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Pulse Spreading in Multimode, Planar, Optical Fibers

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A dielectric slab can keep optical beams confined transversely in its plane if it is tapered, with the slab thickness having a maximum along some straight line. When the square of the local wave number of the slab (k^2) is a quadratic function of the transverse coordinate (y), the rays in the plane of the slab are sinusoids whose optical length is almost independent of the amplitude. For thin slabs ($2d \ll \lambda$) as well as for thick slabs ($2d \gg \lambda$), pulse spreading is large because the ratio of the local phase to group velocity is strongly dependent on the distance (y) from axis. We show that pulse spreading is almost negligible, however, if the thickness of the slab is properly chosen. For example, if the slab thickness on axis is 2.5 micrometers and the refractive index of the slab is 1 percent higher than that of the surrounding medium, pulse spreading is only 0.05 nanosecond per kilometer at a wavelength of 1 micrometer. Pulses in clad fibers having the same width (0.2 millimeter) and carrying the same number of modes (15) spread 50 times faster. Splicing and matching to injection lasers may be easier with planar fibers than with conventional fibers. Low-dispersion planar fibers are therefore attractive when used in conjunction with sources that are multimoded in one dimension. Closed-form expressions are given for square-law and linear-law profiles.

I. INTRODUCTION

This introduction gives first a brief review of the general concepts of pulse transmission in multimode waveguides,[1,2] and subsequently considers the case of planar structures that ensure transverse confinement of the optical beams.

The most important parameters of optical fibers for communication are loss (perhaps a few decibels per kilometer) and pulse spreading (perhaps a few tens of nanoseconds per kilometer). Given these two parameters, the maximum repeater spacing and the transmission capacity of the fiber are pretty much determined, considering the limitations that presently exist in source power (L.E.D. or injection lasers) and detector sensitivity (avalanche photodiodes). If the loss is the limiting factor, a reduction in bandwidth allows an increase in repeater spacing because of the increased receiver sensitivity, but only by a modest distance. Inversely, baseband equalization allows the transmission capacity to be increased at the expense of optical power, but not by a very large factor. In this paper, we consider only the problem of pulse spreading.

Consider first a single-mode waveguide; for instance, a rectangular waveguide whose width is less than a wavelength. The wave number β may be a rapidly varying function of ω, particularly near cut-off. The transit time of a pulse of radiation is equal to the ratio $Ld\beta/d\omega$ of the path length L and the group velocity $d\omega/d\beta$. Because a pulse of small duration has a broad frequency spectrum, some components arrive ahead of the others if $d\beta/d\omega$ varies with ω; that is, if $d^2\beta/d\omega^2 \neq 0$. The pulse duration, τ, is of the order of $(Ld^2\beta/d\omega^2)^{\frac{1}{2}}$. If the waveguide is filled with a material having dispersion, the phenomenon remains essentially the same. Single-mode pulse spreading is small at optical frequencies when the carrier is almost monochromatic (e.g., injection lasers) because, for a given kind of waveguide, single-mode pulse spreading is inversely proportional to the square root of the frequency; that is, it is 100 times smaller at optical frequencies than at microwave frequencies. This effect can therefore be neglected.

A quite different mechanism for pulse spreading is found in multimode waveguides (with modes of the order $m = 0, 1, 2, \cdots$) excited by multimode sources. In most waveguides, different modes have different group velocities. Thus, a pulse decomposes into a train of pulses, one for each mode, having times of arrival $Ld\beta_m/d\omega$, $m = 0, 1, 2, \cdots$. This effect has similarities with the multipath effects observed in open space. Multimode pulse spreading is observed even when a single mode is excited because, soon after, the power is transferred to other modes and back to the first mode, as a result of the irregularities of the fiber or of the bends (see Ref. 2 and references therein). In this paper, we assume that the fiber is perfectly straight and uniform, and investigate ways of minimizing the dependence of $d\beta_m/d\omega$ on m.

To appreciate the magnitude of the problem, let us consider first a nondispersive homogeneous dielectric slab with refractive index n close to unity. By comparing the length of rays at the critical angle (θ_o) to the length of axial rays, we find that the pulse spreading is $\Delta T = (L/c)[(\cos\theta_o)^{-1} - 1] \approx (L/c)(n-1)$. This pulse spreading can be written as a function of M, the number of modes that we want to transmit (a characteristic of the source used) and of the slab width Y: $\Delta T = 400\, M^2(\lambda/Y)^2$ ns/km. For example, if we want to transmit 20 modes and $Y = 70\,\lambda$, pulse spreading is 33 ns/km, a value that seriously limits the transmission capacity for long-distance applications. The guide width Y cannot be increased very much because the bending losses would rapidly increase and because it is difficult to fabricate clad fibers with very small differences in refractive index.

The difficulty is solved in principle if the permittivity ϵ of the medium varies as the square of the transverse coordinate y: $\epsilon(y) = 1 - y^2$. In a square-law medium, the optical length of the rays is almost independent of their amplitude. If the permittivity has the form $\epsilon(y) = (\cosh y)^{-2} \approx 1 - y^2 + \frac{2}{3}y^4 + \cdots$, rays have in fact all *exactly* the same optical length.[1,3–7] Because most glasses have negligible dispersion, such media exhibit very small pulse spreading.* Multimode square-law fibers are certainly attractive. However, it may prove difficult to obtain with sufficient accuracy the desired variation of permittivity. Furthermore, the losses (impurities and scattering) are usually higher for heterogeneous material than for homogeneous material such as fused quartz. It is therefore interesting to investigate whether a dimensional change can replace the continuous change in the refractive index considered above.

