THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 54, No. 6, July–August 1975
Printed in U.S.A.

Polarization-Independent, Multilayer Dielectrics at Oblique Incidence

By A. A. M. SALEH

(Manuscript received January 6, 1975)

This paper gives solutions for multilayer dielectrics that yield polarization-independent operation at a given angle of incidence and at a given frequency. An exhaustive analysis is given for symmetric three-layer dielectrics surrounded by the same medium. Special solutions are also given for symmetric multilayer dielectrics with more than three layers surrounded by the same medium and for multilayer dielectrics surrounded by different media. Rules are presented for obtaining new polarization-independent solutions by cascading given solutions. Finally, for the purpose of demonstration and comparison of various solutions, different design examples are carried out for polarization-independent 3-dB beam splitters to operate at millimeter wavelengths.

I. INTRODUCTION

The state of polarization of a plane wave will, in general, be changed upon transmission through or reflection by a multilayer dielectric at oblique incidence. The reason for this depolarization is that the two eigenmodes of polarization, E-field perpendicular to the plane of incidence and E-field parallel to the plane of incidence, generally have different reflection coefficients and different transmission coefficients. This is not desirable in many applications. For example, such depolarization could cause crosstalk in dual-polarization optical or quasi-optical components such as interference filters,[1] directional couplers[2–5] and diplexers,[6] or mode conversion in beam-splitter-type hybrids[7,5] in oversized waveguides.

This paper provides solutions for multilayer dielectrics whose reflection and transmission coefficients are independent of the state of polarization of the incident wave at a given angle of incidence and a given frequency. Compromising the degree of polarization independence to allow for a broader bandwidth of operation or for variations in the angle of incidence, even though desirable in many practical applications, is not discussed.

The paper is primarily concerned with symmetric, multilayer dielectrics surrounded by the same medium. However, the case of different input and output media and antireflective coatings is also discussed. Some of the results given in this paper have been previously reported by Kard,[8] Baumeister,[9] and Rabinovitch and Pagis.[10] Their work is referred to in appropriate places in the text.

There are other types of structures, besides the ones reported in this paper, that can give polarization-independent operation at oblique incidence. For example, artificial or natural anisotropic dielectric layers or metallic-wire meshes with rectangular, rather than square, cells can, in principle, be constructed to achieve this effect.

II. BASIC EQUATIONS

Consider n adjacent isotropic dielectric layers of infinite transverse extent having relative dielectric constants of $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ and uniform thicknesses of $d_1, d_2, \cdots, d_n$ inserted in a medium of a relative dielectric constant ϵ_r. Define

$$\kappa_i \equiv \epsilon_i/\epsilon_r, \qquad i = 1, 2, \cdots, n \tag{1}$$

as the normalized dielectric constants of the layers. Let a uniform plane wave propagating in the medium be incident at an angle θ on the layers as shown in Fig. 1. The incident, transmitted and reflected waves will have the same state of polarization if the E-field is perpendicular

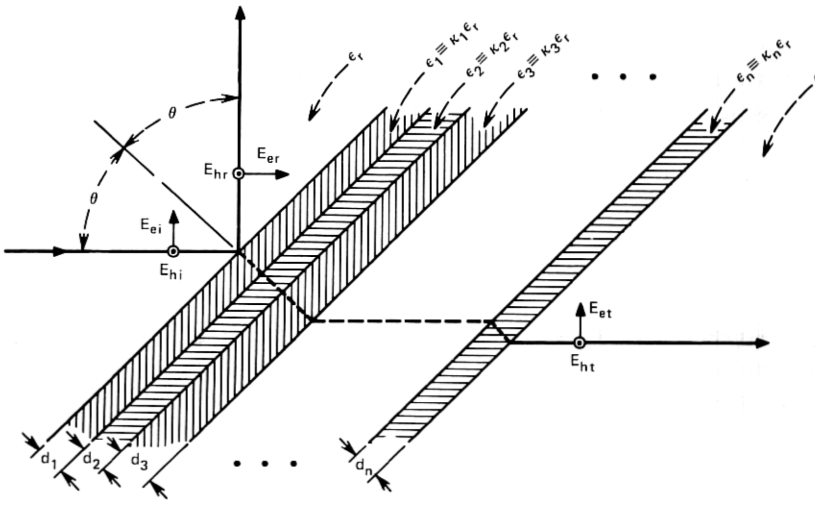

Fig. 1—n-layer dielectric inserted in a medium of a relative dielectric constant ϵ_r. The subscript h refers to the h-mode (E-field perpendicular to plane of incidence), and e refers to the e-mode (E-field parallel to plane of incidence).

to the plane of incidence (h-mode), or if the E-field is parallel to the plane of incidence (e-mode). To calculate the transmission and reflection coefficients for the h- and e-modes, one can represent each layer by an equivalent transmission line.[11,12] The impedance of the line representing the ith layer normalized to that of the line representing the medium in which the layers are inserted is

$$Z_i = Z_{ih} = \cos\theta/(\kappa_i - \sin^2\theta)^{\frac{1}{2}} \tag{2a}$$

for the h-mode, and

$$Z_i = Z_{ie} = (\kappa_i - \sin^2\theta)^{\frac{1}{2}}/(\kappa_i \cos\theta) \tag{3a}$$

for the e-mode. The corresponding electrical length is

$$\phi_i = (2\pi/\lambda_o)d_i\sqrt{\epsilon_r}(\kappa_i - \sin^2\theta)^{\frac{1}{2}} \tag{4a}$$

for both modes; λ_o being the free-space wavelength. If $\kappa_i < \sin^2\theta$, i.e., the total reflection condition is satisfied at the boundary of the ith layer, then the square root of $(\kappa_i - \sin^2\theta)$ is imaginary and one can write

$$Z_{ih} = jX_{ih}, \tag{2b}$$

$$Z_{ie} = -jX_{ie}, \tag{3b}$$

$$\phi_i = -j\alpha_i, \tag{4b}$$

where X_{ih}, X_{ie}, and α_i are positive, real quantities. In this case, coupling is accomplished across that layer by an evanescent wave.

The normalized $ABCD$ matrix corresponding to the ith layer is

$$\mathbf{M}_i = \begin{bmatrix} \cos\phi_i & j\sin\phi_i Z_i \\ j\sin\phi_i/Z_i & \cos\phi_i \end{bmatrix}. \tag{5}$$

The overall, normalized $ABCD$ matrix of the n layers is

$$\begin{bmatrix} A & jB \\ jC & D \end{bmatrix} = \mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_n. \tag{6}$$

The reason for the j in the off-diagonal terms of the matrix of eq. (6) is that A, B, C, and D will be real quantities if the system is lossless. Henceforth, this will be assumed to be the case.

It is worth noting that reciprocity requires that

$$AD + BC = 1, \tag{7}$$

and that for a symmetric structure

$$A = D. \tag{8}$$

The overall field transmission coefficient, t, and field reflection coefficient, r, are given by

$$t \equiv E_t/E_i = 2/[A + D + j(B + C)], \tag{9}$$

$$r \equiv E_r/E_i = [A - D + j(B - C)]/[A + D + j(B + C)]. \tag{10}$$

A subscript of h or e should be added to all the variables in eqs. (5) through (10) to denote the particular mode being considered.

III. CLASSIFICATION OF THE SOLUTIONS

Our goal is to find solutions for the κ's, the d's, and θ to achieve polarization-independent operation. In applications where one is restricted to independent h- and e-modes, or to unpolarized waves, it is sufficient that

$$|t_h| = |t_e|, \qquad |r_h| = |r_e|. \tag{11}$$

To satisfy these conditions, eqs. (7), (9), and (10) require that

$$A_h^2 + B_h^2 + C_h^2 + D_h^2 = A_e^2 + B_e^2 + C_e^2 + D_e^2. \tag{12}$$

For symmetric structures, eqs. (7) and (8) reduce eq. (12) to

$$B_h - C_h = \pm(B_e - C_e). \tag{13}$$

To preserve the state of polarization of an arbitrarily polarized incident wave, condition (11) is not sufficient. Instead, it is necessary that

$$t_h = \pm t_e, \qquad r_h = \pm r_e. \tag{14}$$

The sense of polarization (i.e., the right- or left-handedness) of the transmitted wave will be unchanged if $t_h = +t_e$ and will be reversed if $t_h = -t_e$. On the other hand, the sense of polarization of the reflected wave will be unchanged if $r_h = -r_e$ and will be reversed if $r_h = +r_e$. The latter case is similar to reflection by a perfect, conducting plane.

