculated neglecting the role of incomplete charge transfer. Attention is drawn to the significant differences between the collecting of storage-process noise, which is unsuppressed, transfer-process noise, whose spectral density is nearly totally suppressed at low frequencies, and modulation noise, which is nearly totally suppressed for digital and analog signals. Third, the role of incomplete charge transfer in suppressing the collecting of the noise is shown for digital signals and indicated for analog signals. We conclude with a numerical calculation of the maximum possible signal-to-noise ratio that can be expected from charge-transfer devices. The presentation is sufficiently general and detailed that, with a minimum of background in formal noise theory, one can use the approach to evaluate noise in many novel, solid-state devices.

I. INTRODUCTION

Charge-transfer devices (CTD's), such as the bucket-brigade[1] (BB) and charge-coupled-device[2] (CCD) shift registers, are currently of great interest. These devices consist of a chain of charge-storage elements along which charge packets are transferred from input to output. Noise accompanying each individual transfer of each charge packet will be introduced into each packet. It is the purpose of this article to calculate this noise and its cumulative effects on the output signal.

Noise generated in solid-state devices has been treated extensively,[3–6] and this prior work will be of great assistance to us here. However, there are three major, significant differences between much of this prior work and the present treatment.[7,8] The first difference arises from the nature of the transfer process.[9] The usual treatments of noise discuss situations in which the (noiseless) currents, charge densities, conductances, etc. associated with the signal are time-invariant. Under such conditions, the statistics of the noise are also time-invariant and the noise is said to be stationary. Stationary noise is readily treated in the frequency domain using spectral-density functions. Frequency-domain, linear-circuit analysis of equivalent circuits greatly facilitates the usual treatments. (Sometimes, as in mixer theory, periodic rather than time-invariant situations are considered, and frequency-domain analyses are still convenient.)

In CTD's the situation is quite different. During the transfer of the charge from one storage region to the next, the (noiseless) currents, charge densities, conductances, etc. associated with the transfer of the signal packet are rapidly time-varying.[9] As a result, the noise generated during transfer is nonstationary; that is, the statistics of the current fluctuations that give rise to noise vary appreciably with time. Once the transfer is complete, a certain amount of noise has been introduced into the signal independently of the noise acquired in prior or subsequent transfers. The processes of interest, therefore, are nonperiodic as well as nonstationary. Under such conditions, we have found it best to work in the time domain using correlation functions.

The second basic difference between this treatment and other treatments arises from the physical structure and operation of the device. Ordinarily, we can externally control the noiseless portion of the voltages and currents associated with the signal. In CTD's, however, the size of the charge packets, in addition to the characteristics of the clock voltages, controls the charge transfer. Thus, the noise currents that we usually calculate function as additional driving terms in calculating charge transfer and, of course, result in fluctuations in the sizes of the packets. But it is the reaction of the charge transfer to these currents, and not the currents themselves, from which we must calculate charge-packet fluctuations.[10,11] (Normally, we control either the voltage across, or the current through, the device of interest and calculate fluctuations in the other. Here we control neither directly.)

A third basic difference is that it by no means suffices to calculate only the noise introduced during a single transfer. The collecting[7] of the noise must also be carefully calculated to include correlation effects that lead to a suppression in the cumulative noise added to each packet while being transferred by the device. The number of unexpected effects makes treating noise in CTD's truly interesting.[12]

At first the nonstationary feature of the noise coupled to the reaction of the charge transfer might seem to preclude a reasonably simple, analytic treatment that produces useful results. Indeed, some of our expressions will be a bit complicated. However, owing to certain suppression effects, which will be discussed in detail, the mean-square induced fluctuation in the size of a given charge packet at the end of a single transfer is nearly independent of the size of the packet. Thus, having calculated the magnitude of this fluctuation, it will be possible to obtain a simple, meaningful expression for the spectral density of the accumulated noise at the output which is independent of the signal. This quantity is quite useful for evaluating the effects of noise on the analog performance of a CTD. By contrast, for digital-performance evaluation, only the accumulated mean-square fluctuation in the size of a charge packet at the output is needed. As we shall see, this quantity is arrived at in a straightforward manner using our time-domain analysis without the necessity of working in the frequency domain at all.

In what follows, we shall outline briefly the noise sources whose effect on the output charge packets we shall calculate. Then, following a review of the lumped-charge model of a CTD, which has proven to be so useful in the analysis and calculation of incomplete transfer coefficients, we calculate the statistics of the noise charge introduced in a single transfer in terms of the statistics of the microscopic fluctuations inducing this noise. Thermal, trapping, clock-voltage, emission-current, and leakage-current fluctuations are considered. Three different types of compounding, storage-process, transfer-process, and modulation, are then treated neglecting incomplete-transfer effects. Following this, the general problem of compounding in the presence of incomplete charge transfer is treated and interesting suppression effects are uncovered. The paper ends with calculations of the maximum signal-to-noise ratio expected for state-of-the-art devices. As the techniques employed in calculating nearly all aspects of the noise are of necessity different from those usually encountered, we have chosen to elaborate our methods in some detail. From a knowledge of the noise at the output, the operational limitations[13] and error rates[14] associated with CTD's can be assessed.

The remainder of this article is long and intricate. This arises not from the nature of the noise treated per se, but rather because of the complexity of the devices considered. Nonetheless, familiarity with the physical operation of CTD's as well as acquaintance with Brownian motion and shot noise are the only prerequisites needed. No use is made of highly developed, sophisticated techniques of noise theory. This is not because such techniques are not considered applicable; rather, it is felt that the methods used here are the simplest available for a rigorous treatment. In addition, it is felt that these methods will be useful in treating noise in other dynamic devices as well. Readers well versed in noise theory may feel that this problem is amenable to existing methods. While the contrary is not claimed here, we believe that such treatments will be more involved than might be expected at first sight.

II. SOURCES OF RANDOM NOISE

We should at the outset stress that we are concerned here only with *random* noise. We are not concerned with signal distortion arising from sampling, incomplete transfer,[15,16] direct clock coupling, nonideal regeneration, or any other deterministic process. This is not because we feel that such problems are unimportant. Rather, our philosophy is that random noise is unavoidable, whereas, in principle, deterministic "noise" can be greatly reduced or compensated for by careful design.[13,14] Thus, it is random noise that plays a major role in setting the operational limits of CTD's, which is a problem of general current interest.

Figure 1 indicates schematically the sources of CTD noise with which we shall be concerned. At the input, if the charge packet is created by photon absorption as in imaging application, or if it is injected by an emission-limited mechanism, full shot noise $[\langle (Q - Q_0)^2 \rangle = eQ_0]$ will accompany the signal Q. If, on the other hand, the packet is injected into the input via a resistor-controlled circuit, as in regeneration and storage applications, no shot noise will be introduced.

The CTD itself simultaneously transfers charge and stores charge.[8] In its charge-transfer capacity, thermal noise from the Brownian motion of the carriers composing the transfer current and trapping noise from the fluctuation in occupancy of interface states[17] arise. Care

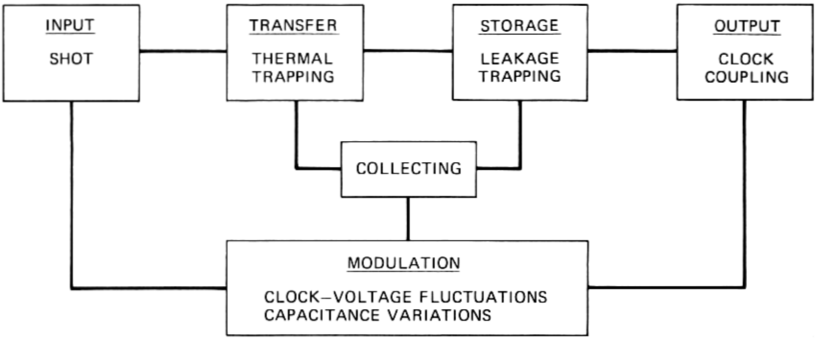

Fig. 1—Sources of noise for a CTD.

must be exercised in calculating this transfer noise because of the tendency of relaxation effects to suppress thermal and trapping fluctuations.[7] Since the transfer of charge from one storage region to the next is controlled by the conductance of the storage regions themselves (or portions thereof), shot noise in the transfer current is totally suppressed for all practical purposes. (Exception: Should such current become barrier-limited, then some shot noise can result. This special case is treated in Section 6.1.) In its capacity of charge storage, noise from leakage-current fluctuations and trap-occupancy fluctuations is introduced. Although intrinsically much smaller than transfer-noise sources, storage noise is unsuppressed, and, hence, it can be important in some cases. At the output, we have the usual problems with detector noise, but we shall not consider these problems at this time.

What is of basic interest is, of course, the cumulative noise in the charge packet by the time it reaches the output of the device. As it turns out, one must be very careful in calculating the collecting of the noise introduced into each packet during each phase of each transfer cycle. For example, if at the end of a transfer phase, a transfer noise of $+\Delta Q$ has been added to the signal, by conservation of charge, a quantity of charge $-\Delta Q$ has been added to the charge left behind. These two contributions to the noise are highly correlated, and this correlation must be taken into account in calculating the noise spectral density at the output.[7] Incomplete transfer of charge distorts the noise as well as the signal, and must also be included in collecting effects. Fluctuations in the clock voltage coupled to fluctuations in the sizes of the storage capacitances of the individual storage cells of the CTD give rise to modulation noise, which collects quite differently from the noise induced by fluctuations in the clock voltage in the absence of cell-parameter variations along the CTD.

These, then, are the aspects of the noise produced by CTD's that we shall discuss. In calculating these various contributions to CTD noise, we were surprised by the variety of the results: the presence or absence of suppression in individual transfer phases, differences in compounding suppression, stationary fluctuations in charge-packet fluctuations induced by nonstationary noise sources, etc. Athough we by no means treat all aspects of CTD noise, the methods we develop should be helpful in calculating the influence of nearly any source of noise on the output of a CTD.

III. MODEL OF CHARGE-TRANSFER DEVICE

All device noise arises ultimately from fluctuations inherent in the transfer of charge carriers between states characterizing the flow of charge through the device.[18] It follows that to calculate CTD noise, we must first understand how a CTD operates under noiseless conditions.[10,11] As this topic has been the subject matter of a large number of papers,[10,11] a detailed elaboration here is not necessary. We shall, however, briefly review a lumped-charge model[10,11] of charge transfer within a single transfer unit of a CTD, which has proven useful in discussing and calculating incomplete charge transfer in CTD's. This model enables us to express the device current as a function of several characteristic voltages, which in turn control the charge passing through the device. Using this model, we can then calculate fluctuations in the sizes of the transferred charge packets in terms of the current fluctuations which accompany the charge transfer and which can be calculated by standard means.[18]

The lumped-charge models we shall use to calculate CTD noise are shown in Figs. 2a and 2b. Charge Q_s stored on capacitor C_s is transferred to capacitor C_d during the transfer cycle. (The clock voltage which determines the duration of the transfer cycle is V_c.) In subsequent cycles, the charge on C_d is transferred to the right step by step to other storage sites, and charge from the left is brought into C_s—both processes are modeled by repetitions of this model for a single transfer unit. For the present, we shall be concerned only with a single transfer unit, either of type (a), if the charge transfer is characterized by a single-step process, or of type (b), if the transfer is characterized by a two-step process. In the latter process, an intermediate capacitor C_b is inserted between C_s and C_d to enhance their

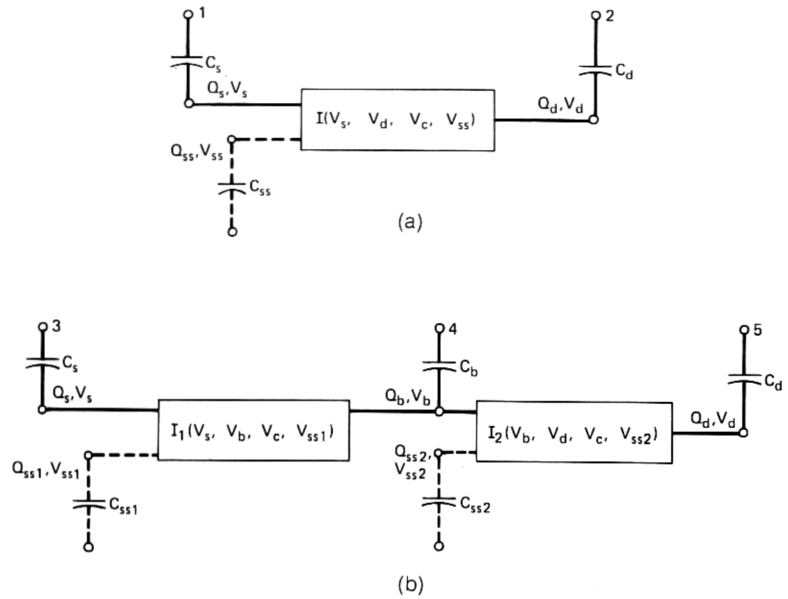

Fig. 2—Lumped-charge model including the effect of interface states for the transfer of charge through a single stage of a CTD. (a) Single-step transfer. (b) Two-step transfer.

mutual isolation so as to reduce the incomplete transfer. As we shall see, this also has the effect of reducing the noise acquired by the transferred packet.

Let us now review the dynamic features of the lumped-charge model of Fig. 2a in some detail. Stored on $C_s = C_s(V_s, V_d, V_c, V_{ss})$ is a time-dependent quantity of stored charge Q_s given by

$$Q_s = \int_{V_{so}}^{V_s} C_s(V_s', V_d, V_c, V_{ss}) dV_s'; \tag{1a}$$

stored on $C_{ss}(V_s, V_d, V_c, V_{ss})$ is a relatively small, time-dependent quantity of interface (trapped) charge Q_{ss} given by

$$Q_{ss} = \int_{V_{sso}}^{V_{ss}} C_{ss}(V_s, V_d, V_c, V_{ss}') dV_{ss}'. \tag{1b}$$

The total charge Q to be transferred through the conductance $I = I(V_s, V_d, V_c, V_{ss})$ is given by

$$Q = Q_s + Q_{ss}. \tag{1c}$$

Stored on C_d is a charge Q_d given by

$$Q_d = Q_o - Q = \int_{V_{do}}^{V_d} C_d(V_s, V'_d, V_c, V_{ss}) dV'_d, \tag{1d}$$

where Q_o is the total charge to be transferred through this single transfer unit. (Q_o is some constant for each transfer event: it is the quantity of charge initially distributed between Q_s and Q_{ss} to be transferred to C_d.) The time dependence of the decay of Q_s during the charge transfer is governed by the equation

$$\dot{Q}_d = -\dot{Q} = I(V_s, V_d, V_c, V_{ss}) + i_n(t), \tag{2}$$

where $i_n(t)$ is the device noise current induced by microscopic fluctuations within the conductance I. To determine CTD noise we must (*i*) solve eqs. (1) and (2) for fluctuations in Q_d induced by $i_n(t)$ (and by fluctuations in V_c and in trap occupancy), and (*ii*) express $i_n(t)$ in terms of the microscopic fluctuations from which it is induced.