A proposal to that effect was first made by Kawakami and Nishizawa.[1] They have shown that optical beams can be confined transversely in the plane of a slab if the slab thickness has a maximum along some straight line (z-axis). This can be understood from a geometrical optics point of view. The slab thickness can be considered a constant over a small interval of the transverse coordinate y. Various modes can propagate in this uniform slab. Let $\mathbf{k}$ denote the wave vector of one of them, e.g., the H_1 mode. Because of isotropy, the magnitude k of $\mathbf{k}$ is the same in all directions. Once the local properties of the waveguide characterized by the wave number $k(y)$ have been obtained, the propagation of optical beams can be found, in the semiclassical approximation. We need deal only with $k(y)$. For instance, if $k^2(y)$ is a quadratic function of y, e.g., $k^2(y) = k_o^2 - \Omega^2 y^2$, the rays are sinusoids and they have almost all the same optical length. Diffraction effects in the

* The properties of graded-index fibers that depart somewhat from a quadratic law have also been investigated (Refs. 8 to 10).

yz plane can be taken into account, to some extent, as the Hamiltonian theory of beam modes shows.[11] For the quadratic variation considered above, for example, the modes of propagation are Hermite-gauss,[1,11] regardless of the physical origin of the variation of k with y (that is, whether the variation of k with y results from a genuine variation in refractive index or from a change in slab thickness). Because we are interested in highly multimoded fibers, we consider only the geometrical optics field. In that approximation, a mode is represented by a manifold of rays $y(z + \zeta)$, $0 < \zeta < Z$, where Z denotes the ray period. The main result of this representation is that the axial propagation constant (k_z) of the guide is the value assumed by k at the turning point $y = \xi$ of the trajectory. Therefore, we need only solve a ray equation.

The preceding discussion is applicable to the propagation of waves at one angular frequency, ω_o. To obtain information concerning the propagation of optical pulses, we need to know, not only $k(y)$, but also the variation of the local group velocity u with y. If the ratio $(\omega_o/k) \cdot (\partial k/\partial \omega)$ of the local phase velocity $(v = \omega_o/k)$ and group velocity $(u = \partial \omega/\partial k)$ happens to be independent of the y coordinate, the time of flight of a pulse along a ray trajectory is proportional to the optical length of that ray. In that case (but only in that case), equal optical lengths imply equal times of flight. The above condition (v/u independent of y) is rather well satisfied for most materials with low dispersion, such as fused quartz, whose refractive index is changed slightly by such processes as ion implantation. (For normal quartz $n = 1.4564$, $dn/d\lambda = -0.27 \times 10^{-5}$ at $\lambda = 0.6563\ \mu$m.) In cases where there is a physical change in the refractive index, it is sufficient to consider the optical lengths of rays with different amplitudes to obtain with good approximation the value of the pulse spreading. For a homogeneous dielectric slab, however, the ratio of the local phase to group velocities is strongly dependent on the slab thickness $(2d)$, and therefore on y, when either $2d \gg \lambda$* or when $2d \ll \lambda$. (The latter approximation is made in Ref. 1; pulse spreading for tapered slabs is not discussed in Ref. 1). We will show that small pulse spreading is obtained only for a precise value of the slab thickness on axis. For simplicity, we have considered only quadratic and linear dependences of k^2 on y. The optimum profile may be different, however. In Section II we give the essential formulas for the ray trajectories and times of flight in structures with

* We are indebted to E. A. J. Marcatili for pointing out that pulse spreading in thick, quadratically tapered slabs is almost as large as in clad slabs. This observation, at first surprising, stimulated our interest in the problem.

known local phase and group velocities. In Section III we consider in detail the case of tapered slabs and given design values for low pulse spreading. General results are given in Appendix A, and analytic solutions for square-law and linear-law tapers are given in Appendix B.

II. GENERAL RESULTS

The local value k of the wave number of a slab mode is given in Section III. In the present section we assume that the local wave number $k \equiv \omega_o v$ and the inverse $\partial k/\partial\omega$ of the local group velocity u are known functions of y at the operating angular frequency (ω_o). We give the general form of the ray equations and the time of flight of a pulse in a mode m, in the geometrical optics (J.W.K.B.) approximation. The derivations are given in Appendix A.

In a medium that is isotropic, time-invariant, and independent of the axial coordinate (z), that is, in a uniform fiber, the ray equations $y(z)$ are most convenient in the form

$$k_z^2 = k^2(\omega, y) - k_y^2, \tag{1a}$$

$$dy/dz = -\partial k_z/\partial k_y = k_y/k_z, \tag{1b}$$

$$dk_y/dz = \partial k_z/\partial y = \tfrac{1}{2}(\partial k^2/\partial y)/k_z, \tag{1c}$$

$$dt/dz = \partial k_z/\partial\omega = \tfrac{1}{2}(\partial k^2/\partial\omega)/k_z. \tag{1d}$$

Because of the t and z invariance of the medium, ω and k_z are constant along any given ray (constants of motion). The x coordinate is ignored. The first equation, (1a), says that, because of local isotropy, $k_z^2 + k_y^2$ is equal to k^2. In (1b) to (1d), k_z is considered a function of k_y, ω, and y. Equations 1(b) and (1c) are the ray equations. They give the increments in ray position (dy) and momentum* (dk_y) for an increment dz of z. As indicated before, k_z characterizes a ray trajectory, that is, it is different from one ray to another, but it remains the same along any given ray. We can eliminate k_y from (1b) and (1c) by differentiation. We obtain

$$d^2y/dz^2 = \tfrac{1}{2}(\partial k^2/\partial y)/k_z^2. \tag{2}$$

We first select, as an initial condition, the angle θ_o that the ray makes with the z axis at the origin of the coordinate system ($y = z = 0$). We then evaluate the constant of motion k_z from

$$k_z = k(0)\cos\theta_o. \tag{3}$$

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* The ray momentum is the transverse component of the wave vector. Ray momenta and photon momenta ($\hbar\mathbf{k}$) are essentially equivalent concepts.

The ray trajectory $y(z)$ is obtained step by step from (1a) and (1b),

$$y_{i+1} = y_i + [k^2(y_i)/k_z^2 - 1]^{\frac{1}{2}}\Delta z, \tag{4}$$

Δz being the increment in z, and $y_0 = 0$. Note that, because of symmetry, it is sufficient to evaluate $y(z)$ from $y = 0$ to the turning point $y = \xi$, with ξ given by $k(\xi) = k_z$.

To any given value of θ_o (or k_z) we can associate a mode number m. The mode number is the area enclosed in phase space (k_y, y) by a ray trajectory, divided by 2π minus $\frac{1}{2}$ (see Appendix A). Thus, if the integration is stopped at the turning point $y = \xi$ (one-fourth of the ray trajectory), we have

$$m = (2/\pi)\int_0^\xi k_y dy - \tfrac{1}{2}. \tag{5}$$

Strictly speaking, only those values θ_{om} of θ_o should be considered that make m an integer in (5). However, because we are interested in modes of high order, m can be considered a continuous parameter. An approximate value for m is $\pi\theta_o\xi/\lambda$, where λ denotes the wavelength on axis $[k(0) \equiv 2\pi/\lambda]$.