The four combinations of signs in eq. (11) are listed in Table I where they are referred to as Cases 1 through 4. The required relations

Table I — Four cases for polarization-independent operation

Case	1	2	3	4
$t_h =$	t_e	t_e	$-t_e$	$-t_e$
$r_h =$	r_e	$-r_e$	r_e	$-r_e$
$A_h =$	A_e	D_e	$-A_e$	$-D_e$
$B_h =$	B_e	C_e	$-B_e$	$-C_e$
$C_h =$	C_e	B_e	$-C_e$	$-B_e$
$D_h =$	D_e	A_e	$-D_e$	$-A_e$

between A_h, B_h, C_h, D_h and A_e, B_e, C_e, D_e to satisfy each of these cases can be deduced from eqs. (9) and (10) and are also given in Table I.

In addition to classifying the solutions in the manner of Table I, another classification based on the numerical values of the κ's is useful. To be concise, let $\kappa_{\min}$ be the smallest value of the κ's and let it be at the mth layer, i.e.,

$$\kappa_{\min} \equiv \min(\kappa_1, \kappa_2, \cdots, \kappa_n) = \kappa_m. \tag{15}$$

If $\kappa_{\min} \geqq 1$, then none of the κ's is smaller than unity and the solution can be realized by dielectric layers, $\epsilon_i \geqq 1$, inserted in air, $\epsilon_r = 1$. This will be referred to as a Type A solution. On the other hand, if $\kappa_{\min} < 1$, the surrounding medium can no longer be air since this would require that at least ϵ_m be less than unity (recall that m is the layer with the smallest value of κ); thus, a dielectric-prism realization as shown in Fig. 2 is necessary. In this case, the mth layer can be an air separation $\epsilon_m = 1$, and thus, from eq. (1)

$$\epsilon_i = \kappa_i/\kappa_{\min} \geqq 1, \qquad \epsilon_r = 1/\kappa_{\min} > 1. \tag{16}$$

The prism realization will be referred to as a Type B solution when $1 > \kappa_{\min} \geqq \sin^2\theta$, and a Type C solution when $\kappa_{\min} < \sin^2\theta$. The dis-

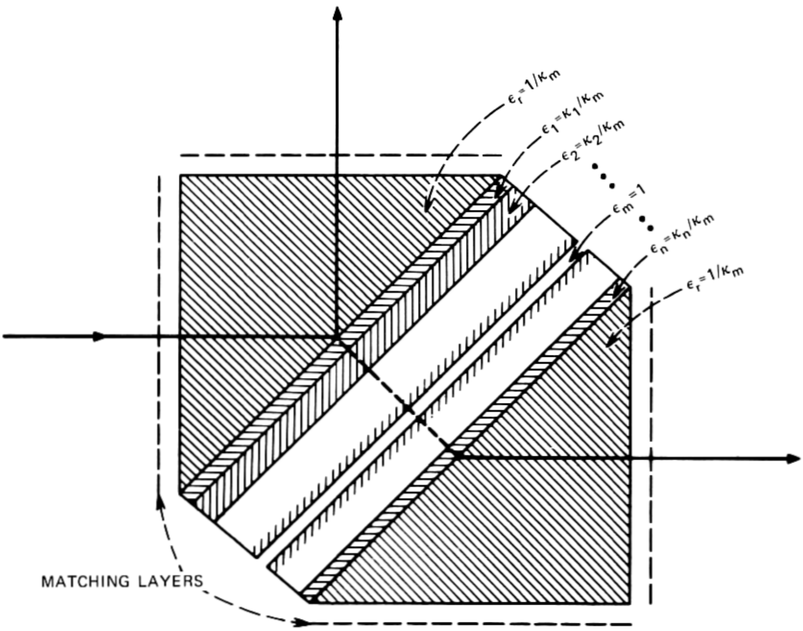

Fig. 2—A prism realization having the mth layer (the layer with the smallest dielectric constant) made of an air separation. Note that $\kappa_m = \kappa_{\min} < 1$. Matching layers are required at the external surfaces of the prisms.

Table II — The three types of solutions

Type	A	B	C
Condition	$\kappa_{\min} \geqq 1$	$\sin^2\theta \leqq \kappa_{\min} < 1$	$\kappa_{\min} < \sin^2\theta$
Description	Dielectric slabs in air ($\epsilon_r = 1$) is possible.	Prism realization ($\epsilon_r > 1$) is necessary. No evanescent waves.	Prism realization ($\epsilon_r > 1$) is necessary. Evanescent waves exist.

tinction is made because in a Type C solution, the condition for total reflection is satisfied at the boundary of the mth layer and coupling across that layer is accomplished by an evanescent wave, as mentioned in Section II. This is not the case for a Type B solution. Table II summarizes the conditions for the three types of solution.

Dielectric-prism devices have been used for many optical[13,14] and quasi-optical[2–5,15–18] applications. When the total reflection condition is satisfied, Type C realization, the device is often said to be of the frustrated-total-reflection type.

Fresnel reflections at the outer surfaces of the prisms might cause undesirable effects.[19] However, this can be avoided by matching each of these surfaces by the familiar $\lambda/4$ transformer, or by other means.[2,20–22] In any case, for the purpose of this paper, the method of matching should be independent of the state of polarization of the incident wave. Thus, Brewster angle matching[17,18] would not be suitable here.

In the remainder of the paper, a combined reference to the four cases of Table I and the three types of Table II is adopted. For example, Case 1A refers to Case 1 of Table I and Type A of Table II.

IV. SINGLE-LAYER SOLUTIONS

It can be shown by using the procedure outlined in Section II that no solution exists for a single layer to satisfy any of the four cases indicated in Table I. However, it is possible to satisfy the weaker condition given in eq. (11) by requiring that

$$\kappa_1 \equiv \epsilon_1/\epsilon_r = \sin^2\theta/(1 + \cos^2\theta), \tag{17}$$

where the symbols are defined in Fig. 1 with $n = 1$. Substituting eq. (12) into eqs. (2) through (10), one obtains

$$t_h = t_e^*, \qquad r_h = r_e^*, \tag{18}$$

where the asterisk denotes complex conjugation.

It is clear from eq. (17) that $\kappa_1 < \sin^2 \theta$ and, hence, from Table II, the solution is of Type C, and a prism realization is necessary.

It is interesting to note that condition (17) is independent of the thickness, d_1, of the layer. Thus, varying d_1 will change the reflection and transmission coefficients without affecting the polarization independence. If $R \equiv |r|^2$ is the required power reflection coefficient, then eqs. (17) and (2) through (10), with $n = 1$, give

$$d_1 = [\lambda_o/(2\pi\sqrt{\epsilon_1})][(1 + \kappa_1)/(1 - \kappa_1)]^{\frac{1}{2}} \times \sinh^{-1} \{2[\kappa_1 R/(1 - R)]^{\frac{1}{2}}/(1 + \kappa_1)\}. \quad (19)$$

A continuous variation of d_1, and hence of R, can be accomplished most conveniently if the layer is simply an air separation, $\epsilon_1 = 1$. In this case, if the angle of incidence is $\theta = 45°$, which is convenient for mechanical considerations, eq. (17) requires the dielectric constant of the prisms to be $\epsilon_r = 3$. On the other hand, if the prisms were made of polystyrene, $\epsilon_r = 2.54$, eq. (17) requires that $\theta = 48.73°$.

V. SYMMETRIC THREE-LAYER SOLUTIONS

It was mentioned in the previous section that no single-layer solution exists that satisfies any of the four cases of Table I. The same is also true for two layers. On the other hand, one can satisfy any of the four cases with three or more layers. In this section, we give all the solutions for symmetric three-layer structures, i.e.,

$$\kappa_3 = \kappa_1, \qquad d_3 = d_1. \quad (20)$$

Even though unsymmetric three-layer structures can have polarization-independent solutions, they will not be discussed here because of the algebraic complexities involved.