For two-step transfer processes, as modeled in Fig. 2b, the dynamics of the charge transfer are more complicated. One must now be concerned with Q_s, Q_b, Q_d, Q_{ss1}, and Q_{ss2} defined by

$$Q_s = \int_{V_{so}}^{V_s} C_s(V'_s, V_b, V_c, V_{ss1}) dV'_s, \tag{3a}$$

$$Q_b = \int_{V_{bo}}^{V_b} C_b(V_s, V'_b, V_d, V_c, V_{ss1}, V_{ss2}) dV'_b, \tag{3b}$$

$$Q_d = \int_{V_{do}}^{V_d} C_d(V_b, V'_d, V_c, V_{ss2}) dV'_d, \tag{3c}$$

$$Q_{ss1} = \int_{V_{ss10}}^{V_{ss1}} C_{ss1}(V_s, V_b, V_c, V'_{ss1}) dV'_{ss1}, \tag{3d}$$

$$Q_{ss2} = \int_{V_{ss20}}^{V_{ss2}} C_{ss2}(V_b, V_d, V_c, V'_{ss2}) dV'_{ss2}. \tag{3e}$$

In addition, we have

$$Q = (Q_s + Q_{ss1}) + (Q_b + Q_{ss2}) \equiv Q_1 + Q_2, \tag{3f}$$

and

$$Q_d = Q_o - Q = Q_o - (Q_1 + Q_2). \tag{3g}$$

The corresponding dynamic equations governing the time dependence of Q_s, Q_b, and Q_d are

$$\dot{Q}_1 = \dot{Q}_s + \dot{Q}_{ss1} = -I_1(V_s, V_b, V_c, V_{ss1}) - i_{n1}(t), \tag{4a}$$

$$\dot{Q}_2 = \dot{Q}_b + \dot{Q}_{ss2} = I_1(V_s, V_b, V_c, V_{ss1}) + i_{n1}(t) - I_2(V_b, V_d, V_c, V_{ss2}) - i_{n2}(t), \tag{4b}$$

$$\dot{Q}_d = I_2(V_b, V_d, V_c, V_{ss2}) + i_{n2}(t), \tag{4c}$$

where again the noise currents i_{n1} and i_{n2} are the device noise currents induced by microscopic fluctuations within the conductances I_1 and I_2, respectively. To calculate the fluctuations in Q_d, we shall solve eqs. (3) and (4) for the influence of i_{n1}, i_{n2}, and fluctuations in V_c and in trap occupancy Q_{ss1} and Q_{ss2}, on Q_d.

IV. NOISE INTRODUCED IN ONE CHARGE-TRANSFER CYCLE

Let us now inquire into the influence of device noise on the size Q_d of a charge packet transferred through one single-step transfer element of a CTD. (Owing to the additional complexity associated with a two-step transfer element, we shall relegate the noise treatment of this case to Appendix A). It is the purpose of the remainder of this section to write the mean-square fluctuation in Q_d, q_d, in terms of the independently fluctuating voltages and currents characterizing the transfer and storage process. By expressing all dynamic quantities in terms of a minimum number of independent ones at the outset, we can greatly reduce the number of cross-correlations which we must eventually include.

If we assume that the fluctuations themselves are sufficiently small, then we can linearize eqs. (1) and (2) about their time-dependent, noiseless solutions. The form taken by eq. (2) upon expanding to lowest order in the fluctuations (and subtracting out the noiseless portion) is

$$\dot{q}_d = -\dot{q} = \frac{\partial I}{\partial V_s} v_s + \frac{\partial I}{\partial V_d} v_d + \frac{\partial I}{\partial V_c} v_c + \frac{\partial I}{\partial V_{ss}} v_{ss} + i_n, \tag{5}$$

in which lower-case letters indicate the fluctuation portion of the quantity of interest, e.g., $Q = Q^o + q$, $Q_d = Q_d^o + q_d$, etc. Since v_s and v_d depend upon q, we must also linearize eq. (1), which will yield this dependence. In calculating incomplete charge transfer, it was necessary to pay special attention to the dependence of C_s and C_d on V_s, V_d, V_c, V_{ss}. This was because most terms in the small-signal expansion of Q_s and Q_d contributed to the incomplete transfer of charge. For our purposes here, however, we can recognize such terms as small; that is, we can drop terms of the order of incomplete transfer times noise compared to terms of the order of unity times noise. As incomplete transfer per transfer is at most 3×10^{-3} in devices of interest, this is certainly warranted. Proceeding in this manner, we obtain

$$\begin{aligned} q = {} & C_s(V_S^o, V_d^o, V_c^o, V_{SS}^o)v_s - C_s(V_{SO}^o, V_d^o, V_c^o, V_{SS}^o)v_{so} \\ & + C_{ss}(V_S^o, V_d^o, V_c^o, V_{SS}^o)v_{ss} - C_{ss}(V_S^o, V_d^o, V_c^o, V_{SSO}^o)v_{sso} \\ & \equiv C_s v_s - C_S^o v_{so} + C_{ss}v_{ss} - C_{SS}^o v_{sso} \end{aligned} \tag{6a}$$

and

$$\begin{aligned} q_o = -q = {} & C_d(V_S^o, V_d^o, V_c^o, V_{SS}^o)v_d - C_d(V_S^o, V_{do}^o, V_c^o, V_{SS}^o)v_{do} \\ & \equiv C_d v_d - C_d^o v_{do}. \end{aligned} \tag{6b}$$

It is these equations that must be solved for v_s and v_d in terms of q, v_c, and v_{ss}.

At this point a minor subtlety enters. In Fig. 2a, contact 1 is tied to the clock voltage V_c. Since a fluctuation v_c in V_c cannot instantaneously alter the amount of charge stored on C_s, fluctuations v_s and v_{so} must be correlated, and indeed this correlation is contained in eq. (1a). A similar argument also applies to contact 2, v_d and v_{do}, and eq. (1d). (Since the capacitance C_{ss} is not tied directly to the clock, we need not concern ourselves with trapped charge at this point.) It follows from eqs. (1a) and (1d) that

$$v_{so} = \frac{\partial V_{so}}{\partial V_c} v_c = \frac{\partial V_s}{\partial V_c} \frac{C_s}{C_S^o} v_c = (+1) \frac{C_s}{C_S^o} v_c \tag{7a}$$

and that

$$v_{do} = \frac{\partial V_{do}}{\partial V_c} v_c = \frac{\partial V_d}{\partial V_c} \frac{C_d}{C_d^o} v_c = (-1) \frac{C_d}{C_d^o} v_c. \tag{7b}$$

[In (7b), $\partial V_d / \partial V_c = -1$ because it is $-V_c$ that is connected to contact 2.] With respect to the traps, since V_{ss} is the effective level to which traps are occupied during the transfer cycles, V_{sso} is unaffected by the dynamics of the operation of the device and, hence, $v_{sso} = 0$. We may now solve eq. (6a) for v_s, (6b) for v_d, and insert into eq. (5) to obtain

$$\begin{aligned} \dot{q}_d = {} & -\left(\frac{g_m}{C_s} + \frac{g_r}{C_d}\right) q_d + \left(g_m + g_r + \frac{\partial I}{\partial V_c}\right) v_c \\ & - \left(g_m \frac{C_{ss}}{C_s} - \frac{\partial I}{\partial V_{ss}}\right) v_{ss} + i_n, \end{aligned} \tag{8a}$$

from which it follows at once that

$$q_d(t) = \int_0^t dt' \exp\left[-\int_{t'}^t dt''/\tau(t'')\right]\left[\left(g_m + \frac{\partial I}{\partial V_c}\right)\bigg|_{t'} v_c(t') - \left(g_m \frac{C_{ss}}{C_s} - \frac{\partial I}{\partial V_{ss}}\right)\bigg|_{t'} v_{ss}(t') + i_n(t')\right], \quad (8b)$$

where we have taken $g_r \ll g_m$ and defined the time-dependent relaxation time τ by

$$1/\tau \equiv g_m/C_s. \quad (8c)$$

(The capacitances C_s and C_d are of comparable size.) In eq. (8), g_m is the forward conductance $(\partial I/\partial V_s)$ and g_r is the reverse conductance $(-\partial I/\partial V_d)$. Because of the inherent undirectionality, $g_m \gg g_r$. The statistical distribution of $q_d(t)$, which we seek, can be determined from (8b) and the statistical properties of v_c, v_{ss}, and i_n.

Our expression for the noise fluctuation in the transferred charge given in eq. (8) has several interesting features which should be carefully noted. The charge fluctuation that accumulates is not simply the integral of the noise current over the time interval. Rather, it is the integral of the noise current times a damping (relaxation) factor. Physically, this arises because if an excess quantity of charge $-q(=q_d)$ has been transferred, then the subsequent current (which depends most strongly on q_s) is reduced from what it would have been in the absence of the fluctuation. The reduced current causes less charge to flow from C_s to C_d, which in turn tends to partially compensate (null out) the effect of the previous fluctuation. This leads to a suppression of the device noise, which is shown explicitly in eq. (8b). During the initial portion of the charge-transfer cycle, the transfer current is largest, as is the conductance g_m. Hence, although the larger the current, the more noise is present in the charge transfer, because the damping is also largest initially, the effect of this noise on the transferred charge can be expected to be greatly suppressed. On the other hand, near the end of the charge-transfer cycle, while the noise from the transfer of charge and its accompanying noise is much reduced in size, so is the damping. As it turns out, it is the noise produced during the end of the transfer cycle that is most important. This is convenient, because it means that the size of the noise produced by the end of the transfer cycle is independent of the size of the initial charge packet (as long as some charge is transferred).

Of primary interest is the mean-square fluctuation in the transferred charge $\langle q_d(t_f)^2 \rangle$, where t_f is the time at the end of the transfer cycle.

Since the fluctuations v_c, v_{ss}, and i_n are mutually independent, as we shall see when we determine their statistical distribution, it follows that

$$\langle q_d(t_f)^2 \rangle = \int_0^{t_f} dt_1 \int_0^{t_f} dt_2 \exp\left[-\int_{t_1}^{t_f} dt_1'/\tau(t_1')\right]$$
$$\times \exp\left[-\int_{t_2}^{t_f} dt_2'/\tau(t_2')\right]\left[\left(g_m + \frac{\partial I}{\partial V_c}\right)\bigg|_{t_1}\left(g_m + \frac{\partial I}{\partial V_c}\right)\bigg|_{t_2}\right.$$
$$\times \langle v_c(t_1)v_c(t_2)\rangle + \left(\frac{1}{\tau} - \frac{\partial I}{\partial Q_{ss}}\right)\bigg|_{t_1}\left(\frac{1}{\tau} - \frac{\partial I}{\partial Q_{ss}}\right)\bigg|_{t_2}$$
$$\left.\times \langle q_{ss}(t_1)q_{ss}(t_2)\rangle + \langle i_n(t_1)i_n(t_2)\rangle\right]. \quad (9)$$

In (9), we have used $C_s v_{ss} = q_{ss}$ and $g_m/C_s = 1/\tau$. Thus, to determine $\langle q_d^2 \rangle$, we must calculate the autocorrelation function of v_c, v_{ss}, and i_n. For stationary noise, such quantities are well known.[3–6] The purpose of the next section will be to calculate these autocorrelation functions for the nonstationary conditions that enter the present problem. The reader should note carefully at this point, moreover, that the factors multiplying these correlation functions involve time-dependent quantities characteristic of the detailed (but noiseless) solutions to the nonlinear device equations [eq. (1)]. Although further simplifications can be made in some cases, it should be evident that, in general, one must understand the noiseless problem in order to do the problem with noise.

Two other important aspects of our results (8b) for $q_d(t)$ and (9) for $\langle q_d(t_s)^2 \rangle$ are these. First, as a result of the suppression factor, $q_d(t_f)$ and, hence, $\langle q_d(t_f)^2 \rangle$ are for all practical purposes independent of the size of the signal. Owing to the suppression, $q_d(t_s)$ depends most strongly on details of the charge transfer for $t \approx t_f$. However, considering the typical size of α, the coefficient of incomplete charge transfer ($\alpha \leqq 10^{-3}$), the details of the charge transfer for $t \approx t_f$ can deviate by only about 10^{-3}, which for our purposes is quite insignificant. Second, $q_d(t_f)$ is a stationary random variable, even though its statistics must be derived from the nonstationary distributions of v_c, v_{ss}, and i_n. These two results greatly simplify our treatment of compounding of the noise in Section V.

(Note that, to avoid undue complication, we have left out leakage current into the storage regions, as well as the noise associated with this current. As this noise source is uncorrelated with the other sources considered above, we can treat it in a separate section. When this component of noise is included, it is no longer the case that $q_d = -q$

[eq. (6b)], and, hence, greater care must be exercised when we sum the noise added with each transfer. This effect also shows up in the distinction between transfer process and storage noise, which is discussed in more detail in Section V.)

V. NOISE INTRODUCED BY MICROSCOPIC FLUCTUATIONS

In the preceding section, we expressed the mean-square fluctuation in the charge transferred during a single transfer cycle in terms of the autocorrelation functions of the various contributions to the device noise. In this section, we shall calculate these correlation functions and then use the results to estimate the noise of each type introduced during a single transfer cycle. Our task is eased considerably because of the extensive effort that has already gone into the study of noise in solid-state devices.[3–6]

Although at first sight the character of the charge transfer from C_s to C_d in a CCD appears to be rather different from that in an IGFET bucket-brigade device, in fact, the two types of charge transfer can be treated in a similar manner. We shall carry over this similarity in treating the device noise associated with $i_n(t)$: we shall make use of the understanding available of noise in IGFET's and then apply these results to the CCD as well. In so doing, we must (and shall) be careful to make note of certain important differences between CCD-mode and BB-mode transfers which can affect the noise calculated.

5.1 Thermal noise

Of the primary sources of noise present in IGFET's—thermal noise at high frequency,[19] generation-recombination ($g - r$) noise at intermediate frequency,[20] and $1/f$ noise at low frequency[21]—by far the most important source of noise associated with $i_n(t)$ is the thermal noise arising from the Brownian motion of the charge carriers in the inverted region of the semiconductor, which forms the conductance I. (This is evident because the spectral densities of $1/f$ and $g - r$ noise have dropped considerably from their peak values by 10^5 Hz, the lower bound on CTD transfer frequencies.) (In CCD's the proximity to the semiconductor-insulator interface of the charge being stored as well as transferred greatly enhances the role of interface trapping on $\langle q_d^2 \rangle$. In fact, for CCD's, it appears that interface noise is the dominant form of device noise in these devices. For reasons that are apparent in the next subsection, we shall treat this contribution as a trapping noise associated with v_{ss} rather than as $1/f$ noise associated with i_n.)

Of the early theoretical work on thermal noise in IGFET's, the treatment by Jordan and Jordan[22] seemed clearest to us on first reading. These authors note that a spontaneous current fluctuation $i_s(x, t)$ at x along the channel of the IGFET $(0 \leqq x \leqq L)$ will give rise to a voltage fluctuation $v_s(x, t) = i_s(x, t)dx/\mu\sigma(x, t)$, where μ is the mobility of the carriers and $\sigma(x, t)$ is the mobile charge per unit length of channel at time t. The spontaneous voltage fluctuation $V_s(x, t)$ will in turn induce an x-dependent voltage fluctuation all along the channel, which in its turn induces a fluctuation in the source-to-drain current i_d given by

$$i_d(t) = \frac{\mu}{L}\sigma(x, t)v_s(x, t). \tag{10a}$$

Making use of the above relation between v_s and i_s, we obtain

$$i_d(t) = i_s(x, t)dx/L. \tag{10b}$$

[The same result can be obtained using the impedance field method (IFM).[23] Alternatively, one can develop a current-current method analogous to the current-voltage method employed in the IFM. This is outlined briefly in Appendix B.] The contribution to the noise current $i_n(t)$ due to $i_d(t)$ induced by fluctuations all along the channel is from (10b)

$$i_n(t) = \int_0^L i_s(x, t)dx/L, \tag{11a}$$

so that

$$\langle i_n(t_1)i_n(t_2)\rangle = \int_0^L dx_1 \int_0^L dx_2 \langle i_s(x_1, t_1)i_s(x_2, t_2)\rangle/L^2. \tag{11b}$$

The autocorrelation function of $i_s(x, t)$ for thermal noise can be found from that of the current density $j_s(\mathbf{x}, t)$ obtained from microscopic noise theory [18] and given by

$$\langle j_s(\mathbf{x}_1, t_1)j_s(\mathbf{x}_2, t_2)\rangle = 2kT\mu\rho(\mathbf{x}_1, t_1)\delta(\mathbf{x}_1 - \mathbf{x}_2)\delta(t_1 - t_2), \tag{12}$$

where $\rho(\mathbf{x}_1)$ is the charge density at $\mathbf{x}_1$. Inserting (12) into (11b) and noting that $\rho(\mathbf{x}) = \sigma(x)/A(x)$, $A(x)$ being the cross-sectional area of the channel at x, we obtain

$$\langle i_n(t_1)i_n(t_2)\rangle = 2kT\delta(t_1 - t_2) \int_0^L \mu\sigma(x, t_1)dx/L^2. \tag{13a}$$

The integral in (13) must be found from a knowledge of the noiseless operation of the device at time t_1. In general, the integral is directly proportional to the forward conductance[22] so that

$$\langle i_n(t_1) i_n(t_2) \rangle = 2kT\delta(t_1 - t_2) g_m(t_1) H_n(t_1), \tag{13b}$$

where $H_n = 1$ for the IGFET operated in the linear region, $H_n = \frac{2}{3}$ for the IGFET operated in saturation, and $H_n = \frac{1}{2}$ for diffusion-limited current.[11] In most cases of interest, the CTD is operated in saturation so that $H_n = \frac{2}{3}$. For very long clock periods, $H_n = \frac{1}{2}$, appropriate near the end of the cycle, can be used. As the 30 percent uncertainty between $\frac{1}{2}$ and $\frac{2}{3}$ is tolerable at present for noise calculations in these devices, we shall not concern ourselves with the additional dependence of H_n on the time t_1. (Setting $H_n = \frac{2}{3}$ and $H_n = \frac{1}{2}$ gives adequate upper and lower bounds on the thermal noise.)