The time of flight T of a pulse is, for a unit length, the inverse $1/v_g$ of the axial group velocity. We show in Appendix A that T is obtained most easily by integrating along a ray ds/u, where $ds = (k/k_y)dy = (k/k_z)dz$ denotes the elementary ray arc length, and $1/u = \partial k/\partial\omega$ the inverse of the local group velocity. Thus,

$$T = Z^{-1}\oint ds/u = (2/Z)\int_0^\xi (\partial k^2/\partial\omega)(k^2 - k_z^2)^{-\frac{1}{2}}dy. \tag{6}$$

Near the turning point $(k = k_z)$, the integrand in (6) is singular. It is therefore preferable from a computational point of view to set $ds = (k/k_z)dz$ and integrate over z rather than over y. We have [also directly from (1d)]

$$T = (2/Zk_z)\int_0^{Z/4} (\partial k^2/\partial\omega)dz. \tag{7}$$

The purpose of this paper is to find ways to minimize the variation ΔT of T for $0 < m < M$, where m is given in (5) and M is the number of modes that we want to transmit. It is interesting to compare this variation to the variation ΔT_c for a clad fiber having the same width $Y \equiv 2\xi$ and the same number of modes M. The latter is, as we have seen in the introduction,

$$\Delta T_c = (1/32)M^2(\lambda/\xi)^2c. \tag{8}$$

Thus, we want to maximize a quality factor Q defined as

$$Q \equiv \Delta T_c/\Delta T = (1/32)M^2(\lambda/\xi)^2/c\Delta T. \tag{9}$$

Note that, since ΔT and ΔT_c are times of flight for unit lengths, they have the dimensions of inverses of velocities. For given $k(y)$ and $(\partial k/\partial\omega)(y)$, integration of (4), (5), and (7) gives $Q(\theta_o)$ in (9). As θ_o is increased, Q increases and reaches a maximum Q_{max}, which characterizes the pulse spreading properties of an optical waveguide for a given profile. The best profile is the one that maximizes Q_{max}, provided other specifications (number of modes, channel width, $\cdots$) are met.

III. TAPERED DIELECTRIC SLABS

Let us now consider the tapered dielectric slab shown in Fig. 1b. We consider only the H_1 mode of the slab. A similar discussion would be applicable to the E_1 mode (and to higher-order modes if the slab is thick enough to support them). Of course, a profile that is optimum for the H_1 mode need not be optimum for the E_1 mode, for example, unless $\epsilon = n^2$ is very close to unity. Let us first give expressions applicable to slabs with constant thicknesses. We assume that the medium is the same on both sides of the slab. (For dissymmetrical media, the formulas in Ref. 13 would be helpful.)

The dispersion equation $k(\omega)$ for H_1 modes in a slab with relative permittivity ϵ and thickness $2d$ is, as is well known,

$$(kd)^2 - \left(\frac{\omega}{c}\,d\right)^2 = \phi^2 \tan^2 \phi, \tag{10a}$$

$$\phi^2 \equiv \epsilon\left(\frac{\omega}{c}\,d\right)^2 - (kd)^2. \tag{10b}$$

From (10) we obtain at $\omega/c = 2\pi$ (that is, $\lambda = 1\ \mu$m, using the μm as the unit of length), by straightforward substitutions and differentiations,

$$d = (1/2\pi)(\epsilon - 1)^{-\frac{1}{2}}\phi/\cos\phi, \tag{11a}$$

$$k^2 = (2\pi)^2[1 + (\epsilon - 1)\sin^2\phi], \tag{11b}$$

$$D \equiv \frac{c}{2}\frac{\partial k^2}{\partial\omega} = \frac{2\pi(\epsilon\phi\tan\phi + \epsilon\sin^2\phi + \cos^2\phi)}{(\phi\tan\phi + 1)}. \tag{11c}$$

Thus, the quantities k^2 and $\frac{1}{2}\partial k^2/\partial\omega$ that enter in our previous expressions are explicit functions of the parameter ϕ, related to d by (11a). The parameter ϕ varies from $\pi/2$ for $d = \infty$ to 0 for $d = 0$. The varia-

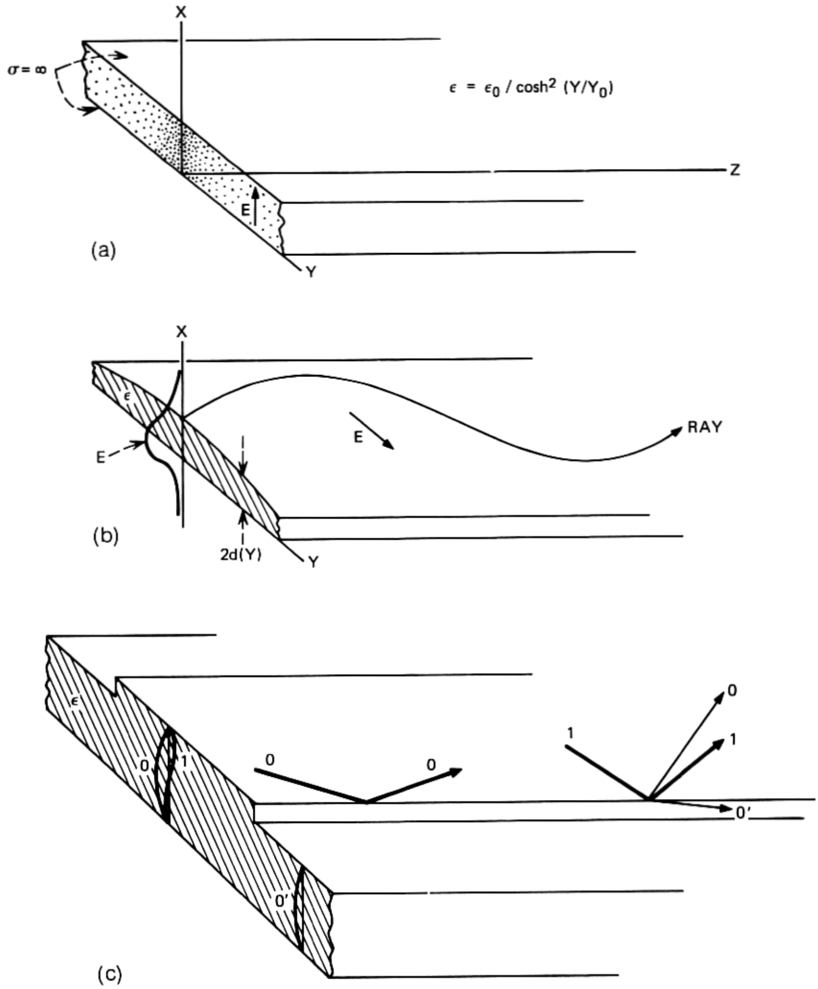

Fig. 1—Planar fibers. (a) Fiber with constant thickness and variation of the permittivity of the form $1/\cosh^2(y)$. (b) Tapered dielectric slab. The field is shown for the H_1 slab mode. (c) Coupling between the various slab modes (H_1, H_2, $\cdots$, E_1, E_2, $\cdots$) cannot be neglected when the thickness $2d(y)$ varies abruptly. This coupling can eliminate the higher-order modes (H_2, E_2, $\cdots$) for suitable dimensions (see Ref. 12).