Using the terminology of Section II, it can be shown that the normalized $ABCD$ parameters of a symmetric three-layer structure are given by

$$A = D = \cos \phi_2 \cos (2\phi_1) - \tfrac{1}{2}[Z_1/Z_2 + Z_2/Z_1] \sin \phi_2 \sin (2\phi_1) \quad (21)$$

$$B = Z_1 \cos \phi_2 \sin (2\phi_1) + Z_2 \sin \phi_2 \cos^2 \phi_1 - [Z_1^2/Z_2] \sin \phi_2 \sin^2 \phi_1 \quad (22)$$

$$C = Z_1^{-1} \cos \phi_2 \sin (2\phi_1) + Z_2^{-1} \sin \phi_2 \cos^2 \phi_1 - [Z_2/Z_1^2] \sin \phi_2 \sin^2 \phi_1. \quad (23)$$

Combining these equations with eqs. (2) through (4), (9), and (10), one can obtain solutions for the cases listed in Tables I and II. In Cases 3 and 4 of Table I, this process involves considerable algebraic manipulation. The results are given below. For simplicity, expressions for ϕ_1 (or α_1) and ϕ_2 (or α_2) are given instead of expressions for d_1 and

d_2. Also, we define

$$p_i \equiv (\kappa_i - \sin^2 \theta)^{\frac{1}{2}}, \qquad i = 1, 2 \tag{24a}$$

$$q_i \equiv (\sin^2 \theta - \kappa_i)^{\frac{1}{2}}, \qquad i = 1, 2 \tag{24b}$$

$$f(x, y) \equiv 2xy - (x + y) \sin^2 \theta, \tag{25}$$

$$R \equiv |r_h|^2 = |r_e|^2, \tag{26}$$

$$S \equiv [R/(1 - R)]^{\frac{1}{2}}, \tag{27}$$

$$U \equiv [2\sqrt{R}/(1 + \sqrt{R})]^{\frac{1}{2}}, \tag{28}$$

$$V \equiv (1 - \sqrt{R})/(1 + \sqrt{R}). \tag{29}$$

In each case, necessary upper and/or lower bounds on κ_1, κ_2, θ, and R are given. Because of the algebraic complexity involved, the bounds on κ_1 and κ_2 in Cases 3 and 4 are looser than those given in Cases 1 and 2.

Case 1A:

$$\phi_1 = (2m + 1)\pi/2, \qquad m = 0, 1, 2, \cdots. \tag{30}$$

$$\kappa_2 = \kappa_1^2/[1 - (\kappa_1 - 1)^2/\cos^2 \theta]. \tag{31}$$

$$\sin \phi_2 = \pm 2S/[\sqrt{\kappa_2}/\kappa_1 - \kappa_1/\sqrt{\kappa_2}]. \tag{32}$$

$$1 + U \cos \theta \leqq \kappa_1 < 1 + \cos \theta. \tag{33}$$

$$(1 + U \cos \theta)^2/V \leqq \kappa_2 < \infty. \tag{34}$$

This solution exists for all R, θ.

Case 1B:

ϕ_1, κ_2, ϕ_2 are the same as in eqs. (30) through (32).

$$\sin^2 \theta < \kappa_1 \leqq 1 - U \cos \theta. \tag{35}$$

$$\sin^2 \theta < \kappa_2 \leqq (1 - U \cos \theta)^2/V. \tag{36}$$

$$1 < \kappa_1/\kappa_2 \leqq \max\,[2/(1 + \sin \theta), \quad V/(1 - U \cos \theta)]. \tag{37}$$

This solution exists if and only if

$$\sin \theta < V^{\frac{1}{2}}, \tag{38a}$$

or, equivalently,

$$R < [\cos^2 \theta/(1 + \sin {}^2\theta)]^2. \tag{38b}$$

Case 1C:

No solution exists.

Case 2A:

$$\phi_1 = (2m + 1)\pi/2, \qquad m = 0, 1, 2, \cdots. \tag{39}$$

$$\kappa_2 = \kappa_1^2. \tag{40}$$

$$\sin \phi_2 = \pm 2S \cos \theta p_1^2 p_2/[(\kappa_1 - 1) \sin \theta]^2. \tag{41}$$

$$\kappa_1 > \kappa_a \quad (>1), \tag{42}$$

$$\kappa_2 > \kappa_a^2 \quad (>1), \tag{43}$$

where κ_a is the root larger than unity of the quartic equation

$$(W - 1)\kappa^4 + 2(2 - W \sin^2 \theta)\kappa^3 - (6 + W \sin^2 \theta \cos^2 \theta)\kappa^2 + 2(2 + W \sin^4 \theta)\kappa - (1 + W \sin^6 \theta) = 0, \tag{44}$$

where

$$W \equiv [2S \cos \theta/\sin^2 \theta]^2. \tag{45}$$

This solution exists if and only if

$$\sin \theta > U, \tag{46a}$$

or, equivalently,

$$R < [\sin^2 \theta/(1 + \cos^2 \theta)]^2. \tag{46b}$$

Case 2B:

ϕ_1, κ_2, ϕ_2 are the same as in eqs. (39) through (41).

$$\sin \theta < \kappa_1 \leqq \kappa_b \quad (<1), \tag{47}$$

$$\sin^2 \theta < \kappa_2 \leqq \kappa_b^2 \quad (<1), \tag{48}$$

where κ_b is the positive root less than unity of the quartic equation (44).

This solution exists for all R, θ.

Case 2C:

ϕ_1, κ_2 are the same as in eqs. (39) and (40).

$$\sinh \alpha_2 = 2S \cos \theta p_1^2 q_2/[(1 - \kappa_1) \sin \theta]^2. \tag{49}$$

$$\sin^2 \theta < \kappa_1 < \sin \theta. \tag{50}$$

$$\sin^4 \theta < \kappa_2 < \sin^2 \theta. \tag{51}$$

This solution exists for all R, θ. It has been previously found by Kard.[8]

Cases 3A, 3B:

No solutions exist.

Case 3C:

$$\cos(2\phi_1) = f(\kappa_1, 1) f(\kappa_1, \kappa_2)/[(\kappa_1 - 1)(\kappa_1 - \kappa_2) \sin^4 \theta]. \tag{52}$$

$$\sinh \alpha_2 = \frac{2\sqrt{\kappa_1} q_2 f(\kappa_1, 1)}{(1 + \kappa_1)(\kappa_1 - \kappa_2) \sin^2 \theta [Y - (\kappa_2 + X^2) \cos^2 \theta]^{\frac{1}{2}}}, \tag{53}$$

$$S = \sqrt{\kappa_1}(\kappa_2 - X) \cos \theta / [Y - (\kappa_2 + X^2) \cos^2 \theta]^{\frac{1}{2}}, \tag{54}$$

where

$$X \equiv f(\kappa_1, \kappa_2)/[(1 + \kappa_1) \sin^2 \theta], \tag{55}$$

$$Y \equiv \kappa_2(1 - \kappa_2)(\kappa_1 - 1)^2/(1 + \kappa_1). \tag{56}$$

$$\kappa_1 > \{1 + 3 \cos^2 \theta + [8 \cos^2 \theta (1 + \cos^2 \theta)]^{\frac{1}{2}}\}/\sin^2 \theta. \tag{57}$$

$$\sin^2 \theta \cos^2 \theta/(1 + \cos^2 \theta) \leqq \kappa_2 \leqq \sin^2 \theta/(1 + \cos^2 \theta). \tag{58}$$

This solution exists if and only if

$$\sin \theta > (2/S)^{\frac{1}{3}}\{[(1 + 32/27S^2)^{\frac{1}{2}} + 1]^{\frac{1}{3}} - [(1 + 32/27S^2)^{\frac{1}{2}} - 1]^{\frac{1}{3}}\}^{\frac{1}{2}}, \tag{59a}$$

or, equivalently,

$$R > 8 \cos^2 \theta/(8 \cos^2 \theta + \sin^6 \theta). \tag{59b}$$

Cases 4A, 4B:

No solutions exist.

Case 4C:

$$\cosh 2\alpha_1 = (1 + \kappa_1)(\kappa_2 + \kappa_1)/[(1 - \kappa_1)(\kappa_2 - \kappa_1)]. \tag{60}$$

$$\tan \phi_2 = \frac{2\sqrt{\kappa_1}\, \kappa_2(1 + \kappa_1) q_1 p_2}{f(\kappa_1, \kappa_2)[(1 + \kappa_2)(\kappa_1^2 + \kappa_2)]^{\frac{1}{2}}}. \tag{61}$$

$$S = \sqrt{\kappa_1}(\kappa_2 - X)/[\cos \theta\, q_1(\kappa_2 + X^2)^{\frac{1}{2}}], \tag{62}$$

where X is given in eq. (55).

$$\kappa_1 < \sin^2 \theta. \tag{63}$$

$$\kappa_2 > \sin^2 \theta. \tag{64}$$

This solution exists for all R, θ.