[Although (13b) was derived for the bucket-brigade type of charge transfer, we shall assume that it is valid for CCD-type transfer as well, where, of course, one uses g_m appropriate to the CCD device.[10,11] This assumption clearly breaks down, however, when $\langle q_d^2 \rangle^{\frac{1}{2}}$ becomes comparable to or larger than the amount of *free* charge incompletely transferred in a single transfer phase. The reason for this difficulty goes back to the assumption of linearity in eq. (5). This, of course, also applies, *mutatis mutandis*, to all other noise contributions as well. Treating these interesting nonlinear problems is, unfortunately, beyond the scope of this paper.]

Returning now to (9), we can calculate the contribution to $\langle q_d^2 \rangle$ arising from thermal noise during a single transfer cycle. If we set $H_n = \frac{2}{3}$, ignore the time dependence of C_s (a 0.1 percent effect if $\alpha = 10^{-3}$), and recall that $\tau^{-1} = g_m/C_s$, inserting (13b) into (9) and integrating over t_1 and t_2, we obtain

$$\langle q_d^2 \rangle |_{kT} = \tfrac{2}{3} kTC_s \tag{14a}$$

in the limit that

$$\exp\left[-\int_0^{t_f} dt'/\tau(t') \right] \ll 1,$$

as it must if the incomplete transfer α is to satisfy $\alpha \ll 1$. (In deriving the above, we have not assumed that g_m is independent of time.) A corresponding result was first given by Boonstra and Sangster;[24] however, their $\langle q_d^2 \rangle$ was four times larger than the right-hand side of (14a). We believe that (14a) is in fact correct.

[If one ignores the nonstationarity of the noise, a result similar to that of Boonstra and Sangster can be obtained in the following manner. For thermal noise, the mean-square current fluctuation is given by

$\langle i_d^2 \rangle = 4kT\frac{2}{3}g_m B$, where g_m is the conductance and B is the bandwidth. If $\tau = C_s/g_m$ is the characteristic relaxation time, then $B = \tau^{-1}$ and $\langle q_d^2 \rangle = \langle i_d^2 \rangle \tau^2 = 4kT\frac{2}{3}C_s$. (The quantities $\langle q_d^2 \rangle$ and $\langle i_d^2 \rangle$ are mean-square fluctuations, not Fourier coefficients. The quantity $\langle i_d^2 \rangle$ is obtained from the spectral density of i_d using the Wiener-Khinchin theorem.) While this result is not too bad, attempting to calculate noise in more complicated, nonstationary situations using stationary results can lead to trouble. For example, for a two-step transfer process, we obtain for the thermal contribution to $\langle q_d^2 \rangle$ the result [from eq. (62)]

$$\langle q_d^2 \rangle |_{kT} = \tfrac{2}{3}kTC_b + \tfrac{2}{3}kTC_s(1 + b^{-1})^{-1}, \tag{14b}$$

where $b = (g_{m2}/C_b)(g_{m1}/C_s)^{-1}$, and is assumed to be independent of time. This result is more of a challenge to obtain from arguments based on stationary noise sources.]

5.2 Interface (trapping) noise

We now focus attention on calculating the statistics of

$$q_{ss}(t) = C_{ss}(t)v_{ss}(t),$$

the fluctuations in occupancy of the interface states during charge transfer. To proceed, we must first write down the dynamic equations governing the trapping. These are simple since we are working in the lumped-charge approximation. If greater accuracy is desired, one can be more microscopic and include the position and energy dependence of the trapping states, as well as their capture cross section and thermal-release time. The procedure is similar but much more involved.

The dynamic equation relating the flow of charge into and out of the traps is

$$\dot{Q}_{ss} = I_t(I, Q_{ss}) - I_r(Q_{ss}) + i_{nss}, \tag{15}$$

where the trapping current I_t is given by

$$I_t = I(\sigma_x/A)(Q_{ss}^t - Q_{ss})/Q_{ss}^t, \tag{16a}$$

and where the release current I_r is given by

$$I_r = Q_{ss}/\tau_r(t). \tag{16b}$$

In (15), i_{nss} is the noise current associated with the filling and emptying of traps; in (16a), I is the transfer current, σ_x/A is the ratio of the effective cross section of all the traps to the area A through which I flows, and the factor $(Q_{ss}^t - Q_{ss})/Q_{ss}^t$ is the fraction of the total number of active sites Q_{ss}^t that are empty; in (16b), τ_r is the thermal-release time. Writing $Q_{ss} = Q^o_{ss} + q_{ss}$, expanding to first order in the noise q_{ss}, and subtracting off the zero-order equation, we obtain for the noise

$$\dot{q}_{ss} = \frac{\partial I_t}{\partial I}\frac{\partial I}{\partial Q_{ss}} q_{ss} + \frac{\partial I_t}{\partial Q_{ss}} q_{ss} - \frac{\partial I_r}{\partial Q_{ss}} q_{ss} + i_{nss} \tag{17a}$$

$$= -\left[-\frac{I_t}{I}\frac{\partial I}{\partial Q_{ss}} + \frac{1}{\tau_t(t)} + \frac{1}{\tau_r(t)}\right] q_{ss} + i_{nss} \tag{17b}$$

$$= -q_{ss}/\tau_{ss}(t) + i_{nss}, \tag{17c}$$

where τ_t is the effective trapping time $(\partial I_t/\partial Q_{ss})$, and where τ_{ss}^{-1} equals the bracketed quantity in (17b). (Ordinarily $I_T \ll I$, so that the first term in τ_{ss}^{-1} can be neglected. Under these circumstances, the noise contribution from different types of traps will be independent, and the traps may be treated independently.)

Proceeding now, we solve (17c) for $q_{ss}(t)$, obtaining

$$q_{ss}(t) = \int_0^t dt' \exp\left[-\int_{t'}^t dt''/\tau_{ss}(t'')\right] i_{nss}(t'). \tag{18}$$

To obtain the correlation function of $q_{ss}(t)$, we must calculate $\langle q_{ss}(t_1)q_{ss}(t_2)\rangle$. The resulting expression involves the correlation function of $i_{nss}(t)$, which, since i_{nss} is an elementary microscopic process, is given by[18]

$$\langle i_{nss}(t_1)i_{nss}(t_2)\rangle = e[I^o_t(t_1) + I^o_r(t_1)]\delta(t_1 - t_2), \tag{19}$$

where e is the charge on an elementary carrier. (The above is not obtained from a spectral density, since for a time-dependent current, a nonstationary process, spectral density is not defined.) For $t_2 > t_1$, it follows that

$$\langle q_{ss}(t_1)q_{ss}(t_2)\rangle = \exp\left[-\int_{t_1}^{t_2} dt/\tau_{ss}(t)\right] \cdot \int_0^{t_1} dt \exp\left[-\int_t^{t_1} dt'/\tau_{ss}(t')\right] e[I^o_t(t) + I^o_r(t)]. \tag{20}$$

If this result is inserted into (9), summation over all trapping states yields the contribution to the transfer noise arising from these states. As is evident from (20), in general, one must know the noiseless solution, especially $\tau_{ss}(t)$, $I^o_t(t)$ and $I^o_r(t)$, in order to actually calculate the noise. A few simplifications, however, permit us to recover Tompsett's result[17] for a single-step, CCD-mode transfer, which is valid in the limit of $I_t \ll I_r$, i.e., near the end of a transfer cycle.

To calculate the contribution to $\langle q_d^2 \rangle$ arising from interface-state trapping, we must, according to (9), know not only $\langle q_{ss}(t_1) q_{ss}(t_2) \rangle$, which we found above (20), but also $(\partial I / \partial Q_{ss})$. This quantity entered into a previous discussion[10] of the role of surface states on incomplete transfer. There it was noted that one could write

$$\partial I / \partial Q_{ss} = -a(t)/\tau(t), \tag{21}$$

where for CCD-mode transfer $a = 0$ and for bucket-brigade-mode transfer a falls from $a = C_s/C_{ch}$ during most of the transfer cycle (C_{ch} being the channel capacity) to $a \approx 0$ toward the end of the cycle, where the IGFET current becomes emission limited. [The vanishing of a for CCD-mode operation is due to the fact that the field felt by a mobile carrier due to another carrier is independent of whether the other carrier is free or trapped. The large value of a for BB-mode transfer arises from the fact that relatively small changes in the threshold voltage, induced by changes in trap occupancy in the channel region, can result in relatively large changes in transfer current. If a is large, it enters (9) as a^2, while, as we shall see, $\langle q_{ss}^2 \rangle$ is proportional to C_{ss} ($\ll C_{ch}$ for the IGFET channel). This implies that the ratio (R) of the surface-state-noise contribution to $\langle q_d^2 \rangle$ for bucket-brigade to that for CCD is C_s/C_{ch}. If, on the other hand, the BB-mode is not turned off until the channel current becomes emission limited, then $a \approx 0$. If we assume that the suppression factors in (9) damp out the noise introduced while $a \gg 1$, then the above ratio (R) becomes C_{ch}/C_s, which means that the contribution to $\langle q_o^2 \rangle$ of q_{ss} can be ignored for BB-mode transfer. We shall assume that this is the primary operational region of interest, pointing out, however, that if the BB-transfer mode is terminated while a is large, one should expect an increase in that portion of $\langle q_d^2 \rangle$ arising from trapping.]

Returning to CCD-mode transfer where we expect interface states to play the largest role, if we assume in eq. (9) that $\langle q_{ss}(t_1) q_{ss}(t_2) \rangle$ varies in time slowly compared to $\tau(t)$, then we can integrate over t_1 and t_2 to obtain

$$\langle q_d(t_f)^2 \rangle \approx \langle q_{ss}(t_f)^2 \rangle. \tag{22}$$

If, in addition, we focus attention on long transfer cycles (10^{-6} second or longer), taking $I_r^o \gg I_t^o$ and $\tau_t \gg \tau_r$, then from (20) we obtain

$$\langle q_{ss}(t_f)^2 \rangle = e \int_{t_c}^{t_f} dt \cdot \exp\left[-2 \int_t^{t_f} dt' / \tau_r(t') \right] Q_{ss}^o(t)/\tau_r(t) + \langle q_{ss}(t_c)^2 \rangle. \tag{23}$$

If we make one more assumption that at time $t_c (0 < t_c < t_f)$, $Q_{ss} = Q_{ss}^o = Q_{ss}^t$, and for subsequent $t(t_c < t < t_f)$, Q_{ss}^o decays as if $I_t = 0$, then

$$Q_{ss}^o(t) = Q_{ss}^t \exp \left[- \int_{t_c}^{t} dt'/\tau_r(t') \right], \tag{24a}$$

$$\langle q_{ss}(t_c)^2 \rangle = 0. \tag{24b}$$

Inserting these into (23), we find that

$$\langle q_{ss}(t_f)^2 \rangle = qQ_{ss}^t \exp \left[- \int_{t_c}^{t_f} dt'/\tau_r(t') \right] \left\{ 1 - \exp \left[- \int_{t_c}^{t_f} dt'/\tau_r(t') \right] \right\}, \tag{24c}$$

as found by Tompsett[17] using other reasoning. Should conditions be such that any of these assumptions are unwarranted, then, of course, the result [(20) in (9)] is more complicated.

The result given in (24c) has the following significance. If on the average $\tau_r \gg (t_f - t_c)$ or $\tau_r \ll (t_f - t_c)$, then $\langle q_{ss}^2 \rangle$ for such states is negligible. In the former case, the transfer process occurs too rapidly for the traps to respond; in the latter case, the trap occupancy can follow the transfer current quite closely; in either case, the noise is greatly suppressed as a consequence. Thus, only when $\tau_r \approx t_f - t_c$ can the traps influence the charge transfer.

If we now sum over the distribution of traps, assumed uniform in energy E, and take τ_r^{-1} proportional to $\exp(-E/kT)$, then following Strain[25] we obtain

$$\langle q_{ss}^2 \rangle = e^2 kTN_{ss}A \ln 2, \tag{24d}$$

Tompsett's result,[17] where N_{ss} is the number of interface states per unit energy per unit area, and A is the active trapping area. Like Tompsett,[17] we conclude that the noise introduced into Q_d from trapping has a mean-square value of

$$\langle q_d^2 \rangle = e^2 kTN_{ss}A \ln 2, \tag{25}$$

that is,

$$\langle q_d^2 \rangle = \langle q_{ss}^2 \rangle.$$

5.3 Clock-voltage noise

The influence of fluctuations in clock voltage on the noise added to the transferred charge in a single transfer can be treated very quickly, especially if it is white as we shall assume. If the spectral density of the noise at zero frequency is $S_v(0)$, then [18]

$$\langle v_c(t_1)v_c(t_2)\rangle = \tfrac{1}{2}S_v(0)\delta(t_1 - t_2). \tag{26}$$

From (9) it follows that the contribution to $\langle q_d^2\rangle$ from clock-voltage fluctuations is

$$\langle q_d^2\rangle|_{\text{clock}} = \tfrac{1}{2}S_v(0)\int_0^{t_f} dt \cdot \exp\left[-2\int_t^{t_f} dt'/\tau(t')\right](g_m + \partial I/\partial V_c)^2|_t. \tag{27a}$$

We recall that $g_m = C_s/\tau$. The direct dependence of I on V_c depends strongly on the type of device, and one must be careful not to include in $\partial I/\partial V_c$ terms already included in $(\partial I/\partial V_s)(\partial V_s/\partial V_c)$. Thus, for a single-step bucket brigade, $\partial I/\partial V_c = +g_m$, while for a single-step CCD, $\partial I/\partial V_c = 0$. If we set $\partial I/\partial V_c = bg_m$, then

$$\langle q_d^2\rangle|_{\text{clock}} = \tfrac{1}{2}S_v(0)C_s^2(1+b)^2\int_0^{t_f} dt \cdot \exp\left[-2\int_t^{t_f} dt'/\tau(t')\right]\tau^{-2}(t). \tag{27b}$$

In general, (27b) must be evaluated from a knowledge of $\tau(t)$ obtained from the noiseless charge-transfer characteristic. If, however, we assume that we can write $\partial I/\partial Q_s = dI/dQ_s$, then

$$dt/\tau(t) = g_m dt/C_s = dt\partial I/\partial V_s/C_s = dt\partial I/\partial Q_s = dtdI/dQ_s = -dI/I.$$

It follows that

$$\langle q_d^2\rangle|_{\text{clock}} = \tfrac{1}{2}S_v(0)C_s^2(1+b)^2\int_{I_f}^{I_o}\frac{dI}{I}\left(\frac{I_f}{I}\right)^2\frac{1}{\tau(I)}, \tag{27c}$$

where $I_f = I(t = t_f)$ and $I_o = I(t = 0)$. In this form it is clear how the integral yields the effective bandwidth B of the white clock-voltage noise. One can replace the τ^{-1} in (27c) by dI/dQ_s, which in turn may be calculated as a function of I using eq. (8) of Ref. 11 ($Q_s = C_sV_s$). For our purposes, we shall be content with

$$\langle q_d^2\rangle|_{\text{clock}} = \tfrac{1}{2}S_v(0)BC_s^2(1+b)^2. \tag{27d}$$

Of course, if the clock-voltage noise is not white, (26) should be replaced with the actual correlation function, which then introduces an additional time-dependent function in the integrand of (27). Relation (27d) will be useful in discussing modulation noise.