tion of

$$\frac{v}{u} \equiv \left(\frac{2\pi}{k^2}\right)\frac{c}{2}\frac{\partial k^2}{\partial \omega} = \frac{\epsilon\phi\tan\phi + \epsilon\sin^2\phi + \cos^2\phi}{[1 + (\epsilon - 1)\sin^2\phi](\phi\tan\phi + 1)} \tag{12}$$

is plotted in Fig. 2 as a function of ϕ for various values of ϵ. For quadratic $k^2(y)$, the optimum value ϕ on axis if close to the maximum of the curves, shown by a dotted line, because, near this maximum, times of flight are proportional to optical lengths (see Section II). Thus, we have for that case a rule for the selection of the slab thickness on axis, $2d_o \equiv 2d(0)$. The optimum value of d_o may be slightly different, however, than the one given by the maximum of the curves in Fig. 2, because we want to minimize the variations of T over a finite range of m.

Instead of specifying the slab profile $d(y)$ or the square of the wave number law $k^2(y)$, we find it convenient, for the ease of computations, to specify $\phi(y)$. If ϕ is quadratic in y, both $k^2(y)$ and $d(y)$ are quadratic in y to first order. Thus, we set

$$\phi = \phi_o - Ky^2, \tag{13}$$

where K denotes a constant, in (11), and substitute in the ray equations, (1b) and (1c), eq. (5) for m and eq. (7) for T.

The variation of the time of flight as a function of the angle θ_o that the ray makes with the z axis at the origin is shown in Fig. 3 for $\phi_o = 1.5$ to 0.2 and $n = 1.45$, $\lambda = 1\ \mu$m. Large pulse spreading is observed

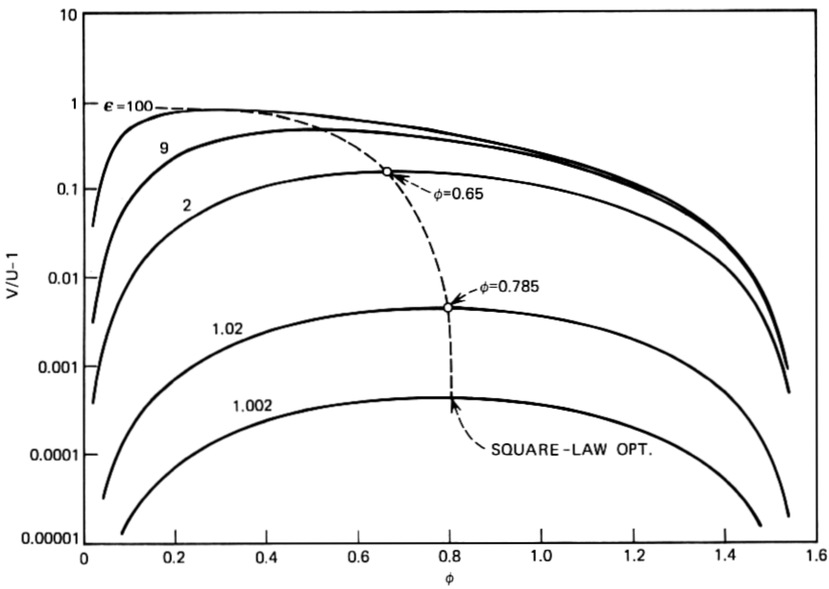

Fig. 2—Variation of the ratio of phase to group velocity in a dielectric slab for different relative permittivities, as a function of the characteristic angle ϕ. The optimum points of operation for low pulse spreading in square-law tapered slabs are shown by a dashed line ($\lambda = 1\ \mu$m).

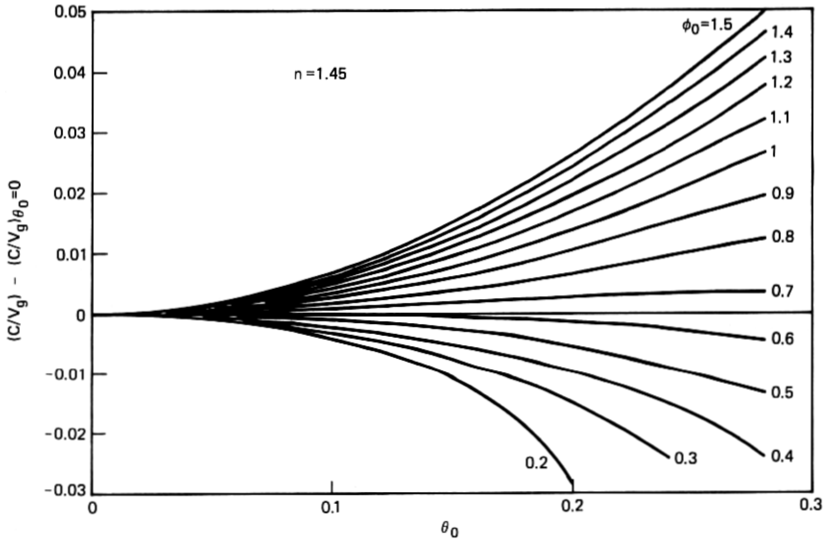

Fig. 3—Ratio of vacuum to axial group velocities (c/v_g) as a function of the ray angle (θ_o) at the origin, for a tapered dielectric slab with $n = 1.45$ and a quadratic variation of the characteristic angle $\phi : \phi = \phi_o - 4 \times 10^{-5}y^2$, for various values of ϕ_o. Group delay is related to c/v_g by $T = (10^4/3)c/v_g$ ns/km. The characteristic angle on axis $\phi_o = 0.65$ is seen to give a small variation of c/v_g over a large range of values of θ_o ($\lambda = 1\ \mu$m).

when the slab is very thick on axis ($\phi_o = 1.5$) or very thin ($\phi_o = 0.2$). Optimum values are between 0.6 and 0.7. Detailed results will be given for the case $n = 1.01$ (refractive index of the slab is 1 percent higher than that of the surrounding medium), which seems of greater practical importance.