Discussion:

The above exhausts all possible solutions for a symmetric three-layer dielectric with no polarization dependence. The choice of the particular solution to implement in a given problem depends on the desired values of R and θ, the available dielectric materials, and whether having dielectric layers surrounded by air (Type A solution) or a prism realization (Type B or C solution) is more suitable.

Cases 1A and 2A, which are the only possible Type A solutions, require large values for the dielectric constants and/or the angle of incidence to achieve moderate or large values of R. This point is illustrated for $R = 0.5$ in the design examples given in Section VIII.

In Case 2, the required relation between κ_1 and κ_2 given in eq. (40) has an advantage for the prism realizations in Cases 2B and 2C. In these two cases, since $\kappa_2 < \kappa_1 < 1$, eqs. (16) and (40) give $\epsilon_2 = 1$, $\epsilon_1 = \kappa_1/\kappa_2$, and $\epsilon_r = 1/\kappa_2 = \epsilon_1^2$. Thus, ϵ_1 can also be used as a quarter-wave transformer to match the outer surfaces of the prisms.

As in the single-layer prism solution given in Section IV, one can vary R in Cases 1 and 2 without affecting the polarization independence by simply changing d_2. This is particularly convenient in the prism realizations in Cases 1B, 2B, and 2C with $\epsilon_2 = 1$ (air) since, in this case, d_2 can be changed continuously. Case 2C is unique in that one can obtain any value of R from zero to unity by varying d_2 from zero to infinity. In Cases 1A, 1B, 2A, and 2B, varying d_2 results in a periodic variation of R between zero and a maximum value less than unity. Varying d_1 or d_2 in Cases 3 or 4 results in a polarization-dependent operation.

Cases 1A, 2B, 2C, and 4C have solutions for all R and θ. This is not true for the remaining three cases, 1B, 2A, and 3C, as indicated by eqs. (38), (46), and (59). However, one should note that the numerical values of ϵ_r, ϵ_1, and ϵ_2, and not necessarily the limitations on R and θ, often determine whether or not any particular case can be used in practice.

In Case 4C, eq. (46) indicates that one can have $\kappa_2 = 1$. In this case, given that $\kappa_1 < 1$ from eq. (63), eq. (16) gives $\epsilon_1 = 1$ and $\epsilon_r = \epsilon_2 = 1/\kappa_1$. Thus, the prism solution can be realized by using one dielectric material with air separations. This is not true for any of the other cases.

VI. SYMMETRIC MULTILAYER SOLUTIONS

When there are more than three layers, an exhaustive analysis similar to that given in the previous section becomes quite involved. However, two relatively simple classes of solutions for a symmetric multilayer dielectric with no polarization dependence can be obtained by generalizing Cases 1 and 2 of the symmetric three-layer dielectric. First, one should note that symmetry requires that the number of layers n be odd, i.e.,

$$n = 2l + 1, \qquad l = 1, 2, \cdots. \tag{65}$$

Thus, from symmetry

$$\kappa_i = \kappa_{n+1-i}, \qquad d_i = d_{n+1-i}, \qquad i = 1, 2, \cdots, l. \tag{66}$$

Let us assume that the electrical lengths of the first, and thus also the last, l layers are odd multiples of $\pi/2$; i.e.,

$$\phi_i = (2m_i + 1)\pi/2; \qquad m_i = 0, 1, 2, \cdots, \qquad i = 1, 2, \cdots, l. \tag{67}$$

While this is a necessary condition to obtain solutions for Cases 1 and 2 of a symmetric three-layer dielectric, as indicated in the previous section, eqs. (30) and (39), it is not necessary in general. However, this assumption is employed here because it simplifies the analysis considerably.

Define the function g_l of the vector $\mathbf{x} = \{x_1, x_2, \cdots, x_{l+1}\}$ as

$$g_l(\mathbf{x}) \equiv x_{l+1}^{[(-1)^l]}\Big(\prod_{i \text{ odd} \leq l} x_i^2\Big)\Big/\Big(\prod_{i \text{ even} \leq l} x_i^2\Big). \tag{68}$$

Thus, $g_1(\mathbf{x}) = x_1^2/x_2$, $g_2(\mathbf{x}) = x_1^2 x_3/x_2^2$, $\cdots$, etc. Each x_i can assume the value of Z_{ih}, Z_{ie}, κ_i, or ν_i (which will be defined later). From eqs. (66) and (67) and Section II, it can be shown that the normalized $ABCD$ parameters of the symmetric multilayer dielectric are given by

$$A = D = (-1)^l \cos(\phi_{l+1}), \tag{69}$$

$$B = (-1)^l \sin(\phi_{l+1}) g_l(\mathbf{Z}), \tag{70}$$

$$C = (-1)^l \sin(\phi_{l+1})/g_l(\mathbf{Z}). \tag{71}$$

Since ϕ_{l+1} is the same for both modes, eq. (69) shows that $A_h = D_h = A_e = D_e$. Thus, from Table I, the above equations can give solutions for Cases 1 and 2 provided that the conditions for the B's and the C's are met. Before satisfying these conditions, however, we note from eqs. (2) and (3) that

$$Z_{ih}Z_{ie} = 1/\kappa_i \tag{72}$$

and define

$$\nu_i \equiv Z_{ie}/Z_{ih} = (\kappa_i - \sin^2\theta)/(\kappa_i \cos^2\theta). \tag{73}$$

As κ_i assumes the increasing values of zero, $\sin^2\theta$, 1, and $+\infty$, then ν_i assumes the increasing values of $-\infty$, 0, 1, and $\sec^2\theta$, respectively.

From eqs. (9), (10), and (65) through (73), and from Table I, one can obtain solutions for Cases 1 ($B_h = B_e$, $C_h = C_e$), and 2 ($B_h = C_e$, $C_h = B_e$). The results are given below. Because of algebraic complexity, each case has not been subdivided into Type A, B, and C solutions. Also, for the same reason, no bounds are given on the κ's, θ, or R.

Case 1:

$$g_l(\boldsymbol{\nu}) = 1. \tag{74}$$

$$\sin\phi_{l+1} = \pm 2S/\{[g_l(\boldsymbol{\kappa})]^{\frac{1}{2}} - [g_l(\boldsymbol{\kappa})]^{-\frac{1}{2}}\}. \tag{75}$$

Case 2:

$$g_l(\boldsymbol{\kappa}) = 1. \tag{76}$$

$$\sin \phi_{l+1} = \pm 2S/\{[g_l(\boldsymbol{\nu})]^{\frac{1}{2}} - [g_l(\boldsymbol{\nu})]^{-\frac{1}{2}}\}. \tag{77}$$

In eqs. (75) and (77), S is the same function of R defined in eq. (27). Note that the roles of the κ's and the ν's are interchanged in the two cases.

For a solution of any particular case to be valid, the value of $\sin \phi_{l+1}$ should either be real with a magnitude not exceeding unity, which leads to a Type A or a Type B solution, or be imaginary, which leads to a Type C solution. The latter solution is not possible in Case 1 because the κ's are real positive quantities, and thus the right-hand side of eq. (75) is always real. On the other hand, since the ν's, and in particular ν_{l+1}, can be negative, a Type C solution is possible in Case 2.

As in the previous section, solutions for Cases 1A, 2B, and 2C in the present section are possible for all values of R and θ.

In Cases 2B and 2C, when κ_{l+1} does not exceed $\kappa_1, \kappa_2, \cdots, \kappa_l$, or unity, eq. (16) shows that one can have a prism realization with $\epsilon_{l+1} = 1$, $\epsilon_l = \kappa_l/\kappa_{l+1}, \cdots, \epsilon_1 = \kappa_1/\kappa_{l+1}$, and $\epsilon_r = 1/\kappa_{l+1}$. Using eq. (76), one can show that l quarter-wave layers with dielectric constants $\epsilon_1, \epsilon_2, \cdots, \epsilon_l$ will match the outer surfaces of the prisms.