VI. OTHER NOISE SOURCES

In the preceding section, we discussed the three primary sources of noise that contribute to the fluctuations in the size of a single charge packet as the result of a single transfer. In this section we examine two other sources of noise which do not readily fit into (9) without undue complication. For simplicity, we shall not include the thermal, trapping, or clock-voltage fluctuations discussed in Section V, as these can be superimposed linearly on the results discussed below.

6.1 Emission-limited-current shot noise

In describing the types of noise expected to be generated in CTD's, we noted that shot noise would *not* play a significant role under ordinary circumstances. This is because, in simplest terms, a single transfer unit of a CTD is like two capacitors connected by a resistor, and, owing to strict charge neutrality in the resistor, shot noise is not present. If, however, the clock period t_0 of an IGFET CTD becomes long ($t_0 > 10^{-6}$ second), the channel current becomes partially emission limited at the source towards the end of the transfer cycle.[11] In this case the shot noise associated with the emitted current will not be totally suppressed. We shall now show how this can be treated. At the same time it will become clear why "ideal" resistors totally suppress shot noise.

In Fig. 3 we represent the barrier region between the diffused source and the channel by a conductance with current I_e and the channel by a conductance with current I. The voltage V_s at the source end of the channel is set V_a as in Ref. 11. In Section V we ignored I_e, assuming its conductance was much greater than that of I. We now consider the more realistic situation in which the conductances of I_e and I are comparable near the end of the transfer cycle. (In such circumstances, the g_m introduced in Section IV and used extensively in Section V, must be replaced by the series conductance of I_e and I in the expressions for thermal, trapping, and clock-voltage noise.)

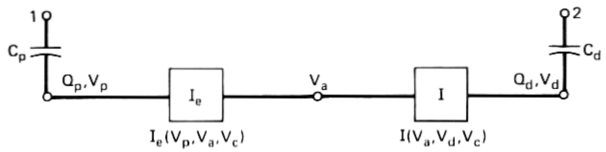

Fig. 3—Lumped-charge model for a CTD including a barrier conductance I_e.

What is happening physically can be outlined as follows. Voltages V_a as well as V_p and V_d can fluctuate in response to fluctuations induced in the conductances. If a small, spontaneous current fluctuation i_s (positive, say) occurs through I_e, then V_a will increase, simultaneously inducing a (positive) fluctuation i in I and a (negative) fluctuation i_e in I_e. The net current fluctuation is $i = i_s + i_e$. If the conductance of I_e is much larger than that of I, then V_a will change very little, $i \approx 0$, and hence $i_e \approx -i_s$. In other words, a current i_e is induced by i_s which nearly cancels i_s; with the larger conductance of I_e, a small v_a induces an i_e sufficient to suppress the i_s fluctuation, while with the smaller conductance of I, v_a induces a much smaller i and I. In the limit of an "ideal" resistor, since it is the bulk which controls the current and not the contacts, $i \rightarrow 0$ in the limit of zero contact resistance, and the shot noise is totally suppressed.

Let us now calculate the shot noise introduced during emission-limited operation. We start as before with the equations of the model:

$$I_e(V_p, V_a) = I_e(V_p - V_a), \tag{28a}$$

$$I = I(V_a, V_d, V_c), \tag{28b}$$

$$Q_s = \int_{V_{po}}^{V_p} C_s(V_p', V_a, V_c), \tag{28c}$$

$$-\dot{Q}_s = I_e + i_s = I. \tag{28d}$$

In (28a) we have assumed that I_e is primarily a function of $(V_p - V_a)$, anticipating that for barrier current, I_e is well-described by a diode equation. Since the variations of I with V_d or V_c and of C_s with V_a or V_c are small, we shall ignore these dependences as we did in Section IV. Setting $I_e = I_e^o + i_e$ and $I = I^o + i$, it follows from (28) that

$$i_e = \frac{\partial I_e}{\partial V_p}(v_p - v_a), \tag{29a}$$

$$i = \frac{\partial I}{\partial V_a} v_a, \tag{29b}$$

$$q_s = C_p v_p, \tag{29c}$$

$$-\dot{q}_s = i_e + i_s = i. \tag{29d}$$

Solving for q_s in terms of i_s, we obtain

$$-\dot{q}_s = \frac{\partial I}{\partial V_a} \frac{(\partial I/\partial V_p)(q_s/C_p) + i_s}{\partial I/\partial V_a + \partial I/\partial V_p}, \tag{30a}$$

or letting $g_m = \partial I/\partial V_a$, $g_b = \partial I/\partial V_p$ and solving for q_s,

$$q_d = -q_s = \int_0^t dt' \exp\left[-\int_{t'}^t dt''/\tau_b(t'')\right] [i_s(1 + g_b/g_m)^{-1}]|_{t'}, \quad (30b)$$

where $\tau_b = [(g_m^{-1} + g_b^{-1})C_p]$. This expression tells us the fluctuation induced in the transferred charge packet q_d resulting from the shot noise associated with the current emitted into the channel from the source. In the $g_b \to \infty$ limit (zero barrier resistance), $q_d \to 0$. In the $g_b \to 0$ limit (infinite barrier resistance), $q_d \to \int i_s dt$, full, unsuppressed shot noise.

In deriving (30b) our primary goal was to determine $\langle q_d^2 \rangle$. Since $\langle i_s(t_1) i_s(t_2) \rangle = eI^o(t_1)\delta(t_1 - t_2)$,

$$\langle q_d^2 \rangle|_{\text{shot}} = \int_0^{t_f} dt \exp\left[-2\int_t^{t_f} dt'/\tau_b(t')\right] eI^o(t) \cdot [1 + g_b(t)/g_m(t)]^{-2}. \quad (31)$$

To evaluate (31) requires, in general, a knowledge of the noiseless solution to the charge-transfer equations. We can obtain a feel for the result if we recall van der Ziel's expression for the current spectral density of diode shot noise:[6]

$$S_i(f) = 2e(I^o + I_o) = 2kTg_b \frac{I^o + 2I_o}{I^o + I_o}. \quad (32a)$$

(I_o is the diode leakage current that we have neglected.) Although derived for stationary noise, it is straightforward to redo the derivation for nonstationary noise to obtain an autocorrelation function of the form

$$\langle i_s(t_1) i_s(t_2) \rangle = e[I^o(t) + I_o)\delta(t_1 - t_2] = kTg_b(t_1) \frac{I^o(t_1) + 2I_o}{I^o(t_1) + I_o} \delta(t_1 - t_2) \quad (32b)$$

or

$$eI^o(t)\delta(t_1 - t_2) \approx kTg_b(t_1)\delta(t_1 - t_2), \quad (32c)$$

where in (32c) we have (again) ignored the diode leakage current I_0. Now then, given the barrier, $\langle q_d^2 \rangle$ will be largest if $g_b \ll g_m$. Inserting (32c) into (31) and integrating on t, it follows that

$$\langle q_d^2 \rangle|_{\text{shot}} < \tfrac{1}{2}kTC_p, \quad (33)$$

which is comparable with the thermal noise produced by I ($\tfrac{2}{3}kTC_p$). If one may assume that towards the end of the cycle $g_b(t) = dg_m(t)$, then

$$\langle q_d^2 \rangle|_{\text{shot}} = \tfrac{1}{2}kTC_p(1+d)^{-1}. \tag{34}$$

While (33) and (34) should be sufficient for estimating the contribution to q_d of shot noise, if more accuracy is desired (31) should be used.

6.2 Leakage-current noise

Although leakage currents are small and, hence, fluctuations in them even smaller, it is of interest to briefly analyze leakage-current noise. This is because (*i*) a portion of the contribution to q_d from leakage-current fluctuations is not suppressed and (*ii*) no longer does $q_d = -q_s$. We shall restrict ourselves to the role of leakage current which flows into the source, drain, and channel of an IGFET, bucket-brigade, single-step transfer CTD. Leakage current into the source and sink of a CCD can probably be treated in a similar manner.

If we assume that a leakage current per unit length of the channel $J_{ch}(x, t)$ enters the channel, following Jordan and Jordan,[22] we find that a fraction $(L - x)/L$ flows towards the source and x/L flows towards the drain. We have assumed, of course, that $\int J_{ch}(x, t)dx \ll I$. From the standpoint of noise this means that

$$\dot{q}_s = -q_s/\tau + \int_0^L dx j_{ch}(x, t)(L - x)/L + \int_s dx j_s(x, t), \tag{35a}$$

and

$$\dot{q}_d = \frac{q_s}{\tau} + \int_0^L dx \frac{j(x, t)x}{L} + \int_d dx j_d(x, t), \tag{35b}$$

where $J = J^o + j$. Solving (35a) for q_s and (35b) for q_d, we obtain

$$q_s(t) = \int_0^t dt' \exp\left[-\int_{t'}^t dt''/\tau(t'')\right] \cdot \left[\int_0^L dx \frac{L - x}{L} j_{ch}(x, t') + \int_s dx j_x(x, t')\right] \tag{35c}$$

and

$$q_d(t) = \int_0^t dt' \left[\frac{q_s(t')}{\tau(t')} + \int_0^L dx \frac{j(x, t')x}{L} + \int_d dx j(x, t')\right]. \tag{35d}$$

Comparing (35c) and (35d), we note that while the fluctuation j is suppressed in q_s and, therefore, in the first contribution to q_d, it is *not* suppressed in the second or third contributions to q_d. Thus, even though leakage currents may in themselves be small, since a portion of their noise is *not* suppressed, one may expect to see some contribution to q_d from this source. As a rough estimate, if after many transfers the size of Q_d is increased by Q_e (leakage charge), one may expect $\langle q_d^2 \rangle$ due to leakage to be about qQ_e. The autocorrelation and cross-correlation functions of q_d and q_s can be calculated at once from (35b, c) if one notes that for full shot noise,

$$\langle j(x_1, t_1) j(x_2, t_2) \rangle = eJ(x_1, t_1)\delta(t_1 - t_2)\delta(x_1 - x_2). \qquad (36)$$

Since an accurate calculation of the results again requires some detailed knowledge of the noiseless charge transfer and the role of leakage current, we shall not pursue this topic further.

VII. STORAGE-PROCESS AND TRANSFER-PROCESS NOISE

Up to this point in our discussion, we have been concerned solely with the various contributions to q_d that result from a single charge transfer of a single charge packet. A charge packet reaching the output of a CTD has, however, been transferred typically 10^2 to 10^3 times. The noise in the output packet is an accumulation not only of the noise acquired by the packet of interest during each transfer, but also the noise contained in incompletely transferred portions of preceding packets. In addition, there is correlation between the noise in successive packets at the output. Some of this correlation arises, of course, from the incompletely transferred portions picked up along the line.[7] However, even in the absence of incomplete charge transfer, there is substantial correlation from packet to packet (see Fig. 4). For example, for thermal and interface-state noise, we noted that throughout a single transfer cycle $q_d = -q$. While q_d accompanies the packet of interest, $q(= -q_d)$ is picked up by the next packet.[8] At the output the correlation will affect the spectral density of the total noise accompanying the signal. In this section, we consider this correlation and in the next section, we consider modulation noise, both in the absence of incomplete transfer; in Section IX, we discuss the output noise including incomplete transfer effects. (Finally we remind the reader that all along we have been concerned with *random* noise which is generated in addition to the signal distortion resulting from incomplete transfer. In many cases, physical processes which contribute to random noise also contribute to incomplete transfer. However, it should be kept in mind that while the noise is random and characterized by stochastic processes, the incomplete transfer is deterministic and characterized by a specific transfer function for the entire device.)

We noted in the case of contributions to q_d arising from thermal, trapping, clock-voltage and barrier-current fluctuations that conserva-

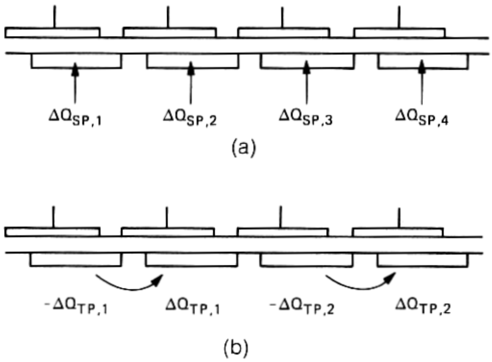

Fig. 4—Schematic illustrating the difference between the origin of storage-process and transfer-process noise.

tion of charge led to the relation $q_d + q = 0$, where q_d is the noise added to the transferred charge packet and q, which is picked up by the following packet, is the noise added to the untransferred charge. Using the notation of Ref. 8 in which $q_{s,\sigma}^{m,\mu}$ is the transfer-noise charge introduced during the σth phase of the sth transfer cycle as charge flows from the $(\mu - 1)$th to the μth phase of the mth CTD cell, the accumulated noise ΔQ_r^N in the charge at the last phase of the Nth cell at the end of the rth transfer cycle is given by

$$\Delta Q_r^N = \sum_{m=1}^{N} \sum_{\mu=1}^{p} (q_{r-(N-m),\mu}^{m,\mu} - q_{r-(N-m)-1,\mu}^{m,\mu}). \tag{37}$$

[In (37), p is the number of charge transfers per cell. If n is the total number of charge transfers in the CTD, then $n = Np$. During the $(r+1)$th transfer cycle, $Q_r^N (= {}_oQ_r^N + \Delta Q_r^N)$ flows from the device to the output through a low-pass filter. If by $g(t)$ we designate the output current per total charge transferred for a single packet, then the output noise current is given by

$$i(t) = \sum_r \Delta Q_r^N g(t - rt_o), \tag{38}$$

where t_o is the clock period. One can calculate the spectral density of $i(t)$, obtaining

$$S_i(f)|_{\text{TP}} = 4n\langle q^2\rangle_{\text{TP}} f_o |g(f)|^2 (1 - \cos 2\pi f/f_o), \tag{39a}$$

where TP denotes transfer process and f_o is the clock frequency $(f_o = t_o^{-1})$. [Here we have assumed that $\langle q^2\rangle$ is independent of (s, σ, m, μ) or, equivalently, of the signal. This is quite reasonable since the suppression effect renders the noise dependent only on the final portions of the charge transfer. And owing to the smallness of the coefficient of incomplete transfer, the dependence of the final portion of charge transfer on the signal is negligible for noise purposes.] For $f \ll f_o/2$, $S_{i\mathrm{TP}}$ is greatly suppressed below its mean value, $S_i(f_o/4)$ (see Fig. 5). This effect results from the fact that at low frequencies one is averaging over such long times that nearly all the q_d are cancelled by their corresponding $q = -q_d$. For $f \approx f_o/2$ (f is constrained by the relation $|f| < f_o/2$), $S_{i\mathrm{TP}}$ is in fact enhanced by the strong, mutual correlation between adjacent packets. The suppression of the transfer process noise at low frequencies is advantageous, since by increasing f_o one can increase the signal-to-noise ratio (S/N).