For $n = 1.01$, $\lambda = 1\ \mu$m, and $\phi = \phi_o - 10^{-5}y^2$, we see in Fig. 4 that the tapered slab can be 50 times superior to the equivalent clad fiber (factor Q). The profile of this fiber is shown in Fig. 6 (curve a), the thickness on axis being equal to 2.5 μm. The results for the case of a linear law $\phi = \phi_o - 5 \times 10^{-3}|y|$ are shown in Fig. 5 and the corresponding profile in Fig. 6 (curve b). For both quadratic and linear laws, we note that a trade-off has to be made between the quality factor Q and the mode number M. (Note that the results are meaningful only when M is large compared with unity.)

In conclusion, tapered dielectric slabs can exhibit very low pulse spreading if properly dimensioned. If the slab material has a refractive index 1 percent higher than that of the surrounding medium, the thickness should be of the order of $2.5 \pm 0.2\ \mu$m at a wavelength of 1 μm.

The waveguide width would be in that case of the order of 0.2 mm. Pulse spreading does not exceed 0.05 ns/km for 15 modes. These optical waveguides are attractive because they can be stacked for multichannel operation (a possible arrangement is shown in Fig. 7) and splicing would perhaps be easier than with conventional fibers (a good angular alignment, however, is required for planar fibers). Further technological researches are needed to settle this point.

IV. ACKNOWLEDGMENT

The author expresses his thanks to E. A. J. Marcatili for stimulating discussions.

APPENDIX A

Times of Flight in the J.W.K.B. Approximation

The purpose of this appendix is to derive the ray equations and the time-of-flight equations from general principles. We start from the Hamilton equations in space-time both for conceptual clarity and to

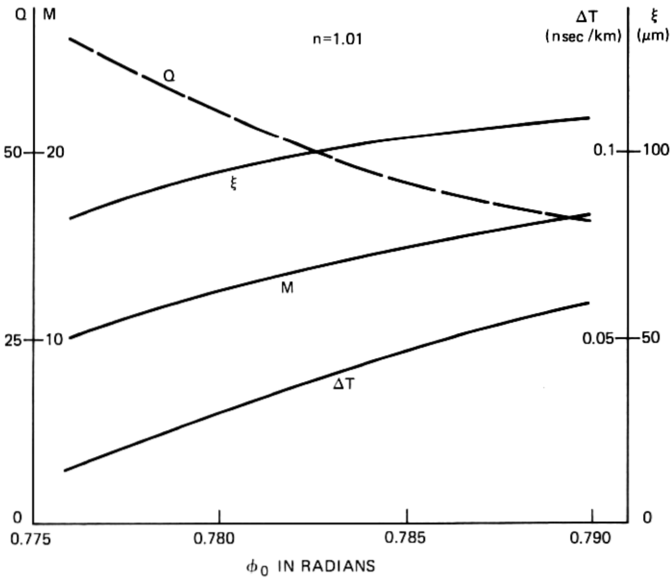

Fig. 4—Variation with the characteristic angle on axis ϕ_o (or slab thickness on axis $2d_o$) of the quality factor Q (defined as the ratio of pulse spreading for an equivalent clad fiber ΔT_c to the actual pulse spreading ΔT) for $\epsilon = 1.02$ ($n = 1.01$) and $\phi = \phi_o - 10^{-5}y^2$. ξ denotes the maximum ray excursion, M the total number of modes, and ΔT the pulse spreading. The ray period is 14 mm and θ_o is equal to 2.6° for $\phi_o = 0.785$ ($\lambda = 1$ μm).

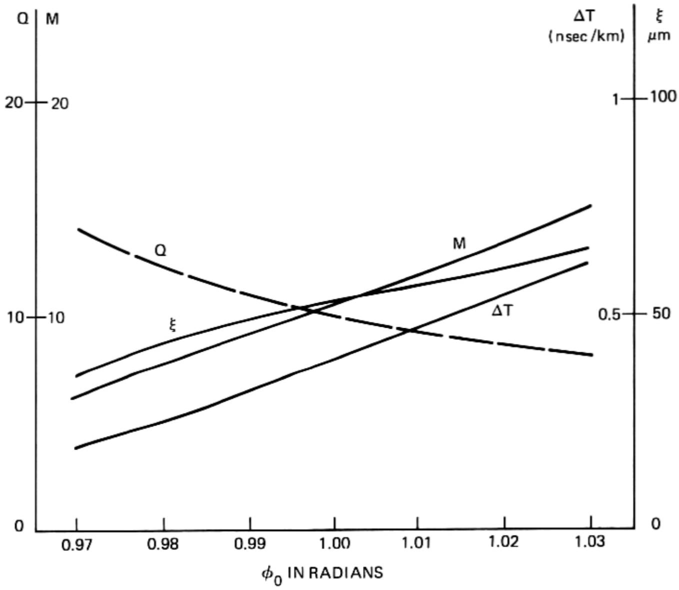

Fig. 5—Variation with the characteristic angle on axis ϕ_o (or slab thickness on axis $2d_o$) of the quality factor Q for $\epsilon = 1.02$ ($n = 1.01$) and $\phi = \phi_o - 5 \times 10^{-3}|y|$. The variations of ξ, M, and ΔT are also given. The ray period is 5.6 mm and θ_o is equal to 4° for $\phi_o = 1$.

facilitate generalizations to anisotropic or time varying media (which are not discussed in detail in the main text, but are of potential interest).