In practice, it is desirable to construct the multilayer dielectric reflector with the least number of different dielectrics. Thus, let us assume that two dielectrics with normalized dielectric constants κ_1 and κ_2 are used alternatively; i.e.,

$$\kappa_i = \kappa_1, \quad i \text{ odd}, \tag{78a}$$

$$\kappa_i = \kappa_2, \quad i \text{ even}. \tag{78b}$$

Combining this assumption with those in eqs. (65) through (67), the general solutions given in eqs. (74) through (77) simplify considerably. Because of its practical importance, we restrict ourselves to Type A solution, where κ_1 and κ_2 are larger than unity. No solution exists if either κ_1 or κ_2 is equal to unity.

Case 1A:

$$\kappa_2 = \sin^2 \theta/\{1 - [(\kappa_1 - \sin^2 \theta)/\kappa_1]^{(l+1)/l} (\cos \theta)^{-2/l}\}. \tag{79}$$

$$\sin \phi_{l+1} = \pm 2S/[(\kappa_2^l/\kappa_1^{l+1})^{\frac{1}{2}} - (\kappa_1^{l+1}/\kappa_2^l)^{\frac{1}{2}}]. \tag{80}$$

$$\kappa_1 < \sin^2 \theta/[1 - (\cos \theta)^{2/(l+1)}]. \tag{81}$$

$$\kappa_1^{l+1}/\kappa_2^l < V. \tag{82}$$

This solution exists for all R, θ.

Case 2A:

$$\kappa_2 = \kappa_1^{(l+1)/l} \tag{83}$$

$$\sin \phi_{l+1} = \pm 2S/[(\nu_1^{l+1}/\nu_2^l)^{\frac{1}{2}} - (\nu_2^l/\nu_1^{l+1})^{\frac{1}{2}}]. \tag{84}$$

$$\nu_2^l/\nu_1^{l+1} < V. \tag{85}$$

This solution exists if and only if

$$\sin \theta > U, \tag{86a}$$

or, equivalently,

$$R < [\sin^2 \theta/(1 + \cos^2 \theta)]^2. \tag{86b}$$

The quantities S, U, and V used in the above equations are defined in eqs. (27) through (29). Of course, when $l = 1$, i.e., $n = 3$, the above solutions reduce to Cases 1A and 2A given in the previous section for symmetric three-layer dielectric.

VII. CASCADING SOLUTIONS

It may happen that some required values of the reflection coefficient cannot be obtained by any of the solutions given in the previous two sections because of limitations on the values of the dielectric constant available in practice, or because the resulting angle of incidence is not suitable for a particular application. In this case, the desired multilayer dielectric may be obtained by cascading two or more solutions, provided that they have the same ϵ_r and the same θ.

We now proceed to find the rules for proper cascading to maintain polarization-independent operation. Let the two solutions to be cascaded be labeled x and y. They are not necessarily symmetric or identical. Let $\phi = 2\pi d \cos \theta/\lambda$ be the electric length between x and y (see Fig. 3). Dropping the subscripts h and e, which denote the particular mode being considered, we define r_x, r_x', t_x, r_y, r_y', and t_y as the reflection and transmission coefficients of x and y in the directions specified in the figure. The transmission coefficients t_x and t_y do not depend on the direction of propagation because of reciprocity and because the medium is the same on both sides of each of x and y. Because of losslessness, the difference between r_x and r_x' or between r_y and r_y' is a phase factor. Of course, if x is symmetric then $r_x = r_x'$, and if y is symmetric then $r_y = r_y'$.

Using the method of successive reflections,[23] it can be shown that for a wave incident on the x side of the cascade, the overall field transmission coefficient, t, and field reflection coefficient, r, are given by

$$t = t_x t_y \exp(-j\phi)/[1 - r_x r_y \exp(-j2\phi)], \tag{87}$$

$$r = r_x' + t_x^2 r_y \exp(-j2\phi)/[1 - r_x r_y \exp(-j2\phi)], \tag{88}$$

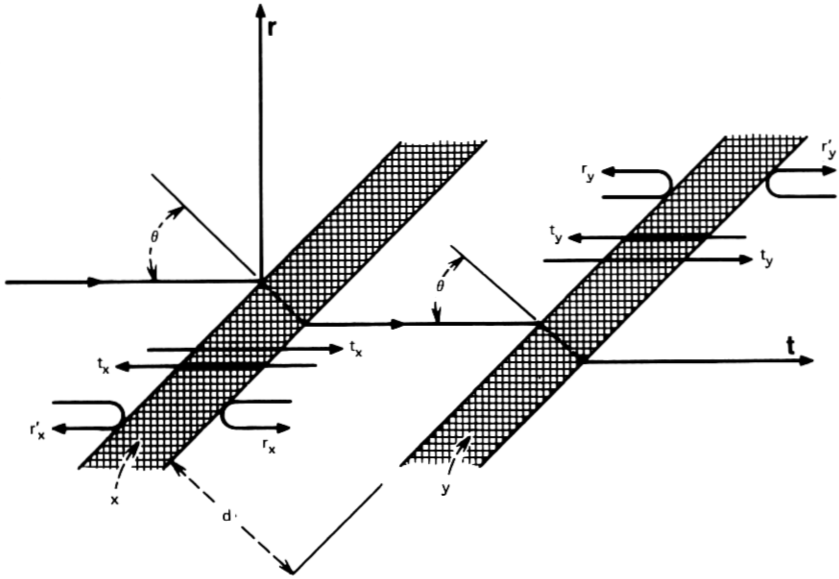

Fig. 3—Cascading of two polarization-independent solutions x and y to obtain an overall polarization-independent solution.

where we recall that a subscript of h or e has been dropped from all the t's and r's.

Since x and y are polarization-independent solutions, then, from eq. (14) or Table I, $r_{xh}r_{yh} = \pm r_{xe}r_{ye}$. Only the positive sign is acceptable, otherwise the magnitude of the denominators in eqs. (87) and (88) would be different for the h- and e-modes, which will result in a polarization-dependent operation for the cascade. Thus, the sign of r_e/r_h should be the same for all the cascaded sections; i.e., one can only have a combination of Cases 1 and 3 or Cases 2 and 4.

Let the abbreviation $i*j = k$ mean: "Cascading Cases i and j gives Case k." It can be shown from eqs. (87) and (88) and Table I that

$$1*1 = 3*3 = 1, \qquad 1*3 = 3*1 = 3, \tag{89a}$$

$$2*2 = 4*4 = 2, \qquad 2*4 = 4*2 = 4. \tag{89b}$$

Let the required overall power reflection coefficient of the cascade be

$$R \equiv |r|^2, \tag{90}$$

and let it be realized by two identical cascaded sections each having a power reflection coefficient R_x. Then, one can write

$$r_x = r_y \equiv R_x^{\frac{1}{2}} e^{j\phi_x}, \tag{91}$$

where ϕ_x is the phase angle of r_x. This can be found given the ϵ's, d's, and θ by using eqs. (1) through (10). Substituting eq. (91) in eqs. (87) and (88), and defining

$$\psi \equiv \phi_x - \phi, \tag{92}$$

one obtains

$$R = [1 + (1 - R_x)^2/(4R_x \sin^2 \psi)]^{-1}. \tag{93}$$

Thus, for a given value of R_x, one can obtain any value of R satisfying

$$0 \leqq R \leqq 4R_x/(1 + R_x)^2. \tag{94}$$

Note that the upper bound on R is larger than R_x. Conversely, for a given value of R, one can use any value of R_x satisfying

$$[1 - (1 - R)^{\frac{1}{2}}]^2/R \leqq R_x \leqq 1. \tag{95}$$

The higher the value used for R_x, the smaller is the overall bandwidth.