For the moment, we ignore the transfer-process noise associated with leakage and consider only storage-process (SP) noise, which we define as fluctuations that influence the size of each packet independently during each transfer phase. Since, under such circumstances, there is no correlation between the noise in different packets, the spectral density of the filtered current at the output is independent of frequency (white) and is given by

$$S_i(f)|_{\mathrm{SP}} = 2n\langle q^2\rangle_{\mathrm{SP}} f_o |g(f)|^2 \tag{39b}$$

[$|g(f)| \approx 1$ for $|f| < f_o/2$]. If $\langle q^2\rangle_{\mathrm{TP}} = \langle q^2\rangle_{\mathrm{SP}}$, the integral of $S_i(f)$ over $0 < f < f_o/2$ is twice as large for TP as for SP noise. This is because in the case of TP noise each fluctuation contributes to two charge packets, whereas in the SP case, only one packet is affected. We note that $S_i(f)$ for SP noise is *not* suppressed for $f \ll f_o/2$. Thus, although leakage noise is expected to be small, since neither is a portion of it suppressed in forming $\langle q^2\rangle_{\mathrm{SP}}$ nor is $S_i(f)$ suppressed by packet-packet correlation, the role of leakage-current noise may in some cases be more important than is usually appreciated.

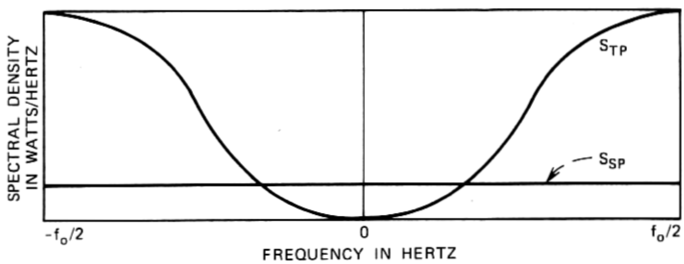

Fig. 5—Noise spectral densities S_{SP} and S_{TP} plotted versus frequency f.

In discussing TP and SP noise above (and in Ref. 8), we have assumed that different, or rather independent, mechanisms contribute to $\langle q^2 \rangle_{SP}$ and $\langle q^2 \rangle_{TP}$, and, for TP noise, that $q_d = -q$. For leakage-current noise, we found that in fact the same fluctuation could contribute to both SP- and TP-like noise, and that $q_d \neq -q = -q_s$. The question is raised: What is the spectral density of such noise?

To calculate the spectral density of the noise in cases where TP- and SP-like effects are correlated, we proceed as follows. Let the noise accumulated on the source in a single transfer phase be $q_s(s, \sigma, m, \mu)$ and the noise on the drain be $q_d(s, \sigma, m, \mu)$, where (s, σ, m, μ) is defined above. Then

$$\Delta Q_r^N = \sum_{m=1}^{N} \sum_{\mu=1}^{p} \{q_d[r - (n - m), \mu, m, \mu] + q_s[r - (N - m) - 1, \mu, m, \mu]\}, \quad (40)$$

where we continue to ignore incomplete transfer effects. It follows that the spectral density of $i(t)$ defined by (38) is given by

$$S_i(f) = 2nf_o|g(f)|^2[\langle q_d^2 \rangle + \langle q_s^2 \rangle + 2\langle q_d q_s \rangle \cos(2\pi f/f_o)]. \quad (41a)$$

[For pure TP noise, $q_s = -q_d$, and we recover (39a); for pure SP noise, $q_s \ll q_d$ owing to suppression, and we recover (39b).] By rewriting (41a) slightly, it separates into SP and TP portions:

$$S_i(f) = 2nf_o|g(f)|^2\langle (q_d + q_s)^2 \rangle + 4nf_o|g(f)|^2\langle -q_s q_d \rangle [1 - \cos(2\pi f/f_o)]. \quad (41b)$$

Thus, we find that even when the sources of noise leading to TP and SP noise are correlated and, hence, more complicated than those treated in Ref. 8, we still obtain an unsuppressed, white-noise contribution and a suppressed contribution. If incomplete transfer effects are included, the spectral densities become much more complicated. We take up this matter again in Section IX.

VIII. MODULATION NOISE

In Section 5.3, we calculated the noise introduced into the signal during a single transfer resulting from fluctuations in the clock voltage. To calculate the effect that this modulation noise has on the output of the CTD, we must notice that each clock-voltage fluctuation is felt simultaneously by each transferring packet throughout the entire device. If the elements of the CTD are physically quite similar, each fluctuation will induce nearly the same noise contribution to the $(q_s, q_d = -q_s)$ pair generated in each CTD element during each cycle.

This means that modulation noise can be expected to be substantially suppressed, which is in fact what we find. In calculating modulation noise, we shall first assume that the transfer parameters of each pair of transfer elements of the CTD are identical, and then we shall include some variation in these parameters.

If by q we represent that portion of q_d introduced by clock-voltage fluctuations, since q is purely TP noise, it follows from (37) that

$$\Delta Q_r^N = \sum_{\mu=1}^{p} \left[-q_{r-N,\mu}^{1,\mu} + q_{r,\mu}^{N,\mu} + \sum_{n=1}^{N-1} \left(q_{r-(N-n),\mu}^{n,\mu} - q_{r-(N-n),\mu}^{n+1,\mu} \right) \right] \tag{42a}$$

$$= \sum_{\mu=1}^{p} \left(q_{r,\mu}^{N,\mu} - q_{r-N,\mu}^{1,\mu} \right). \tag{42b}$$

The simplification attained in (42b) results from the above observation concerning the similarity of contributions to the noise throughout the CTD, which stated quantitatively is that

$$q_{s,\mu}^{m,\mu} = q_{s,\mu}^{n,\mu} \tag{43}$$

for all cells m, n during a given cycle s. It follows from (42b) that the contribution to the total mean-square fluctuation in the output due to clock-voltage fluctuations is simply

$$\langle (\Delta Q_r^N)^2 \rangle |_{\text{clock}} = 2p \langle q_d^2 \rangle |_{\text{clock}}, \tag{44}$$

where $\langle q_d^2 \rangle$ is given by (27d) and p is the number of phases per cycle. The most interesting feature of (44) is that it is independent of n, the number of transfers each charge packet is subjected to in the CTD. Thus, this portion of modulation noise is not compounded and as such can be expected to be small.

If we introduce the possibility that the physical parameters characterizing each transfer stage of the CTD are slightly different, then (43) will not hold, and, as a consequence, the clock-voltage fluctuations will not essentially cancel. Let us assume that these physical parameters are distributed according to some probability distribution. Then, in place of q in (42a), let us write $(\bar{q} + q')$ where $\bar{q}$ is the charge fluctuation averaged over the distribution of the device parameters, and q' is the deviation from this average. Inserting $(\bar{q} + q')$ into (42a) for q, expressing ΔQ as $(\Delta\bar{Q} + \Delta Q')$, a straightforward calculation leads to

$$\langle \Delta\bar{Q}_r^N \Delta\bar{Q}_s^N \rangle = 2p \langle \bar{q}^2 \rangle [\delta_{r,s} - \tfrac{1}{2}(\delta_{r,s-N} + \delta_{r,s+N})] \tag{45a}$$

and

$$\langle \Delta Q_r'^N \Delta Q_s'^N \rangle = 2n \langle \overline{q'^2} \rangle [\delta_{r,s} - \tfrac{1}{2}(\delta_{r,s-1} + \delta_{r,s+1})], \tag{45b}$$

in which the *bar* denotes averaging over the device parameters. To calculate $\langle\overline{q'^2}\rangle = \langle\overline{(q - \bar{q})^2}\rangle = \langle\overline{q^2}\rangle - \langle\bar{q}^2\rangle$ for clock-voltage fluctuations, it is perhaps simplest to carry out the average first over the clock-voltage fluctuations obtaining (27), and then average over the device parameters. If we assume that the primary variation from cell to cell is the size of the storage capacitance C_s, then (27d) yields

$$\langle(\Delta Q_r'^N)^2\rangle = 2n\cdot\tfrac{1}{2}S_V(0)B(1 + b)^2\langle(C_s - \bar{C}_s)^2\rangle. \qquad (45c)$$

This result expresses the effect on the size of the charge packet of the coupling between fluctuations in clock voltage and deviations in storage capacitance. Unlike $\langle(\Delta\bar{Q}_r^N)^2\rangle$ given in (44), $\langle(\Delta Q_r'^N)^2\rangle$ is proportional to n, expressing the fact that during each transfer event, noise is added to the signal. Finally, we note from (45b) that $\Delta Q_r'^N$ has the character of transfer process noises so that its spectral density is suppressed for $f \ll f_o/2$. The spectral density of $\Delta\bar{Q}_r^N$, which can be obtained from (45a), is weird. It oscillates once between 0 and $4p\langle\bar{q}^2\rangle$ each time f changes by only f_o/N. It is probably best approximated as white of size $2p\langle\bar{q}^2\rangle$.

IX. INFLUENCE OF INCOMPLETE CHARGE TRANSFER ON COLLECTING

In the two preceding sections we have ignored the influence of incomplete charge transfer on the noise accumulated in charge packets by the time they reach the output. In Section VII, we took $\alpha = 0$ to illustrate as simply as possible the suppression in the spectral density of transfer-process noise at low frequencies. In Section VIII, we took $\alpha = 0$ to treat the multicorrelated charge fluctuations induced by the clock voltage in a straightforward manner. We now include incomplete transfer and find that, even though it alters only a small fraction of the signal and, hence, of the noise on each transfer, typically 10^{-3} to 10^{-4}, its accumulated effects are extremely important in some cases.[7]

The effects of incomplete transfer on a charge packet are not simple, even when linearity is assumed. The key is to write down a general expression for the charge, seen as a function of time at the output, which arises from a given charge packet introduced at an earlier time on an arbitrary cell. Once this is done, one can combine the effects on the output of fluctuation-induced noise charge which is created on each stage during each phase of each cycle. The result is a complex, combinatorial expression which for $n\alpha \geqq 1$ is nontrivial to evaluate numerically. Despite these complications, the results obtained are worth the effort needed to obtain them.[7]

To calculate the effect of incomplete charge transfer on the collecting of noise, we first calculate the portion of the noise q_r^m added to the charge packet in the mth cell during the rth clock cycle which is observed at the output during the sth clock cycle. As defined, q_r^m is related to the $q_{r,\sigma}^{m,\mu}$ of Section VII according to

$$q_r^m = \sum_{\mu=1}^{p} q_{r,\mu}^{m,\mu}. \tag{46}$$

Thus, for purposes of simplicity, we shall ignore for the moment the fact that the transfer of charge within each cell involves p distinct (independent) transfers. The error involved in so doing is of the order of $n\alpha^2$, which for devices of interest is less than 0.01. Since the linear model we shall use to treat incomplete transfer is probably not this accurate, this approximation is justified. However, we must be careful to use (46) when we calculate terms such as $\langle q_s^m q_r^n \rangle$ so that we do not neglect correlation effects.

The dynamic equation that governs the transfer of the charge $q(r, m)$ on the mth cell during the rth cycle is

$$q(r+1, m+1) = q(r, m) - \epsilon q(r, m) + \epsilon q(r, m+1), \tag{47}$$

where ϵ is the coefficient of incomplete transfer per cell.[15] [One can relate ϵ to α, the coefficient of incomplete transfer per transfer,[16] through $\epsilon = p\alpha$, or more accurately, through $(1-\epsilon) = (1-\alpha)^p$.] Using (47), it is straightforward to calculate the charge $q_m(s, N+1)$ observed during the sth cycle at the output, the $(N+1)$th stage of an N-cell register, as a function of charges $q_o(r, m)$, which are added to the packet present in the mth cell during the rth cycle. The result is

$$q_m(s, N+1) = (1-\epsilon)^{N-m+1} \sum_{r=0}^{\infty} \binom{r+N-m}{r} \cdot \epsilon^r q_o[s-(r+N-m+1), m]. \tag{48}$$

[If $\epsilon = 0$, the limit of negligible incomplete transfer $q_m(s, N+1)$ becomes

$$q_m(s, N+1) = q_o[s-(N-m+1), m].$$

Thus, the additional output charge seen during the sth cycle is just the charge added to the mth cell, $N-m+1$ cycles earlier.] Using (48), we calculate the total noise charge ΔQ_s^N observed at the output of an N-cell CTD during the sth cycle by replacing $q_o(r, m)$ by the noise charge q_r^m and summing overall cells m. The result is

$$\Delta Q_s^N = (1-\epsilon)\sum_{m=1}^{N}(1-\epsilon)^{N-m}\sum_{r=0}^{\infty}\binom{r+N-m}{r} \cdot \epsilon^r q^m_{s-(r+N-m+1)}, \quad (49)$$

where

$$q_r^m = q_d(r, m) + q_s(r-1, m). \quad (50)$$

In (50), $q_d(r, m)$ is the noise added to the charge on drain during the cycle of interest, and $q_s(r-1, m)$ is the noise added to the charge remaining on the source during the preceding cycle and picked up by the charge packet coming by on the next cycle. [The expression in (49), as well as our treatment of collecting, is valid whether or not the statistics of the individual q_d's and q_s's must be calculated using a nonlinear approach.] From eq. (49) we can calculate nearly all compounding effects of interest, a few of which we now consider in some detail.

For digital purposes, the most important quantity of interest is $\langle(\Delta Q_s^N)^2\rangle$, the mean-square fluctuation in the size of the output charge packet. This quantity can be calculated from (49) keeping in mind that the only nonzero, cross-correlation function that enters is $\langle q_d(r, m)q_s(r, m)\rangle$ for all r and m. The result is

$$\langle(\Delta Q^N)^2\rangle = \langle(q_d+q_s)^2\rangle H_{\mathrm{SP}}(p, N) + 2\langle -q_s q_d\rangle H_{\mathrm{TP}}(p, N), \quad (51a)$$

where q_d and q_s are noise added to the charge in the drain and source, respectively, during a single transfer cycle, the statistics of which we calculated in Section V and Section VI, H_{SP} is the collecting factor for storage-process noise $q_s = 0$, and H_{TP} is that for transfer-process noise $(-q_s = q_d)$. The analytical expressions for H_{SP} and H_{TP}, ignoring an unimportant factor of $(1-\epsilon)^2$, are

$$H_{\mathrm{SP}}(p, N) = p\sum_{m=1}^{N}(1-\epsilon)^{2(N-m)}\sum_{r=0}^{\infty}\binom{r+N-m}{r}^2\epsilon^{2r} \quad (51b)$$

and

$$H_{\mathrm{TP}}(p, N) = p\sum_{m=1}^{N}(1-\epsilon)^{2(N-m)} \cdot \sum_{r=0}^{\infty}\binom{r+N-m}{r}^2\epsilon^{2r}\left(1-\epsilon\frac{r+1+N-m}{r+1}\right). \quad (51c)$$

These can be evaluated exactly (Appendix C) as well as approximately (Appendix D). [The latter is necessary because the former, although exact, is difficult to evaluate for the large N and small ϵ of greatest interest. These calculations are nontrivial. For example, except when $N\epsilon \ll 1$, attempting to calculate (51b) or (51c) using Stirling's formula to approximate the factorials is doomed to failure.] One may note at the outset that for $\epsilon = 0$, $H_{SP} = H_{TP} = pN = n$ the number of transfers, as expected. This result is also approximately valid for $N\epsilon = n\alpha \ll 1$.