A general medium is described by a function of ω, $\mathbf{k}$, t, $\mathbf{x}$

$$H(\omega, \mathbf{k}, t, \mathbf{x}) = 0. \tag{14a}$$

The space-time trajectories (world lines) of particles or wave packets, $[t(\sigma), \mathbf{x}(\sigma)]$ or $\mathbf{x}(t)$, are obtained by integrating the Hamilton equations

$$\begin{aligned} dt/d\sigma &= -\partial H/\partial\omega, \\ d\mathbf{x}/d\sigma &= \partial H/\partial\mathbf{k}, \\ d\omega/d\sigma &= \partial H/\partial t, \\ d\mathbf{k}/d\sigma &= -\partial H/\partial\mathbf{x}, \end{aligned} \tag{14b}$$

where σ denotes an arbitrary parameter.* These equations are in a suit-

* If we define $\mathbf{X} \equiv \{\mathbf{x}, ict\}$, $K \equiv \{\mathbf{k}, i\omega/c\}$, the Hamilton equations (14b) are: $d\mathbf{X}/d\sigma = \partial H/\partial\mathbf{K}$ and $d\mathbf{K}/d\sigma = -\partial H/\partial\mathbf{X}$. The latter follows from the first (see Ref. 11) because $H = 0$ and $\mathbf{K} = \nabla S$. The dynamical significance of the Hamilton equations follows from the expression of the canonical stress-energy tensor (Ref. 14):

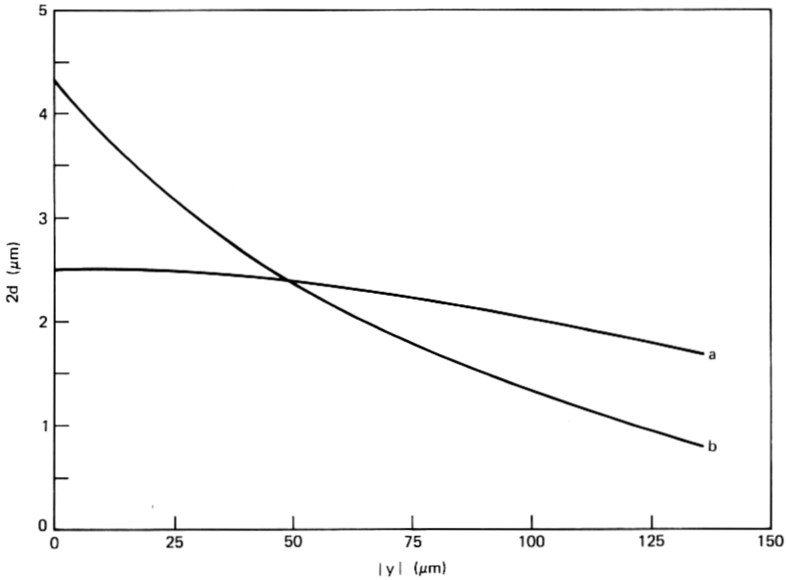

Fig. 6—Slab profiles for $n = 1.01$. (a) Quadratic case $\phi = 0.785 - 10^{-5}y^2$. (b) Linear case $\phi = 1 - 5 \times 10^{-3}|y|$.

able form for numerical integration. The initial conditions must, of course, be consistent with (14a). Then (14a) remains satisfied at all σ because, from (14b), $dH/d\sigma = 0$.

For time-invariant media, the form

$$\omega = \omega(\mathbf{k}, \mathbf{x}) \tag{15}$$

is more useful. The motion $\mathbf{x}(t)$ of a wave packet is a solution of the Hamilton equations

$$\begin{aligned} d\mathbf{x}/dt &= \partial\omega/\partial\mathbf{k}, \\ d\mathbf{k}/dt &= -\partial\omega/\partial\mathbf{x}. \end{aligned} \tag{16}$$

If we are interested only in ray trajectories at some fixed ω, we can rewrite (15)

$$h(\mathbf{k}, \mathbf{x}) = 0, \tag{17a}$$

$\mathbf{T} = \mathbf{K}\partial\mathcal{L}/\partial\mathbf{K}$, where $\mathcal{L}$ denotes the averaged Lagrangian density. $\partial\mathcal{L}/\partial\mathbf{K}$ is the (conserved) wave action, and $\mathbf{T}$ is conserved in time-invariant homogeneous media. The equality of group and energy velocities readily follows from this expression for $\mathbf{T}$. Note that these results are applicable to any linear wave (e.g., matter waves, acoustical waves, or optical waves).

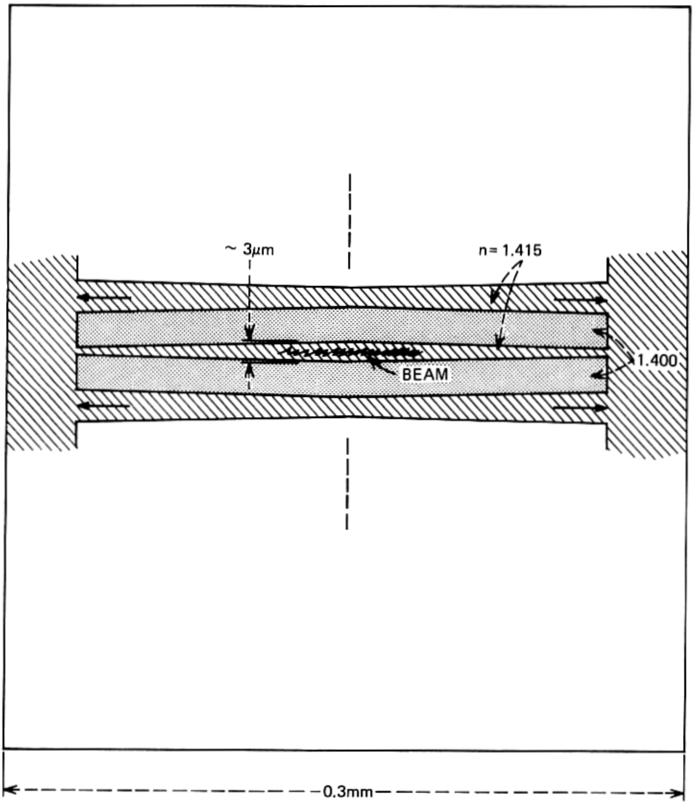

Fig. 7—Stacked tapered dielectric slabs. Adjacent slabs are separated by slabs with inverted slope and high index to minimize crosstalk caused by scattering.

and obtain the rays from

$$\begin{aligned} d\mathbf{x}/d\sigma &= \partial h/\partial \mathbf{k}, \\ d\mathbf{k}/d\sigma &= -\partial h/\partial \mathbf{x}, \end{aligned} \tag{17b}$$

where σ is again an arbitrary parameter. Equation (17) is the reduction of (14) to three dimensions. Note that the Fermat principle (in three dimensions) is applicable to rays $\mathbf{x}(\sigma)$ at a constant frequency ω. It is unrelated to the time of flight of wave packets, except for nondispersive media. It is important for our study that the time of flight of a pulse be carefully distinguished from the transit time of the crest of a time-harmonic wave (optical length). The latter is the integral of the ray index along the ray path, evaluated at a fixed frequency ω.