VIII. DIFFERENT INPUT AND OUTPUT MEDIA, AND ANTIREFLECTION COATINGS

Often, especially at optical wavelengths, the multilayer dielectric is supported by a substrate having a relative dielectric constant ϵ_s which is, in general, different from that of the input medium, ϵ_r. In this case, general analysis of polarization-independent solutions is complicated. However, it can be shown that special solutions similar to those for the symmetric $(2l + 1)$-layer dielectric given in Section VI can be easily obtained. For this purpose, let θ, as before, be the angle of incidence in the ϵ_r medium. Assume that there are $l(\geqq 1)$ layers with relative dielectric constants $\epsilon_1, \epsilon_2, \cdots, \epsilon_l$ between the ϵ_r and ϵ_s media. The normalized dielectric constants $\kappa_1, \kappa_2, \cdots, \kappa_l$ are defined by eq. (1). In addition, we define

$$\kappa_{l+1} \equiv \epsilon_s/\epsilon_r. \tag{96}$$

Further, as was done in Section VI, assume that each of the l layers is effectively a quarter-wave thick at the given angle of incidence, i.e.,

$$\phi_i = (2m_i + 1)\pi/2; \qquad m_i = 0, 1, 2, \cdots, \qquad i = 1, 2, \cdots, l. \tag{97}$$

With the above definitions and assumptions, it can be shown from eqs. (1) through (6) that the normalized input impedance as seen from the ϵ_r medium is given by

$$Z_{in} = g_l(\mathbf{Z}), \tag{98}$$

where $\mathbf{Z} = \{Z_1, Z_2, \cdots, Z_{l+1}\}$ and the function g_l is defined in eq. (68). Thus, the reflection coefficient in the ϵ_r medium is given by

$$r = (Z_{in} - 1)/(Z_{in} + 1). \tag{99}$$

A subscript of h or e, depending on the particular mode being considered, has been dropped from all the variables in eqs. (98) and (99).

It is clear that if $\kappa_{l+1} \equiv \epsilon_s/\epsilon_r > \sin^2\theta$, then Z_{l+1}, Z_{in} and r are all real quantities. In this case, it can be shown that the reflection coefficient r_s in the ϵ_s medium, and the transmission coefficient t across the l layers, are given by

$$r_s = (-1)^{l+1}r, \tag{100}$$

and

$$t = (-j)^l(-1)^{\sum_{i=1}^{l} m_i}(1 - r^2)^{\frac{1}{2}}, \tag{101}$$

where the m_i's are the integers defined in eq. (97). Since the impedances of the input and output media are different, t is defined here as the scattering transmission coefficient, which is usually referred to as S_{12}. Using the same notation, r and r_s could be referred to as S_{11} and S_{22}, respectively.

The coefficients r_s and t can not be defined if the total reflection condition, $\kappa_{l+1} \equiv \epsilon_s/\epsilon_r < \sin^2\theta$, is satisfied. In this case, Z_{l+1} is imaginary and a nonpropagating evanescent wave exists in the ϵ_s medium. It also follows that Z_{in} is imaginary, and hence, from eq. (99), $|r| = 1$.

It can be seen from eq. (99) that for $r_e = r_h$ (Case 1), $Z_{in,e} = Z_{in,h}$; and that for $r_e = -r_h$ (Case 2), $Z_{in,e} = 1/Z_{in,h}$. Thus, combining eqs. (98), (68), (72), and (73), and defining $R = |r_e|^2 = |r_h|^2$ as the power reflection coefficient, one obtains the following solutions.

Case 1:

$$g_l(\nu) = 1. \tag{102}$$

$$R = \{[g_l(\kappa)]^{\frac{1}{2}} - 1\}^2/\{[g_l(\kappa)]^{\frac{1}{2}} + 1\}^2. \tag{103}$$

Case 2:

$$g_l(\kappa) = 1. \tag{104}$$

$$R = \{[g_l(\nu)]^{\frac{1}{2}} - 1\}^2/\{[g_l(\nu)]^{\frac{1}{2}} + 1\}^2. \tag{105}$$

The above solution of Case 2 has been previously obtained by Baumeister[9] and by Rabinovitch and Pagis.[10] Both papers, however, overlooked the solution of Case 1.

It is observed that eqs. (102) and (104) are identical to eqs. (74) and (76) for the symmetric $(2l + 1)$-layer dielectric. This should not be surprising since the latter can be obtained by cascading, back to back, two identical solutions of the type described in the present section. In fact, this is the reason that the conditions for polarization-independent operation given in Sections V and VI for Cases 1 and 2 are independent of the thickness of the middle layer.

It is clear from eqs. (102) through (105) that if

$$g_l(\boldsymbol{\kappa}) = g_l(\boldsymbol{\nu}) = 1, \tag{106}$$

one obtains $R = 0$. Thus, eq. (106) gives the conditions for polarization-independent antireflection coatings at inclined incidence. It has been previously reported by Baumeister.[9] Rabinovitch and Pagis[10] have incorrectly stated that no such solution exists. It is interesting to note that the condition $g_l(\boldsymbol{\kappa}) = 1$, which is independent of the angle of incidence, is the same condition on the dielectric constants that yields an antireflection coating at normal incidence.

At least two layers, $l \geqq 2$, are required to satisfy eq. (106). For $l = 2$, $\epsilon_r = 1$ and for a given finite θ, eq. (106) gives the unique solution.

$$\epsilon_1 = [\epsilon_s^{\frac{1}{2}} + \sin^2\theta + (\epsilon_s - \sin^2\theta)^{\frac{1}{2}}\cos\theta]/(\epsilon_s^{\frac{1}{2}} + 1), \tag{107a}$$

$$\epsilon_2 = \epsilon_1\epsilon_s^{\frac{1}{2}}. \tag{107b}$$

For $l > 2$, the solution of eq. (106) is not unique, and we have extra degrees of freedom in choosing the dielectric constants of the layers. However, as pointed out by Baumeister,[9] the values of dielectric constants available in practice will often force a compromise of the effectiveness of the antireflection coatings.

IX. DESIGN EXAMPLES

For the purpose of demonstration and comparison of various solutions, let us design polarization-independent, multilayer, dielectric beam splitters with a power reflection coefficient $R = 0.5$ at 50 GHz, i.e., $\lambda_o = 6$ mm. (The frequency is relevant only for the selection of the dielectric materials.) Such 3-dB beam splitters may be used in circular-waveguide, channel-dropping filters.[7]

All the solutions considered are symmetric, and all, with the exception of the last solution, are of Type A with air as the surrounding medium; i.e.,

$$\epsilon_r = 1,$$

and thus, from eq. (1) and Table II, it follows that

$$\kappa_i = \epsilon_i \geqq 1, \qquad i = 1, 2, \cdots, n.$$

In three of the solutions, the frequency response of $|r_h|^2$ and $|r_e|^2$, in dB, and the phase difference

$$\begin{aligned} \Delta\phi &\equiv \text{phase}(r_h) - \text{phase}(r_e) \qquad (\text{mod } 180°) \\ &= \text{phase}(t_h) - \text{phase}(t_e) \qquad (\text{mod } 180°) \end{aligned}$$

are given. The second equality follows from losslessness and symmetry

Table III — Some low-loss dielectrics at millimeter wavelengths

Dielectric	Dielectric constant	Loss tangent	Frequency range of measurements	Reference No.
Kearfott High-Purity Alumina	9.4	0.00017	14 to 50 GHz	24
Polystyrene	2.54	0.0012	10 to 25 GHz	25
Teflon	2.1	0.0005	50 to 70 GHz	Unpublished report
Eccofoam PS*	1.02 to 2.0	0.0004	10 GHz	26

* This is a CO_2 foamed polystyrene whose density can be adjusted to yield any dielectric constant in the range specified. It is not clear how the Rayleigh scattering by the CO_2 bubbles will affect the loss tangent at frequencies higher than 10 GHz.

since, under these conditions, the difference between the phases of r and t is $\pm 90°$ at all frequencies as can be deduced from eqs. (8) through (10).

The dielectric materials employed in our solutions will be chosen from those given in Table III. Because of their low loss, these materials are suitable for use at millimeter wavelengths. Other suitable dielectrics can be found in Refs. 24 and 25.