The approximate results obtained from evaluating the sums in (51b, c),

$$H_{SP}(p, N) = p\chi_N(\epsilon) = p(N + \tfrac{1}{2})t^{-1}e^{-b}[I_o(b) + I_1(b)] - \tfrac{1}{2} \quad (52a)$$

and

$$H_{TP}(p, N) = p\varphi_N(\epsilon) = p\{(1 - \epsilon) + \epsilon^{-1}(x + 1)^{-1}t[\chi_{N-1}(\epsilon) + 1 - \chi_N(\epsilon)]\}, \quad (52b)$$

where

$$t = \frac{(1 - \epsilon)}{(1 + \epsilon)}, \quad (52c)$$

$$b = 2(N + \tfrac{1}{2})\epsilon(1 - \epsilon)^{-2}, \quad (52d)$$

and

$$x = \frac{(1 + \gamma)}{(1 - \gamma)}, \qquad \gamma = \epsilon^2, \quad (52e)$$

are quite interesting. (χ_N and φ_N are given in Appendix D.) For $N\epsilon \ll 1$, where we expect incomplete transfer to play a very minor role in the compounding of noise, indeed we find that H_{SP} and H_{TP} are nearly equal to $pN = n$, the total number of transfers experienced by each packet in the device. This is just what one expects: The cumulative, mean-square noise charge after n independent transfers is just n times the mean-square noise charge following a single transfer. [The factor of two in the TP term of (51a) arises because, as explained in Section VII, for each $+q$ noise contribution there is a $-q$ contribution. Thus for each transfer, two noise terms are produced.] As N increases (ϵ is fixed), however, incomplete transfer plays a more significant role, altering the noise in two important ways. First, of course, the noise is incompletely transferred along with the signal. For $N\epsilon \gg 1$, H_{SP} increases only as $n^{\frac{1}{2}}$, reflecting this attenuation of the noise. Second, and even more important, for TP noise, incomplete transfer enables each $+q$, $-q$ pair created during each transfer of each packet to mix and, hence, null out or suppress the total noise. Thus, for $N\epsilon \gg 1$, H_{TP} approaches constant value, $(2\alpha)^{-1}$, independent of N.

In other words, for $N\epsilon = n\alpha \gg 1$, the collecting saturates and further collecting is totally suppressed (see Fig. 6).[7] Of course, the signal is greatly distorted by the incomplete transfer if $N\epsilon \gg 1$. Nonetheless, a CTD has maximum storage capacity[13] for $1 < N\epsilon < 4$, and by using dynamic detection or optimum linear filtering (transversal filtering), one can greatly suppress signal distortion from incomplete transfer.[13] Thus, calculating the compounding factors, H_{SP} and H_{TP}, for $N\epsilon$ other than $N\epsilon \ll 1$ is not an academic exercise.

For analog purposes it is necessary to calculate the autocorrelation function $\langle \Delta Q_s^N \Delta Q_t^N \rangle$ using (49) and (50), from which the current (or voltage) spectral density of the noise can be obtained as in Section VII. We shall pursue this no further than to point out that since $q_s = -q$ must be transferred one more time than q, the effect of incomplete transfer on each q, q_s pair will be slightly different, and this will reduce their mutual correlation at the output. Thus, total suppression at zero frequency is no longer expected.

Incomplete charge transfer will also affect fluctuations at the output caused by modulation noise. In Section VIII we found that modulation noise was so highly correlated that, in the absence of incomplete transfer, the largest portion of modulation noise was not compounded. Introducing incomplete transfer, however, will destroy

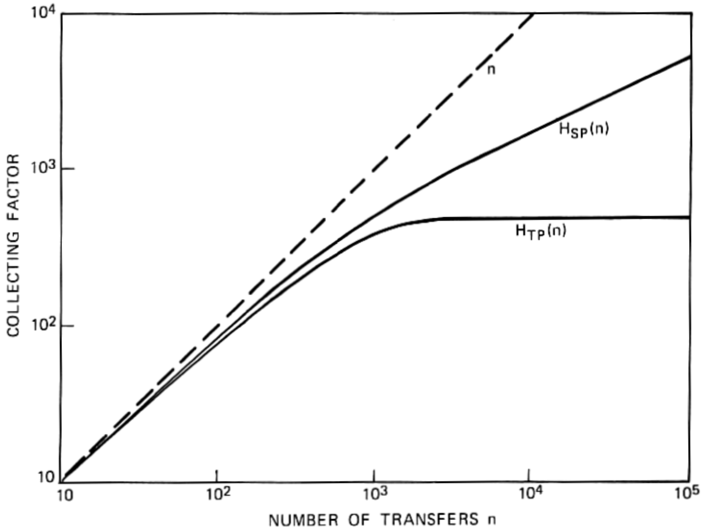

Fig. 6—Suppressed, collecting factors $H_{SP}(n)$ and $H_{TP}(n)$ plotted versus n, the number of transfers, for $\alpha = 10^{-3}$.

the exact correlation which led to this cancellation. Again, if desired, the effect of the collecting of incomplete charge transfer on the modulation noise for digital or analog applications can be calculated using (49) and (50).

Noise present in the input signal can clearly be treated simply as part of the signal with respect to incomplete-transfer effects. A somewhat different situation arises, however, when the signal enters the CTD all along its extent, as is the case in imaging applications. Here full (photon) shot noise results in a signal-dependent contribution to the mean-square, input noise of each packet equal to $eQ^o_{s,m}$, where $Q^o_{s,m}$ is the average size of the packet injected at site m during cycle s. However, in this case, the input packets undergo different numbers of transfers, $p(N - m)$, between the cell in which they are injected and the CTD output. Thus, the influence of incomplete transfer affects the signal and noise originating at each different cell differently. As with the other compounding effects, the collecting of this shot noise in the presence of incomplete transfer can be calculated if desired using (49) and (50).

X. CALCULATIONS

In the foregoing we have discussed a number of sources of noise in CTD's and their effect on the output signal in the absence and presence of incomplete charge transfer. In most cases, inserting the appropriate physical parameters for the noiseless operation of the device of interest suffices to calculate $\langle q^2 \rangle$, the mean-square fluctuation in size of a given charge packet acquired on a given transfer. Then, using (51), the influence of this fluctuation on the output signal can be determined. Such calculations are, in general, difficult, owing to the necessity of evaluating integrals such as those in (27b). However, realistic approximations can be made as indicated to obtain useful results.

It is of interest, however, to determine the minimum amount of noise expected to be present in CTD's assuming one can minimize clock-voltage fluctuations, surface states, incomplete transfer, etc., and operate each device so as to avoid emission-limited currents, etc. In this ideal situation, one is left only with thermal noise, or with thermal noise plus shot noise on the input signal, and incomplete transfer (intrinsic and modulation in the sense of Refs. 10 and 11). The signal-to-noise ratio (S/N) was calculated for two characteristic coefficients of incomplete transfer ($=10^{-3}$, 10^{-4}) and four characteristic capacitances (C = 1, 0.1, 0.01, 0.001 pF) as a function of the number of transfers n from input to output:

(*i*) Using eq. (14a) for the thermal noise acquired per transfer.
(*ii*) Using eq. (51a) for the influence of this noise on the output.
(*iii*) Taking the thermal noise to be purely TP, including shot noise at the input when present.
(*iv*) Including incomplete transfer effects on both the signal and the noise.

The results are plotted in Figs. 7 and 8. The ratio Q_o/C designates the maximum signal level (10 volts), and one-half this amount (5 volts) is the minimum signal level. Since the square of the signal charge is proportional to C^2, while the mean-square of the noise charge is proportional to C for both thermal and input shot noise, the S/N decreases proportionately with smaller C (small CTD cells). As the number of charge transfers is increased, the contribution of device noise to the total noise soon dominates that of the input noise. (From S/N one can also calculate the maximum information storage capacity of the CTD as a function of n, α, and C.) In general, other noise sources are present which reduce S/N from the ideal results shown here. While valid in general for BB-mode transfers, in the case of

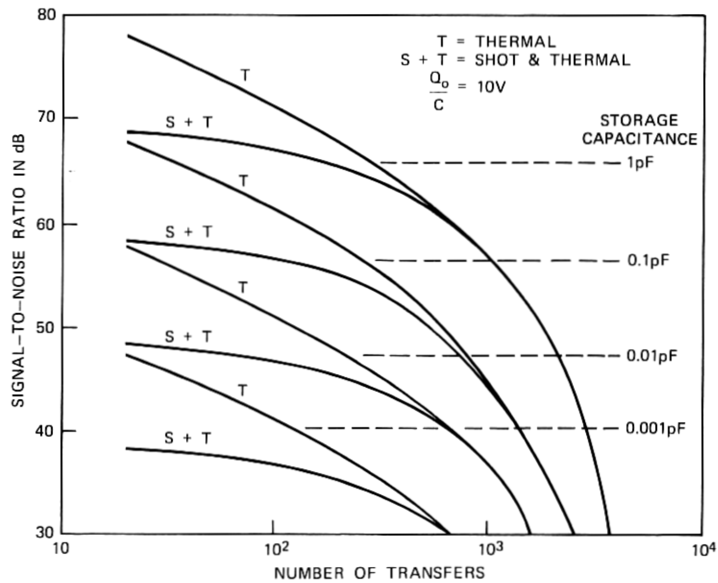

Fig. 7—Signal-to-noise ratio for CTD ($\alpha = 10^{-3}$) with storage capacitance of 1 pF, 0.1 pF, 0.01 pF, 0.001 pF.

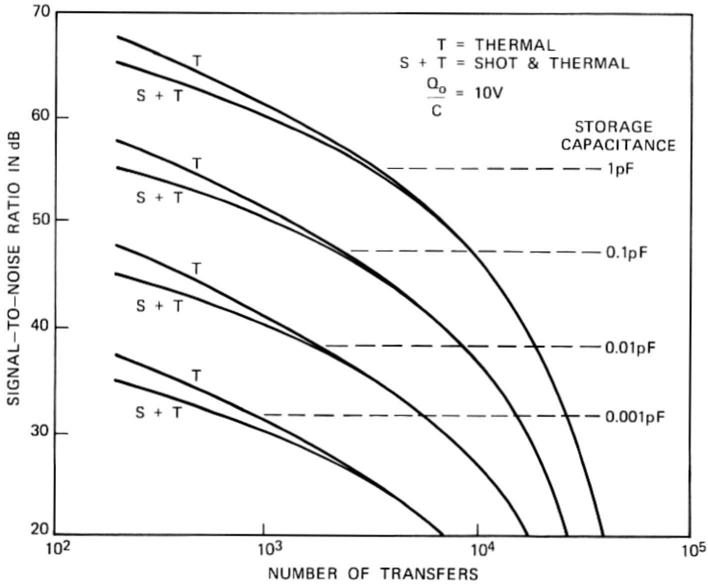

Fig. 8—Signal-to-noise ratio for CTD ($\alpha = 10^{-4}$) with storage capacitance of 1 pF, 0.1 pF, 0.01 pF, 0.001 pF.

CCD-mode transfers, if the rms thermal noise becomes comparable to the free charge incompletely transferred on a single transfer, then the noise predicted by our linear approach will overestimate the true noise.

XI. CONCLUSIONS

In the foregoing we have calculated the influence of several important sources of noise on the output of a CTD in a manner which includes all important relaxation, suppression, and correlation effects. In so doing, we encountered interesting effects, such as nonstationary noise, and we uncovered a number of unexpected results, such as the nearly total suppression of the spectral density of transfer-process noise at low frequencies and the total suppression of the collecting of transfer-process noise after many transfers of a charge packet. Because of the novelty of these and other effects, they were treated in considerable detail in a manner which did not presuppose considerable prior experience with formal treatments of device noise. In noting the results, the proportionate decrease of the S/N with decreasing storage capacitance was illustrated. This unavoidable feature will ultimately limit the practical size of CTD's.

Several of our results, despite the relatively complicated manner in which they were obtained, are remarkably simple, e.g., thermal noise, $\langle q^2 \rangle \propto kTC$, surface-state noise, $\langle q^2 \rangle \propto ekTN_{ss}A$, etc. One wonders whether there exists a general approach which circumvents the necessity of paying such careful attention to the detailed processes that accompany the transfer of charge through a CTD. We feel that there does not exist such an approach. Nonetheless, in Appendix E we outline a very rough method which ignores nearly all details of the charge transfer event. We find the intuitively appealing result that the mean-square noise acquired per transfer is to within a factor of the order of unity equal to full shot noise on the (total and not differential) incompletely transferred portion of the charge. Although providing a rough rule of thumb, since the approach is not totally reliable for calculating incomplete transfer, its accuracy for treating noise is not guaranteed. By contrast, the methods used in the bulk of this paper should be applicable in many types of integrated-circuit, dynamic devices of which CTD's are the first examples.

There are several interesting noise problems that we did not consider here. For example, in discussing modulation noise we indicated the possibility of nonuniformity in the physical parameters of each cell coupling to clock-voltage fluctuations to produce a collecting source of noise. Such cell nonuniformities also, of course, will result in a distribution of α's, the coefficients of incomplete charge transfer. This will, in turn, result in an additional effect on the nature of the compounding of the noise acquired in each transfer. The results are expected to be no less surprising than the effect such a distribution of α's has on the signal. A distribution of α's about their mean α_o leads to *less* signal distortion than if all the α's were a_o.[15,26] (This is actually not too significant for application purposes, since usually all deviations from the desired $\alpha = \alpha_d$ will be to larger α, thereby increasing α_o and enhancing distortion.)

A second problem worthy of attention is how to treat the noise in cases where the clock-voltage waveform does not turn off the flow of transferring charge abruptly.[9] In such cases, the nonlinear terms in the noise fluctuations are not small relative to the linear terms. The noise problem is then nonlinear and much more complicated.

A third problem, straightforward but tedious, is to calculate the S/N's, error rates, and device storage capacities for devices including regenerators, optimum linear filtering, and/or dynamic detection.[13,14] Considering the many new features that have arisen in the present study, as well as the many types of noise that enter in such different ways, we feel such problems will not prove unrewarding.

XII. ACKNOWLEDGMENTS

It is a pleasure to thank G. E. Smith for initially interesting me in this problem and C. N. Berglund, H. J. Boll, J. R. Brews, J. E. Iwersen, D. Kahng, R. H. Krambeck, W. Shockley, G. E. Smith, R. J. Strain, N. S. Thornber, and M. F. Tompsett for many helpful discussions and several crucial comments.

APPENDIX A

Noise Introduced During a Single, Two-Step, Transfer Cycle

The purpose of this appendix is to calculate the noise fluctuations induced in the transferred charge during a single, two-step, transfer cycle. The procedure followed is the same as that in Section IV of the text for a single-step process. Here we shall merely outline the essential steps leading to the desired result.

We begin by linearizing the dynamic equations (4a, b, c) to obtain

$$\dot{q}_1 = -\left(\frac{\partial I_1}{\partial V_s} v_s + \frac{\partial I_1}{\partial V_b} v_b + \frac{\partial I_1}{\partial V_c} v_c + \frac{\partial I_1}{\partial V_{ss1}} v_{ss1}\right) - i_{n1}, \qquad (53a)$$

$$\dot{q}_2 = -\dot{q}_d - \dot{q}_1, \qquad (53b)$$

$$\dot{q}_d = \left(\frac{\partial I_2}{\partial V_b} v_b + \frac{\partial I_2}{\partial V_d} v_d + \frac{\partial I_2}{\partial V_c} v_c + \frac{\partial I_2}{\partial V_{ss2}} v_{ss2}\right) + i_{n2}. \qquad (53c)$$

The constraint equations (3a–e), when linearized and when terms of the order of the incomplete transfer are dropped, yield the following relations among the fluctuating quantities:

$$q_s = C_s v_s - C_s^o v_{so}, \qquad (54a)$$

$$q_b = C_b v_b - C_b^o v_{bo}, \qquad (54b)$$

$$q_d = C_d v_d - C_d^o v_{do}, \qquad (54c)$$

$$q_{ss1} = C_{ss1} v_{ss1} - C_{ss1}^o v_{ss1}^o, \qquad (54d)$$

$$q_{ss2} = C_{ss2} v_{ss2} - C_{ss2}^o v_{ss2}^o. \qquad (54e)$$

In addition, by the same reasoning that led to eqs. (7a, b) in the text, we find that

$$v_{so} = \frac{\partial V_{so}}{\partial V_c} v_c = \frac{\partial V_s}{\partial V_c} \frac{C_s}{C_s^o} v_c = (+1) \frac{C_s}{C_s^o} v_c, \tag{55a}$$

$$v_{bo} = \frac{\partial V_{bo}}{\partial V_c} v_c' = \frac{\partial V_b}{\partial V_c} \frac{C_b}{C_b^o} v_c' = a \frac{C_b}{C_b^o} v_c', \tag{55b}$$

$$v_{do} = \frac{\partial V_{do}}{\partial V_c} v_c = \frac{\partial V_d}{\partial V_c} \frac{C_d}{C_d^o} v_c = (-1) \frac{C_d}{C_d^o} v_c, \tag{55c}$$

where $a = \partial V_b/\partial V_c$. [For a two-step process, transfer cells whose second step is characterized by a CCD-type of charge transfer, $a = -1$ and $v_c' = v_c$. For such cells whose second step is BB-like, the value of a and v_c' will depend on the details of the device. For example, for the C4D, $a = 0$; for the tetrode BB, $a = 1$ and $v_c' = v_{bl}$ is the voltage on the tetrode bias line. Thus, for purposes of generality we must use the form given in (55b).] Also v_{ss1}^o and v_{ss2}^o can be set equal to zero, as discussed in the text.