We need the Hamilton equations in one more form, in which the z axis is singled out. For media that are invariant in the z direction, it is convenient to solve (14a) for k_z. Ignoring the x coordinate, we have

$$H \equiv k_z - k_z(\omega, k_y, y) = 0, \tag{18a}$$

and the ray equations are, from (14b),

$$\begin{aligned} dy/dz &= -\partial k_z/\partial k_y, \\ dt/dz &= \partial k_z/\partial \omega, \\ dk_y/dz &= \partial k_z/\partial y, \end{aligned} \tag{18b}$$

where ω and k_z are constants of motion. If the surface y, z is isotropic, $\mathbf{k}$ enters only through its magnitude k. Thus,

$$k_z^2 = k^2(\omega, y) - k_y^2, \tag{19}$$

and (18) becomes

$$dy/dz = k_y/k_z, \tag{20a}$$

$$dk_y/dz = \tfrac{1}{2}(\partial k^2/\partial y)/k_z, \tag{20b}$$

$$dt/dz = \tfrac{1}{2}(\partial k^2/\partial \omega)/k_z. \tag{20c}$$

These are the expressions used in the main text. Equations (20a) and (20b) give the rate of change of the ray position and momentum as a function of z. Equation (20c) gives the time of flight of a pulse by direct integration. We now show that this result can be obtained from the J.W.K.B. approximation of the wave optics solution.

The scalar Helmholtz equation is obtained from the substitution

$$k_y \rightarrow -i\partial/\partial y \tag{21}$$

in (19). We obtain

$$[\partial^2/\partial y^2 + k^2(\omega, y)]\psi_m = k_{zm}^2\psi_m, \tag{22}$$

where $m = 0, 1, 2, \cdots$, for trapped modes. Given $k(\omega, y)$, we look for solutions of (22) that are square-integrable and obtain the time of flight of a pulse in a mode m over a unit length by differentiating k_{zm} with respect to ω,

$$T = 1/v_g = \partial k_{zm}/\partial \omega. \tag{23}$$

Instead of solving (22) for k_z and differentiating with respect to ω, we may use the Hellmann-Feynman (H.F.) theorem.[15] Let $\mathcal{H}$ be a self-adjoint operator depending on a parameter ω,

$$\mathcal{H}(\omega)\psi_m = E_m\psi_m. \tag{24}$$

Premultiplying both sides of (24) by ψ_m we obtain

$$E_m = \langle \psi_m \mathcal{H}(\omega) \psi_m \rangle / \langle \psi_m \psi_m \rangle, \tag{25}$$

where

$$\langle ab \rangle \equiv \int_{-\infty}^{+\infty} a^*(y) b(y) dy. \tag{26}$$

It is not difficult to show that E_m is stationary with respect to a small change in ψ_m. Thus, when we differentiate (25) with respect to ω (or ω^2), we can ignore the dependence of ψ_m on ω (or ω^2). We have for ψ a real

$$\frac{dE_m}{d\omega^2} = \frac{\langle \psi_m (d\mathcal{H}/d\omega^2) \psi_m \rangle}{\langle \psi_m \psi_m \rangle}. \tag{27}$$

In our case, (22),

$$\mathcal{H}(\omega) \equiv d^2/dy^2 + k^2(\omega, y). \tag{28}$$

Thus, by application of the H.F. theorem we obtain

$$(c/v_g)_m = (k_o/k_{zm})(dk_{zm}^2/dk_o^2) = (k_o/k_{zm}) \times \int_{-\infty}^{+\infty} (\partial k^2/\partial k_o^2) \psi_m^2 dy \Big/ \int_{-\infty}^{+\infty} \psi_m^2 dy \equiv (k_o/k_{zm}) \langle \partial k^2 / \partial k_o^2 \rangle_m. \tag{29}$$

The J.W.K.B. method shows that, for large m, a mode can be represented by a manifold of rays satisfying the Bohr-Sommerfeld condition

$$\oint k_y dy = (m + \tfrac{1}{2}) 2\pi, \tag{30}$$

where the integral on the lefthand side in (30) is the area enclosed in phase space (k_y, y) by a ray trajectory. Equation (30) expresses the uniqueness of the phase of the field. At the turning point, $k_y = 0$, $y = \xi_m$, we have from (19)

$$k_{zm} = k(\omega, \xi_m). \tag{31}$$

An alternative way of obtaining the time of flight of a pulse in a mode m is to integrate ds/u from $z = 0$ to 1 along a ray of the manifold. The arc length is denoted by $ds = (k/k_z)dz$ and $u^{-1} \equiv \partial k/\partial \omega$ is the inverse of the local group velocity. Thus,

$$T = \int_0^1 \left(\frac{\partial k}{\partial \omega} \right) \left(\frac{k}{k_z} \right) dz = c^{-1} \left(\frac{k_o}{k_z} \right) \left\langle \frac{\partial k^2}{\partial k_o^2} \right\rangle. \tag{32}$$

This expression, (32), in which $\langle \ \rangle$ denotes an average taken along a ray period, is the semiclassical analog of the Hellmann-Feynman theorem eqs. (27) and (29), and is used in the main text. It can be obtained alternatively by noting that the group velocity in a waveguide is the ratio of the total energy flow to the energy stored per unit length. (This is a special case of the theorem derived in Ref. 16 for periodic bi-anisotropic media. To obtain the result applicable to open waveguides, we only have to let the periods go to infinity.) The result, (32), follows by integrating the energy density along a ray pencil bounded by the rays $y(z)$ and $y(z + dz)$. Let us sketch the proof. If Pdz denotes the energy flow in this ray pencil, the total energy flow in the waveguide is PZ. The energy density, on the other hand, is $P/u \sin \theta$, where θ is the angle that the ray makes with the z axis. Thus, the energy per unit length is obtained by integrating Pds/u along the ray, in agreement with (32).