9.1 Symmetric three-layer, Case 1A solution ($\epsilon_1, \epsilon_2, \epsilon_1$)

With $R = 0.5$, eqs. (33) and (34) require that $1.64 \leqq \epsilon_1 < 1.71$ and $\epsilon_2 \geqq 15.74$ for $\theta = 45°$; $1.46 \leqq \epsilon_1 < 1.5$ and $\epsilon_2 \geqq 12.34$ for $\theta = 60°$; or $1.23 \leqq \epsilon_1 < 1.26$ and $\epsilon_2 \geqq 8.90$ for $\theta = 75°$. It is difficult to realize a solution at millimeter wavelengths for θ less than about 60° because of the high required value ϵ_2. However, for $\theta \approx 75°$, the value of ϵ_2 can be realized by Kearfott Alumina ($\epsilon = 9.4$). The corresponding low value of ϵ_1 can be obtained by using Eccofoam PS, or foamed rubber such as Buna S Rubber whose dielectric constant can be extrapolated to be 1.26 at 50 GHz.[25] Thus, with $\epsilon_1 = 1.26$ and $\epsilon_2 = 9.4$, eq. (31) gives $\theta = 73.43°$, eqs. (4a) and (30) give $d_1/\lambda_o = 0.428 + m_1 \times 0.856$, and eqs. (4a) and (31) give $d_2/\lambda_o = 0.078 + m_2 \times 0.172$, where m_1 and m_2 are integers.

The frequency response of this solution is given in Fig. 4 for $m_1 = m_2 = 0$. The large required value of θ makes this solution undesirable.

9.2 Symmetric three-layer, Case 2A solution ($\epsilon_1, \epsilon_2, \epsilon_1$)

With $R = 0.5$, eq. (46) requires that $\theta > 65.53°$; and eqs. (40) and (42) through (44) require that $\epsilon_1 > 4.61$ and $\epsilon_2 = \epsilon_1^2 > 21.23$ for $\theta = 70°$; $\epsilon_1 > 2.09$ and $\epsilon_2 = \epsilon_1^2 > 4.38$ for $\theta = 75°$; and $\epsilon_1 > 1.58$

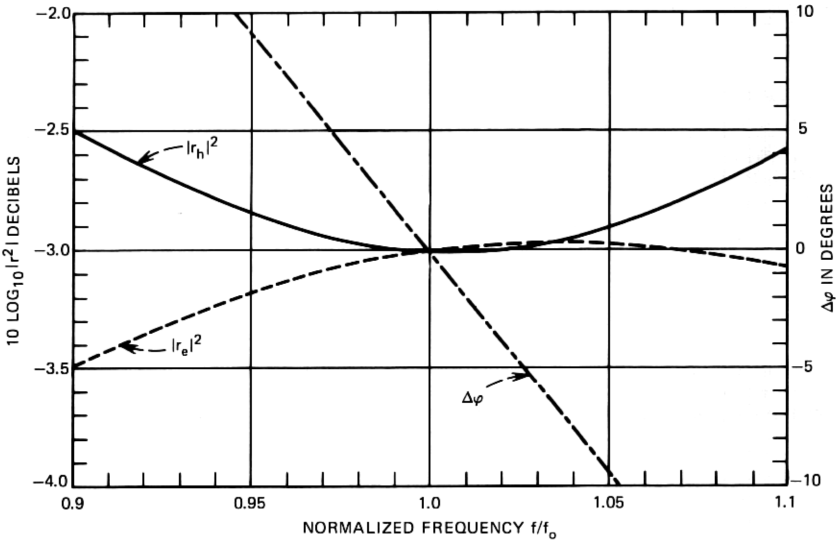

Fig. 4—Frequency response of the solution in 9.1 [three-layer, Type 1A solution with $\theta = 73.43°$, $\epsilon_1 = 1.26$ (Buna S Rubber), $\epsilon_2 = 9.4$ (Kearfott Alumina), $d_1/\lambda_0 = 0.428$, and $d_2/\lambda_o = 0.078$].

and $\epsilon_2 = \epsilon_1^2 > 2.49$ for $\theta = 78°$. In the last case, the solution can be realized by Eccofoam PS ($\epsilon = 1.59$) and polystyrene ($\epsilon = 2.54$). Just as in the solution in Section 9.1, the required large value of θ makes this solution undesirable. For symmetric multilayer, Type-2A solutions with more than three layers of two dielectrics, eq. (86) shows that θ is still required to exceed 65.53°.

9.3 *Symmetric five-layer, Case 1A solution* ($\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_2, \epsilon_1$)

For $R = 0.5$ and $l = 2$, i.e., n = five layers, eqs. (79), (80), and (81) can be satisfied with $\epsilon_1 = \epsilon_3 = 2.1$ (Teflon), $\epsilon_2 = 9.4$ (Kearfott Alumina), and $\theta = 46.91°$, which is a convenient angle of incidence. Further, eqs. (4a) and (67) give $d_1/\lambda_o = 0.200 + m_1 \times 0.399$ and $d_2/\lambda_o = 0.084 + m_2 \times 0.168$, and eqs. (4a) and (80) give $d_3/\lambda_o = 0.103 + m_3 \times 0.399$, where m_1, m_2, and m_3 are integers. The frequency response of this solution is given in Fig. 5 for $m_1 = m_2 = m_3 = 0$.

9.4 *Symmetric seven-layer, Case 1A solution* ($\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_3, \epsilon_2, \epsilon_1$)

For $R = 0.5$ and $l = 3$, i.e., n = seven layers, eqs. (79), (80), and (81) can be satisfied with $\epsilon_1 = \epsilon_3 = 2.54$ (polystyrene), $\epsilon_2 = \epsilon_4 = 9.4$ (Kearfott Alumina), and $\theta = 47.52°$. Further, eqs. (4a) and (67) give $d_1/\lambda_o = d_3/\lambda_o = 0.177 + m_1 \times 0.354$ and $d_2/\lambda_o = 0.084 + m_2 \times 0.168$, and eqs. (4a) and (80) give $d_4/\lambda_o = 0.026 + m_4 \times 0.168$, where m_1, m_2, and m_4 are integers. This solution has a narrower bandwidth than the solution in 9.3.

9.5 Symmetric seven-layer, Case 1A solution obtained by cascading two identical, symmetric three-layer, Case 1A solutions (ϵ_1, ϵ_2, ϵ_1, 1.0, ϵ_1, ϵ_2, ϵ_1)

With $R = 0.5$, eq. (95) shows that R_x, the power reflection coefficient of each cascaded three-layer section, should not be less than 0.1716; thus let us choose $R_x = 0.1716$. In this case, eqs. (31), (33), and (34) can be satisfied with $\epsilon_1 = 1.6$ (Eccofoam PS), $\epsilon_2 = 9.4$ (Kearfott Alumina), and $\theta = 45.30°$. Further, eqs. (4a) and (30) give $d_1/\lambda_o = 0.239 + m_1 \times 0.478$ and eqs. (4a) and (32) give $d_2/\lambda_o = 0.038 + m_2 \times 0.168$.

To find the width d of the air separation, one should first find the phase ϕ_x of the reflection coefficient of each cascaded three-layer section. By substituting the values of ϵ_1, ϵ_2, d_1, d_2, and θ obtained above in eqs. (2) through (10), one finds that $\phi_x = 43.59°$. With $R = 0.5$ and $R_x = 0.1716$, eq. (93) gives $\psi = \pm 90°$ (mod 180°), and hence, the electrical length of the air spacing between the two three-layer sections is found from eq. (92) to be $\phi = 133.59° + m \times 180°$, where m is an integer. Thus, from eq. (4a) with $\epsilon_r = \kappa_i = 1$, $d/\lambda_o = 0.528$

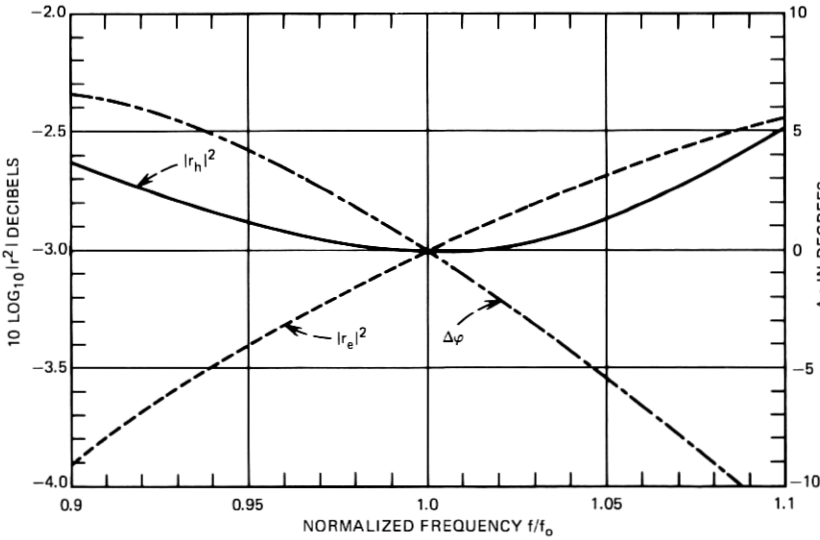

Fig. 5—Frequency response of the solution in 9.3 [five-layer, Type 1A solution with $\theta = 46.91°$, $\epsilon_1 = \epsilon_3 = 2.1$ (Teflon), $\epsilon_2 = 9.4$ (Kearfott Alumina), $d_1/\lambda_o = 0.200$, $d_2/\lambda_o = 0.084$, and $d_3/\lambda_o = 0.103$].