The next step is to solve eqs. (54a–e) for v_s, v_b, etc. and insert the results into (53a–c). The result of this substitution is

$$\dot{q}_1 = -\frac{g_{m1}}{C_s} q_1 - \left(g_{m1} + \frac{\partial I_1}{\partial V_c}\right) v_c + \left(g_{m1} \frac{C_{ss1}}{C_s} - \frac{\partial I_1}{\partial V_{ss1}}\right) v_{ss1} - i_{n1}, \tag{56a}$$

$$q_2 = -q_d - q_1, \tag{56b}$$

$$\dot{q}_d = \frac{g_{m2}}{C_b} q_2 + g_{m2} a v_c' + \frac{\partial I_2}{\partial v_c} v_c + \left(g_{m2} \frac{C_{ss2}}{C_b} - \frac{\partial I_2}{\partial V_{ss2}}\right) v_{ss2} + i_{n2}, \tag{56c}$$

in which we have assumed that the forward conductances

$$g_{m1} \equiv \partial I_1/\partial V_s \quad \text{and} \quad g_{m2} \equiv \partial I_2/\partial V_b$$

greatly exceed the reverse conductances

$$g_{r1} \equiv -\partial I_1/\partial V_b \quad \text{and} \quad g_{r2} \equiv -\partial I_2/\partial V_d.$$

It is now straightforward to solve (56a) for $q_1(t)$, insert into (56b) to obtain $q_2(t)$, and then solve (56c) for $q_d(t)$. The result is

$$q_d(t) = \int_0^t dt' \exp\left[-\int_{t'}^t dt''/\tau_2(t'')\right]\left[\frac{g_{m2}}{C_b}(-q_1) + \left(g_{m2} a + \frac{\partial I_2}{\partial V_c}\right) v_c + g_{m2} a (v_c' - v_c) - \left(g_{m2} \frac{C_{ss2}}{C_b} - \frac{\partial I_2}{\partial V_{ss2}}\right) v_{ss2} + i_{n2}\right]\Bigg|_{t'}, \tag{57a}$$

where

$$-q_1(t') = \int_0^{t'} dt'' \exp\left\{-\int_{t''}^{t'} dt'''/[\tau_1(t''')]\right\}\left[\left(g_{m1} + \frac{\partial I_1}{\partial V_c}\right)v_c - \left(g_{m1}\frac{C_{ss1}}{C_s} - \frac{\partial I_1}{\partial V_{ss1}}\right)v_{ss1} + i_{n1}\right]\Bigg|_{t''}, \quad (57b)$$

$$1/\tau_1 = g_{m1}/C_s \quad \text{and} \quad 1/\tau_2 = g_{m2}/C_b. \quad (57c)$$

If we define the two suppression factors according to

$$S_1(t, t') = \exp\left(-\int_{t'}^{t} dt''/\tau_1(t'')\right) \quad (58a)$$

and

$$S_2(t, t') = \exp\left(-\int_{t'}^{t} dt''/\tau_2(t'')\right), \quad (58b)$$

let

$$n(t, t') = \int_{t'}^{t} dt'' \exp\left(-\int_{t''}^{t} [1/\tau_2(t''') - 1/\tau_1(t''')]dt'''\right)\Big/ \tau_2(t''), \quad (58c)$$

insert (57b) into (57a), and regroup terms, we obtain

$$\begin{aligned} q_d(t) = \int_0^t dt' \Bigg\{ & S_1(t, t')n(t, t')i_{n1}(t') + S_2(t, t')i_{n2}(t') \\ & - S_1(t, t')n(t, t')\left(\frac{C_{ss1}}{\tau_1} - \frac{\partial I_1}{\partial V_{ss1}}\right)\Bigg|_{t'} v_{ss1}(t') \\ & - S_2(t, t')\left(\frac{C_{ss2}}{\tau_2} - \frac{\partial I_2}{\partial V_{ss2}}\right)\Bigg|_{t'} v_{ss2}(t') \\ & + \left[S_1(t, t')n(t, t')\left(g_{m1} + \frac{\partial I_1}{\partial V_c}\right)\Bigg|_{t'} \right. \\ & \left. + S_2(t, t')\left(g_{m2}a + \frac{\partial I_2}{\partial V_c}\right)\Bigg|_{t'}\right] v_c(t') \\ & + S_2(t, t')[g_{m2}a(v_c' - v_c)]|_{t'} \Bigg\}. \end{aligned} \quad (59)$$

Although it is most convenient to keep the $S_1 n$ product as two terms, their combination is of interest:

$$S_1(t, t')n(t, t') = \int_{t'}^{t} dt'' \exp\left(-\int_{t'}^{t''} dt'''/\tau_1(t''') - \int_{t''}^{t} dt'''/\tau_2(t''')\right)\Big/ \tau_2(t''). \quad (60)$$

It is clear that to calculate $\langle q_d^2 \rangle$, it is necessary to know $\langle i_{n1}(t_1)i_{n1}(t_2)\rangle$, $\langle i_{n2}(t_1)i_{n2}(t_2)\rangle$, $\langle v_{ss1}(t_1)v_{ss1}(t_2)\rangle$, $\langle v_{ss2}(t_1)v_{ss2}(t_2)\rangle$, $\langle v_c(t_1)v_c(t_2)\rangle$, $\langle v_c'(t_1)v_c'(t_2)\rangle$, and $\langle v_c'(t_1)v_c(t_2)\rangle$. These are discussed in the text.

It is of interest to calculate the contribution to $\langle q_d^2 \rangle$ that results from thermal noise. From (59) and (13b) ($H_n = \frac{2}{3}$), we obtain

$$\langle q_d^2 \rangle |_{kT} = 2kT \frac{2}{3} \int_0^{t_f} dt [S_1^2(t_f, t)n^2(t_f, t)g_{m1}(t) + S_2^2(t_f, t)g_{m2}(t)]. \quad (61)$$

Now ordinarily a two-step device is constructed so that

$$g_{m1}/C_s = \tau_1^{-1} < \tau_2^{-1} = g_{m2}/C_b.$$

Let us assume that the decay current is such that $\tau_1^{-1}(t)b = \tau_2^{-1}(t)$, where b can be taken independent of time ($b > 1$). Then we can perform the integrals in (61), obtaining

$$\langle q_d^2 \rangle |_{kT} = \tfrac{2}{3}kTC_b + \tfrac{2}{3}kTC_s(1 + b^{-1})^{-1} \quad (62)$$

for the thermal contribution to the noise acquired during a single, two-step transfer. The first term is expected from our result (14a) for an individual, single-step transfer. However, the second term includes a suppression factor whose presence one certainly would not have expected *a priori* using arguments assuming stationarity.

APPENDIX B

Outline and Modification of the Impedance-Field Method

As originally presented,[23] the impedance-field method (IFM) consists of dividing the problem of calculating device noise into two simpler problems. First, it was recognized that a given fluctuation in the velocity of a charge carrier at a given location induces a calculable fluctuation in the open-circuit voltage. Second, it was shown how, from a knowledge of the spectral density of the velocity fluctuations of the individual charge carriers, the spectral density of the open-circuit voltage fluctuations could be calculated. The first problem requires only an understanding of the operation of the device of interest under noiseless conditions, while the second can be deduced from the microscopic behavior of charge carriers in a small region of the device.[18] Thus, as long as the microscopic noise is simple, which it nearly always is, the device-noise problem is reduced to integrating the microscopic noise sources over the device, weighted by the influence of a unit fluctuation in each volume element on the output noise.

Let us now outline the IFM in more detail.[23] If $\mathbf{u}_t$ is the time-dependent fluctuation of carrier velocity $\mathbf{v}_t$ from its mean $\mathbf{v}_{ot}$,

$$\mathbf{u}_t \equiv \mathbf{v}_t - \mathbf{v}_{ot},$$

then the dipole current $\delta \dot{\mathbf{P}}_\alpha$ in a small volume element ΔV_α produced by the carriers in this volume is given by

$$\delta \dot{\mathbf{P}}_\alpha = q \sum_j \mathbf{u}_{jt}, \tag{63}$$

where the j-sum is over all carriers in ΔV_α. To calculate the effect of $\delta \dot{\mathbf{P}}_\alpha$ on the open-circuit voltage of the device, we note that if δI_α is injected into the device at $\mathbf{r}_\alpha$ (and if the device is grounded at $\mathbf{r} = \mathbf{0}$), the voltage induced at the contact labeled N, δV_N, is given by

$$\delta V_N = Z_{N\alpha} \delta I_\alpha. \tag{64}$$

If $Z_{N\alpha}$ is evaluated for all α, one maps out the "impedance field." If now one injects δI_α at $\mathbf{r}_\alpha + \delta \mathbf{r}$ and removes δI_α at $\mathbf{r}_\alpha$, linear superposition of small signals implies that

$$\delta V_N = [Z_{Na}(\mathbf{r}_a + \delta \mathbf{r}) - Z_{Na}(\mathbf{r}_a)]\delta I_\alpha \tag{65a}$$

$$= (\nabla Z_{N\mathbf{r}}) . \delta \mathbf{r} \delta I_\alpha = \nabla Z_{N\mathbf{r}} . \delta \dot{\mathbf{P}}_a. \tag{65b}$$

The last equality follows because the dipole current $\delta \dot{\mathbf{P}}_\alpha$ equals $\delta I_a \delta \mathbf{r}$ if δI_a is chosen appropriately. Since we can relate $\delta \dot{\mathbf{P}}_\alpha$ to δV_N, from a knowledge of the statistics of the former, we can calculate those of the latter. In particular, from (63) and (65b) we can immediately write down the spectral density $S_{V_N}(f)$ of δV_N in terms of that of $\mathbf{u}_\alpha$:

$$S_{V_N}(f) = \sum_{\alpha,i,j} (\nabla_i Z_{N\alpha})(\nabla_j Z_{N\alpha})^* q^2 n_\alpha S_{\alpha u_i u_j}(f), \tag{66}$$

where i, j each run over x, y, z, and n_α is the number of carriers in ΔV_α. Since in the text we work in the time domain, we work directly with (65).

In calculating CTD noise, it is most convenient to work with short-circuit current fluctuations rather than with open-circuit voltage fluctuations. One may redo the above, calculating the short-circuit current fluctuation δI_N induced by δI_α. The result is that

$$\delta I_N = B_{N\alpha} \delta I_\alpha, \tag{67a}$$

where

$$B_{N\alpha} = Z_{N\alpha}/Z_N, \tag{67b}$$

and where Z_N is the impedance of the device between contact N and ground. In terms of dipole currents, one obtains

$$\delta I_N = \nabla B_{N\mathbf{r}} \cdot \delta \mathbf{r} \delta I_\alpha = \nabla B_{N\mathbf{r}} \cdot \delta \dot{\mathbf{P}}_\alpha. \tag{68}$$

For an IGFET,

$$\nabla B_{N\mathbf{r}} = L^{-1}\hat{l}, \tag{69}$$

where L is the channel length and $\hat{l}$ is a unit vector along the channel.[22] In complete analogy to (66) one finds that

$$S_{I_N}(f) = \sum_{\alpha,i,j} (\nabla_i B_{N\alpha})(\nabla_j B_{N\alpha})^* q^2 n_\alpha S_{\alpha u_i u_j}(f). \tag{70}$$

In arriving at (66) and (70), we have taken into account the independence between spontaneous fluctuations which occur in separate regions ΔV_α. Thus, the expression for $\langle u_i(t_1) u_j(t_2) \rangle$ for the IFM, which corresponds to (12) in the text, is that for thermal noise

$$\langle u_i(t_1) u_j(t_2) \rangle = \frac{2kT}{q} \mu \delta(t_1 - t_2) \delta_{ij}, \tag{71}$$

which lacks the spatial delta function. The equivalence between the Langevin method used in the text and the IFM outlined there is discussed in some detail in Refs. 27 and 28.

APPENDIX C

*Exact Calculation of H_{SP} and H_{TP}**

In this appendix, we shall evaluate the following sums exactly:

$$\chi_N(\epsilon) = \sum_{m=1}^{N} \sum_{r=0}^{\infty} \left[(1-\epsilon)^{N-m} \epsilon^r \binom{r+N-m}{r} \right]^2$$

$$\varphi_N(\epsilon) = \sum_{m=1}^{N} \sum_{r=0}^{\infty} \left[(1-\epsilon)^{N-m} \epsilon^r \binom{r+N-m}{r} \right]^2 \cdot \left(1 - \epsilon \frac{r+1+N-m}{r+1} \right). \tag{72}$$

As the first step, set $n = N - m$, $\gamma = \epsilon^2$, and perform the sum on r. This yields

$$\chi_N(\epsilon) = \sum_{n=0}^{N-1} \frac{(1-\epsilon)^{2n}}{n!} \frac{d^n}{d\gamma^n} \left(\frac{\gamma^n}{(1-\gamma)^{n+1}} \right) \tag{73a}$$

$$\varphi_N(\epsilon) = \frac{1-\epsilon}{1-\gamma} + \sum_{n=1}^{N-1} \frac{(1-\epsilon)^{2n}}{n!} \frac{d^n}{d\gamma^n} \left(\frac{\gamma^n}{(1-\gamma)^{n+1}} \right) - \epsilon \frac{d^n}{d\gamma^n} \left(\frac{\gamma^{n-1}}{(1-\gamma)^{n+1}} \right). \tag{73b}$$

* Derivation due to N. S. Thornber.

The second step involves taking the n derivatives with respect to γ using the relation[29]

$$\frac{d^n}{dy^n}(fg) = \sum_{s=0}^{n} \binom{n}{s} \frac{d^s f}{dy^s} \frac{d^{n-s} g}{dy^{n-s}}$$

and recognizing that the resulting sum on s is to within certain simple factors equal to a Legendre polynomial,[29] $P_n(x)$, where $x = (1 + \gamma)/(1 - \gamma)$. This yields

$$\chi_N(\epsilon) = (1 - \gamma)^{-1} \sum_{n=0}^{N-1} t^n P_n(x) \tag{74a}$$

$$\varphi_N(\epsilon) = (1 - \gamma)^{-1} \left[(1 - \epsilon) + \sum_{n=1}^{N-1} t^n P_n(x) - \epsilon t^n \frac{x - 1}{n\gamma} P_n'(x) \right], \tag{74b}$$

where $t = (1 - \epsilon)/(1 + \epsilon)$. In (74b), if we note that[30]

$$P_n'(x)/n = [xP_n(x) - P_{n-1}(x)]/(x^2 - 1),$$

then all that remains is to evaluate $\sum_n t^n P_n(x)$.