APPENDIX B

Square-Law and Linear-Law Media

In this appendix we work out the case of square-law and linear-law media because they lend themselves to exact analytical expressions that are useful for comparison with computed solutions. The case in which the wave number k varies quadratically with y is also useful to obtain first-order solutions. Let us consider this case first.

$$k^2(\omega, y) = k_o^2(\omega) - \Omega^2(\omega)y^2, \tag{33}$$

where the functions $k_o(\omega)$ and $\Omega(\omega)$ are arbitrary. The wave equation, (22), is

$$(\partial^2/\partial y^2 + k_o^2 - \Omega^2 y^2) = k_z^2 \psi, \tag{34}$$

where ψ represents, for instance, the y component of the electric field for H modes in a dielectric slab. This equation has the well-known eigenvalues

$$k_z^2 = k_o^2 - (2m + 1)\Omega. \tag{35}$$

Thus,

$$\begin{aligned} T = 1/v_g = dk_z/d\omega &= [k_o\dot{k}_o - (m + \tfrac{1}{2})\dot{\Omega}][k_o^2 - (2m + 1)\Omega]^{-\frac{1}{2}} \\ &= \dot{k}_o + (\Omega/k_o)(\dot{k}_o/k_o - \dot{\Omega}/\Omega)(m + \tfrac{1}{2}) + (\Omega^2/k_o^3) \\ &\quad \times [(\tfrac{3}{2})\dot{k}_o/k_o - \dot{\Omega}/\Omega](m + \tfrac{1}{2})^2 + \cdots, \end{aligned} \tag{36}$$

where upper dots denote differentiation with respect to ω. The condition for the removal of the first-order terms in (36), $\dot{k}/k_o = \dot{\Omega}/\Omega$, is the same as the condition of stationarity of $v/u \equiv \omega k^{-2}\tfrac{1}{2}(\partial k^2/\partial \omega)$ given in the main text. (Note that m is proportional to θ_o^2. Thus, first-order terms in m correspond to θ_o^2 terms.)

Let us now show that this result can be derived from the ray equations. Equations (19) and (20) are

$$dy/dz = k_y/k_z, \tag{37a}$$

$$dk_y/dz = -\Omega^2 y/k_z, \tag{37b}$$

$$d^2y/dz^2 + (\Omega/k_z)^2 y = 0. \tag{37c}$$

The solution of these equations is straightforward. We obtain

$$y = (k_{yo}/\Omega) \sin [(\Omega/k_z)z], \tag{38a}$$

$$k_y = k_{yo} \cos [(\Omega/k_z)z], \tag{38b}$$

where

$$k_{yo} \equiv k_o^2 - k_z^2, \tag{38c}$$

if we specify, for simplicity, that $y(0) = 0$, and use (33). The quantum condition, (30), is therefore

$$k_{yo}^2 = (2m + 1)\Omega. \tag{39}$$

Thus, setting $k_{yo}/\Omega \equiv \xi$, the axial wave number is given by

$$k_z^2 = k^2(\omega, \xi) = k_o^2 - \Omega^2\xi^2 = k_o^2 - (2m + 1)\Omega, \tag{40a}$$

in exact agreement with (35) (the agreement needs to be exact only for square-law media).

The ratio of the optical length of a ray period (period Z) to the corresponding length on axis is

$$\begin{aligned} R &= (k_o Z)^{-1} \int_0^Z k ds = (k_o k_z Z)^{-1} \int_0^Z k^2 dz \\ &= (k_o k_z Z)^{-1} \int_0^Z \{k_o^2 - k_{yo}^2 \sin^2 [(\Omega/k_z)z]\} dz \\ &= (1 - \tfrac{1}{2} \sin^2 \theta_o)/\cos \theta_o = 1 + \theta_o^4/8 + \cdots, \end{aligned} \tag{40b}$$

where θ_o denotes the angle between the ray and the z axis at the origin. By comparison, we have for a clad slab

$$R_c = 1/\cos \theta_o \approx 1 + \theta_o^2/2 + \cdots. \tag{40c}$$

Thus, for small θ_o, $R - 1$ is much smaller than $R_c - 1$, as discussed in the introduction. The above results, (40b) and (40c), are significant in the problem of pulse spreading in graded-index fibers if the material has low dispersion, but they are not relevant to tapered dielectric slabs. They are given here only for comparison.

Let us now evaluate the group velocity by integrating ds/u along a ray of the manifold, following (32). We have

$$v_g^{-1} = (k_z Z)^{-1} \frac{1}{2} \int_0^Z \left(\frac{\partial k_o^2}{\partial \omega} - \frac{\partial \Omega^2}{\partial \omega} y^2 \right) dz, \tag{41}$$

where $Z = 2\pi k_z / \Omega$ denotes the spatial period, and $y(z)$ is given in (38a). The integration is straightforward. Using (39), a result identical to (36) is obtained. Note that the above results are exact; the paraxial approximation was not made. We have shown in Appendix A that it is legitimate to evaluate pulse spreading by integrating the inverse of the local group velocity along rays representing the modes of propagation, in the limit of large mode numbers. The agreement is now found to be exact for square-law media.

For a linear-law medium with

$$k^2(\omega, y) = k_o^2(\omega) - 2a(\omega)|y|, \tag{42}$$

we shall only give the results. The rays are, from (20),

$$y(z) = \tan \theta_o\, z \mp (a/2k_o^2 \cos^2 \theta_o) z^2 \begin{cases} 0 < z < Z/2 \\ Z/2 < z < Z, \end{cases} \tag{43}$$

with a period

$$Z = 4k_o^2 \sin \theta_o \cos \theta_o / a. \tag{44}$$

The ratio of the optical length of a ray to the length on axis is

$$\int k ds \Big/ \int k_o dz = (1 - \tfrac{2}{3} \sin^2 \theta_o)/\cos \theta_o = 1 - \theta_o^2/6 + \cdots. \tag{45}$$

The situation is opposite to that of a clad fiber: The optical length *decreases* as θ_o increases. Therefore, we may in that case have a small increase of v/u when the slab thickness is reduced, that is, work on the right side of the dotted line in Fig. 2. This leads to a thicker slab than in the case of square-law profiles. These theoretical results are confirmed by the curves in Figs. 5 and 6. We note that the optimum ϕ_o is about 1, the maximum of the v/u curve being at only 0.8. The time of flight is, using (32),

$$T = 1/v_g = (\cos \theta_o)^{-1} dk_o/d\omega - (23/6)(k_o/a)(\sin^2 \theta_o / \cos \theta_o)(da/d\omega).$$

Thus, T is independent of θ_o for small θ_o (no terms in θ_o^2) if $k_o(\omega)$ and $a(\omega)$ in (42) are related by

$$(dk_o/d\omega)/k_o = (23/3)(da/d\omega)/a. \tag{46}$$

It can be shown that this condition corresponds to an increase of v/u with $|y|$, in agreement with the previous discussion.

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