Before proceeding, we should call attention to a potential source of trouble. Since $0 < \gamma = \epsilon^2 < 1$, it follows that $x > 1$ and $t < 1$. While Legendre polynomials for $|x| > 1$ are well-defined, their properties are not nearly so simple as they are for x in the usual region of interest, $|x| \leqq 1$. Thus, for fixed ϵ (and γ), if n becomes large, evaluating $P_n(x)$ for x only slightly larger than 1 is quite tricky. This provides motivation for the approach adopted in Appendix D.

Returning to the remaining sum in (74), we take the generating function[31] for Legendre polynomials, valid for $|t| < 1$ and $|y| < 1$,

$$\sum_{S=0}^{\infty} t^s P_s(y) = (1 - 2yt + t^2)^{-\frac{1}{2}},$$

multiply both sides by $P_m(y)$, and integrate on y from -1 to $+1$. Using the orthogonality of these polynomials, we obtain

$$t^m = \frac{2m + 1}{2} \int_{-1}^{1} dy P_m(y)(1 - 2yt + t^2)^{-\frac{1}{2}}. \tag{75}$$

Using the summation formula[32] for Legendre's polynomials, it follows that

$$\sum_{n=0}^{N-1} t^n P_n(x) = \int_{-1}^{1} dy (1 - 2yt + t^2)^{-\frac{1}{2}} \frac{N/2}{x - y} \cdot [P_N(x)P_{N-1}(y) - P_N(y)P_{N-1}(x)]. \tag{76}$$

To perform the final integral over y we make use of (75), noting that

$2tx = 1 + t^2$, and hence that

$$(1 - 2yt + t^2)^{\frac{1}{2}}(x - y) = 2^{\frac{1}{2}}(x - y)^{\frac{3}{2}}t^{\frac{1}{2}}.$$

Thus, since

$$\int_{-1}^{1} dy(x - y)^{-\frac{3}{2}}P_n(y) = -2\,\frac{d}{dx}\int_{-1}^{1} dy(x - y)^{-\frac{1}{2}}P_n(y)$$

and since from (75)

$$2t^m(2m + 1)^{-1} = \int_{-1}^{1} dy[2t(x - y)]^{-\frac{1}{2}}P_m(y),$$

it follows after some algebra that

$$\sum_{n=0}^{N-1} t^nP_n(x) = \frac{2N}{1 - t^2}\,[P_N(x)t^N - P_{N-1}(x)t^{N+1}]. \tag{77}$$

Using (77) for the sums present in (55a, b) yields exact expressions for χ_N and φ_N in terms of two or three Legendre polynomials, respectively. The difficulties encountered in evaluating these expressions for $N \gg 1$ and $x \gtrsim 1$ made it clear that another form of the result was needed, one in which N and ϵ enter on an equal footing, preferably as a product. Such a result is derived in Appendix D.

APPENDIX D

Approximate Calculation of H_{SP} and H_{TP}

In this appendix, we evaluate the sum

$$\psi_{N-1}(\epsilon) = \sum_{n=0}^{N-1} t^nP_n(x), \tag{78}$$

where, as in Appendix C, $t = (1 - \epsilon)/(1 + \epsilon)$, $x = (1 + \gamma)/(1 - \gamma)$, and $\gamma = \epsilon^2$. The form we obtain will be a good approximation for $N \gg 1$ and will be very easy to evaluate numerically.

If we define $\psi_{-1} = 0$, then we can write

$$\psi_n - \psi_{n-1} = t^nP_n(x) \tag{79}$$

and evaluate the z-transform ψ_z of ψ_n defined by

$$\psi_z = \sum_{n=0}^{\infty} \psi_n z^{-n}. \tag{80}$$

Thus, from (79) we obtain

$$\psi_z - \frac{1}{z}\psi_z = \sum_{n=0}^{\infty}\left(\frac{t}{z}\right)^n P_n(x) = \left[1 - 2x\frac{t}{z} + \left(\frac{t}{z}\right)^2\right]^{-\frac{1}{2}}. \tag{81}$$

Solving for ψ_z, we obtain, using $2xt = 1 + t^2$,

$$\psi_z = (1 - z^{-1})^{-\frac{1}{2}}(1 - t^2 z^{-1})^{-\frac{1}{2}}. \tag{82}$$

Thus,

$$\psi_n = \frac{1}{2\pi i} \oint \psi_z z^{n-1} dz, \tag{83a}$$

where the closed contour in the complex plane includes $z = 0$. Letting $z = e^s$, (83a) becomes ($\sigma = 0^+$)

$$\psi_n = \frac{1}{2\pi i} \int_{\sigma - i\pi}^{\sigma + i\pi} ds\, e^{ns}(1 - e^{-s})^{-\frac{1}{2}}(1 - t^2 e^{-s})^{-\frac{1}{2}} \tag{83b}$$

an exact expression for $\psi_N(\epsilon)$ if $n = N$.

An exact integration of (83b) would recover (77). However, a very useful approximation for large n can be obtained at once if we note that under such circumstances most of the contribution to the integral comes in the vicinity of $s = 0$. Thus, expanding $\exp(-s) \approx 1 - s$ and then, taking the limits of integration to be from $(\sigma - i\infty)$ to $(\sigma + i\infty)$, yields

$$\psi_n \approx \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds\, e^{ns} s^{-\frac{1}{2}} t^{-1}[s + (t^{-2} - 1)]^{-\frac{1}{2}}. \tag{84a}$$

This inverse Laplace transform can be found in the tables.[33] The result is

$$\psi_n \approx t^{-1} n e^{-b}[I_o(b) + I_1(b)], \tag{84b}$$

where

$$b = n(t^{-2} - 1)/2 = 2n\epsilon(1 - \epsilon)^{-2}. \tag{84c}$$

We can match this result, valid for $n \gg 1$, to the $n \ll 1$ limit to obtain an approximate result good for all n. At this point, however, it must be stressed that the major difficulty in approximating ψ_n for large n arises from the necessity to cancel exactly any exponential dependence of ψ_n on n. Clearly, physically compounding can increase with n no more rapidly than linearly. This delicate cancellation is in fact achieved in (84b) since $I_0(b)$ and $I_1(b)$ go as $\exp(b)/(2\pi b)^{-\frac{1}{2}}$ for $b \gg 1$. There are other ways of arriving at (84), but this is one of the simplest.

To determine the form of ψ_n for n sufficiently small ($n\epsilon \ll 1$), that is, such that an expansion of ψ_n in powers of $n\epsilon$ rapidly converges in a few terms we note from (71), (73a), and (78) that we have the relation

$$\psi_{N-1} = (1 - \gamma) \sum_{n=0}^{N-1} \sum_{r=0}^{\infty} \left[(1 - \epsilon)^n \epsilon^r \binom{r + n}{r} \right]^2 \tag{85a}$$

$$= N[1 - \epsilon(N - 1) + 0(N^2\epsilon^2)]. \tag{85b}$$

On the other hand, expanding (84b) to similar order yields

$$\psi_n \approx n(1 + 2\epsilon - \epsilon n). \tag{86}$$

Since (84b) and consequently (86) can be expected to be valid only for $n\epsilon \gg 1$, we are not required to set $n = N - 1$ in (84b). [Nonetheless, it is encouraging to note that even for $n\epsilon \ll 1$, if we do set $n = N - 1$ in (86), then (85b) and (86) agree as to the coefficients of N and $N^2\epsilon$, the two largest terms.] Even further improvement can be obtained if we replace ψ_n by $\psi_n + \delta_1$ and n by $N - 1 + \delta_2$, where δ_1 and δ_2 are of the order of unity. This alteration clearly does not alter significantly our result valid for large n ($n\epsilon \gg 1$), and it improves the result for $n\epsilon \ll 1$. After a little algebra, one obtains $\delta_1 = -\frac{1}{2}$ and $\delta_2 = \frac{3}{2}$, which ensure that terms of order unity and $N\epsilon$ match. We conclude that $\chi_N(\epsilon)$ is given quite well by

$$\chi_N(\epsilon) = (N + \tfrac{1}{2})t^{-1}e^{-b}[I_0(b) + I_1(b)] - \tfrac{1}{2}, \tag{87a}$$

where

$$b = 2(N + \tfrac{1}{2})\epsilon(1 - \epsilon)^2 \quad \text{and} \quad t = (1 - \epsilon)/(1 + \epsilon), \tag{87b}$$

in which we have ignored the $(1 - \gamma)^{-1}$ factor in (73a). (Recall that $\gamma = \epsilon^2 \approx 10^{-6}$.) The expression (87a) enables us to calculate the compounding factor for storage-process noise for arbitrary N and ϵ with relative ease.

The compounding factor for transfer-process noise is obtained from $\varphi_N(\epsilon)$, which can be expressed in terms of $\chi_N(\epsilon)$ in the form

$$\begin{aligned}\varphi_N(\epsilon) &= (1 - \epsilon) + [\chi_N(\epsilon) - 1] \\ &\qquad - \epsilon\gamma^{-1}(x + 1)^{-1}\{x[\chi_N(\epsilon) - 1] - t\chi_{N-1}(\epsilon)\} \qquad (88a) \\ &= (1 - \epsilon) + \epsilon^{-1}(x + 1)^{-1}t[\chi_{N-1}(\epsilon) + 1 - \chi_N(\epsilon)], \qquad (88b)\end{aligned}$$

where we have used (73) and ignored the prefactor of $(1 - \gamma)^{-1}$. Since $\gamma = \epsilon^2$, the prefactor of the last term of (88) is approximately $(2\epsilon)^{-1}$. Nonetheless, this term is well-behaved even where our approximate result (87) is used in (88) to obtain an approximate $\varphi_N(\epsilon)$.

It is of interest to calculate χ_N and φ_N in the limit of $N\epsilon \gg 1$. The former limit is just N in both cases. In the latter limit

$$\chi_N(\epsilon) \to (N/\pi\epsilon)^{\frac{1}{2}} = p^{-1}(n/\pi\alpha)^{\frac{1}{2}}, \quad N\epsilon \gg 1 \tag{89a}$$

and

$$\varphi_N(\epsilon) \to (2\epsilon)^{-1} = (2p\alpha)^{-1}, \quad N\epsilon \gg 1. \tag{89b}$$

The second limit is most easily obtained by noting that $x(x + 1)^{-1}$

$= (1+\gamma)/2$, $(x+1)^{-1} = (1-\gamma)/2$, $\epsilon\gamma^{-1}[(1+\gamma) - t(1-\gamma)]/2 = 1$, and $(\chi_{N-1} + 1 - \chi_N) = 1 + 0(N^{-\frac{1}{2}})$, $N\epsilon \gg 1$.

APPENDIX E

Another Approach to CTD Noise

Throughout our treatment we have paid careful attention to the details of the transfer and storage of charge in CTD's in order to calculate the noise induced by the device. We have noted, however, that by making reasonable approximations, we can often reduce rather complicated, exact results to much simpler, approximate results adequate for most purposes. One cannot help but wonder, therefore, if there does not exist an approximate but adequate short cut for calculating CTD noise. One clue is to compare Tompsett's results[17] for incompletely transferred charge Q_i with his results for mean-square noise charge induced by interface states $\langle q^2 \rangle$. If we ignore factors of the order unity we find that

$$\langle q^2 \rangle = eQ_i, \tag{90}$$

where

$$Q_i = ekTN_{ss}. \tag{91}$$

In other words, the noise induced by interface states is just the shot noise associated with the incompletely transferred portion of the interface charge.[34] While one will never recover thermal noise from such arguments, one expects not to do too badly in estimating the influence on the output of those noise sources, such as interface states or emission-limited (barrier-limited) currents, which are closely tied to incomplete charge transfer. Anyway, the basic idea, that of shot noise on incompletely transferred charge, is appealing, however approximate and/or incomplete it may in fact actually be.

We can derive such a result as follows. We make the assumption that we can approximate the current I which flows from one storage region to the next during a single transfer as $I(Q)$, where Q is the charge to be transferred at any given time during the transfer phase. (If this assumption had been made in calculating the coefficient of incomplete charge transfer,[10,11] then certain of the results obtained would have been erroneous. Nonetheless, without this assumption, details of the charge transfer enter, which we wish to avoid.) As in the text we also assume that we can linearize the equations governing the noise. Thus, writing $Q = Q^0 + q$, the equation of motion

$$\dot{Q} = -I(Q) \tag{92}$$

becomes

$$\dot{Q}^0 = -I(Q^0) = -I^0 \tag{93a}$$

for the noiseless transfer, and for the noise

$$\dot{q} = -\frac{dI}{dQ^0}q + d_q(t), \tag{93b}$$

where $d_q(t)$ is the naturally arising statistical driving term. Since I depends only on Q, the derivative appearing in (93b) is total and is evaluated using $Q^0(t)$ from (93a). Solving for $q(t)$ is straightforward:

$$q(t) = \int_0^t dt' d_q(t') \exp\left(-\int_{t'}^t (dI/dQ^0)_{t''} dt''\right) \tag{94a}$$

$$= \int_0^t dt' d_q(t') \left(\frac{I^0(t)}{I^0(t')}\right), \tag{94b}$$

in which we have used (93a) in going from (94a) to (94b). The $[I(t)/I(t')]$ factor suppresses the shot-like noise associated with the transfer.

The statistics of q follow from those of d_q using eq. (94b). The statistics of d_q are such that

$$\langle d_q(t_1)d_q(t_2)\rangle = eI[Q^0(t_1)]\delta(t_1 - t_2). \tag{95}$$

(This is not an additional assumption, but rather (95) follows from our initial assumption that I depends only on Q.) It follows that at the end of the transfer cycle

$$\langle q^2\rangle = e\int_{Q_1^0}^{Q_0^0} dQ[I^0(Q_1^0)/I^0(Q)]^2, \tag{96}$$

where Q_0 is the initial charge $Q^0(0)$ to be transferred, and Q_1 is the mean charge $Q^0(t_f)$ left behind at the end of the transfer. [We note that in the absence of the expression factor, $\langle q^2\rangle = e(Q_0 - Q_1) \approx eQ_0$, full shot noise.]

To proceed, we must know $I(Q^0)$. If, toward the end of the transfer cycle, $Q \to Q_1 \gg Q_d$, where Q_d is the charge packet size above which the primary force driving the transfer current arises from the packet itself, then $I(Q)$ will be proportional to Q^2, and, using (97), $\langle q^2\rangle \approx eQ_1/3$. If, on the other hand, toward the end of the transfer cycle, $Q \to Q_1 \ll Q_d$, so that the primary force driving the transfer current is diffusion or fringing fields, the $I(Q)$ will be proportional to Q, and, using (96), $\langle q^2\rangle \approx eQ_1$. Taking a more specific example, let us set $CV = Q$ in eq. (8) of Ref. 11, as a realistic approximation. It follows that

$$\langle q^2 \rangle = e \int_{Q_1}^{Q_0} dQ \frac{Q_1^2 \left(Q_1 + \frac{2kT}{e} C\right)^2}{Q^2 \left(Q + \frac{2kT}{e} C\right)^2}. \tag{97}$$

Equation (97) can be integrated exactly. For our purposes, it suffices to consider two limiting cases: $Q_1 \gg 2kTC/e$, in which case $\langle q^2 \rangle \approx eQ_1/3$, and $Q_1 \ll 2kTC/e$, in which case $\langle q^2 \rangle \approx eQ_1$. Thus, we find in fact that $\langle q^2 \rangle$ can be viewed roughly as the shot noise on the incompletely transferred charge. However, it should be noted that Q_1 is the *total*, and not the much smaller differential (αQ_0), charge incompletely transferred. While the results of this appendix are appealing as a short cut, the reader is strongly advised to keep the basic assumption $[I = I(Q)]$ firmly in mind and to use extreme caution in generalizing this approach to other problems.

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