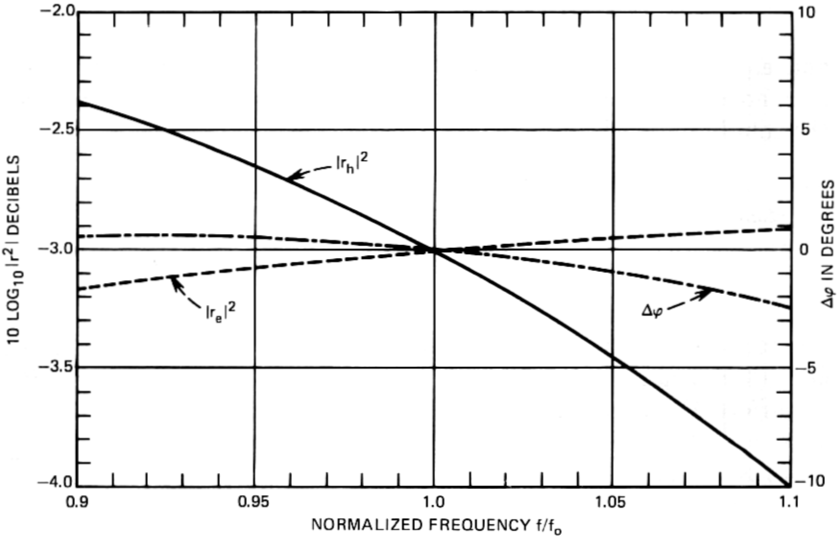

Fig. 6—Frequency response of the solution in 9.6 [three-layer, Type 2C prism solution with $\theta = 45°$, $\epsilon_r = 2.54$ (polystyrene), $\epsilon_1 = 1.59$ (Eccofoam PS), $\epsilon_2 = 1$, $d_1/\lambda_0 = 0.439$, and $d_2/\lambda_o = 0.236$].

$+ m \times 0.711$. This solution has a narrower bandwidth than the solution in 9.3.

9.6 Symmetric three-layer, Case 2C prism solution (ϵ_r, ϵ_1, 1.0, ϵ_1, ϵ_r)

Choosing $\theta = 45°$ and $\epsilon_2 = 1.0$, eqs. (16), (40), (50) and (51) give $1.414 < \epsilon_1 < 2$ and $2 < \epsilon_r = \epsilon_1^2 < 4$. Thus, the solution can be obtained with $\epsilon_1 = 1.59$ (Eccofoam PS) and $\epsilon_r = \epsilon_1^2 = 2.54$ (polystyrene). Further, eqs. (4a) and (39) give $d_1/\lambda_o = 0.439 + m \times 0.879$, where m is an integer, and with $R = 0.5$, eqs. (4a), (4b), and (49) give $d_2/\lambda_o = 0.236$. The frequency response of this solution is given in Fig. 6. for $m = 0$ and under the assumption that the outer surfaces of the prisms are perfectly matched. As mentioned at the end of Section V, the matching can be accomplished by using a quarter-wave layer having a dielectric constant $\epsilon_1 = 1.59$ (Eccofoam PS).

X. ACKNOWLEDGMENT

The author wishes to thank J. A. Arnaud for useful suggestions.

REFERENCES

1. M. Born and E. Wolf, *Principles of Optics*, Oxford: Pergamon Press, 1965.
2. A. F. Harvey, "Optical Techniques at Microwave Frequencies," Proc. IEE, *B106* (March 1959), pp. 141–157.
3. R. Trembley and A. Boivin, "Concepts and Techniques of Microwave Optics," Applied Optics, *5*, No. 2 (February 1966), pp. 249–278.
4. R. Levy, "Directional Couplers," in *Advances in Microwaves, 1*, L. Young, Ed., New York: Academic Press, 1966, pp. 196–205.

5. J. J. Taub, H. J. Hindin, O. F. Hinckelmann, and M. L. Wright, "Submillimeter Components Using Oversized Quasi-Optical Waveguide," IEEE Trans. Microwave Theory Tech., *MTT-11*, No. 5 (September 1963), pp. 338–350.
6. M. Koyama and S. Shimada, "The Quasi-Optical Filters Used for the Domestic Satellite Communication System," Electr. and Commun. in Japan, *56-B*, No. 3 (March 1973), pp. 74–81.
7. E. A. J. Marcatili and D. L. Disbee, "Band-Splitting Filter," B.S.T.J., *40*, No. 1 (January 1961), pp. 197–212.
8. P. G. Kard, "On the Elimination of the Doublet Structure of the Transmission Band in a Total-Reflection Light Filter," Optics and Spectroscopy, (Translation of Optika i Spektroskopiia), *VI*, No. 3 (March 1959), pp. 244–246.
9. P. Baumeister, "The Transmission and Degree of Polarization of Quarter-Wave Stacks at Non-Normal Incidence," Optica Acta, *8*, No. 2 (April, 1961), pp. 105–119.
10. K. Rabinovitch and A. Pagis, "Polarization Effects in Multilayer Dielectric Thin Films," Optica Acta, *21*, No. 12 (December 1974), pp. 963–980.
11. R. B. Adler, L. J. Chu, and R. M. Fano, *Electromagnetic Energy Transmission and Radiation*, New York: John Wiley, 1965, pp. 354–362.
12. J. R. Wait, *Electromagnetic Waves in Stratified Media*, New York: Pergamon Press, 1962, pp. 10–15.
13. P. Leurgans and A. F. Turner, "Frustrated Total Reflection Interference Filters," J. Opt. Soc. Am., *37*, No. 12 (December 1947), p. 983.
14. M. Born and E. Wolf, Ref. 1, pp. 350–351.
15. H. D. Raker and G. R. Valenzuela, "A Double-Prism Attenuator for Millimeter Waves," IRE Trans. Microwave Theory Tech., (Correspondence), *MTT-10*, No. 5 (September 1962), pp. 392–393.
16. H. J. Hindin and J. J. Taub, "Oversized Waveguide Directional Coupler," IRE Trans. Microwave Theory Tech. (Correspondence), *MTT-10*, No. 5 (September 1962), pp. 394–395.
17. T. Makimoto and T. Sueta, "Prism Directional Couplers Making Use of Brewster Angle Transmission," *Memoirs of the Institute of Scientific and Industrial Research*, Osaka University, Japan, *21*, 1964.
18. T. Sueta, N. Kumagai, and S. Kurazono, "Wideband Quasi-Optical Prism Components," Elect. and Commun. in Japan, *49*, No. 3 (March 1966), pp. 100–106.
19. R. G. Fellers and J. Taylor, "Internal Reflections in Dielectric Prisms," IEEE Trans. Microwave Theory Tech., *MTT-12*, No. 6 (November 1964), pp. 584–587.
20. R. E. Collin and J. Brown, "The Design of Quarter-Wave Matching Layers for Dielectric Surfaces," Proc. IEE, *103C* (September 1955), pp. 153–158.
21. T. Morita and S. B. Cohn, "Microwave Lens Matching by Simulated Quarter-Wave Transformers," IRE Trans. Antennas and Propagation, *AP-4*, No. 1 (January 1965), pp. 33–39.
22. E. M. T. Jones and S. B. Cohn, "Surface Matching of Dielectric Lenses," J. of Appl. Phys., *26*, No. 4 (April 1955), pp. 452–457.
23. M. Born and E. Wolf, Ref. 1, pp. 323–329.
24. Lab. for Insulation Research, M.I.T., *Tables of Dielectric Materials*, *5*, April 1957.
25. A. R. Von Hipple, ed., *Dielectric Materials and Applications*, M.I.T. Press and John Wiley, New York, 1954.
26. Emerson and Cuming, Inc., Dielectric Materials Division, "Eccofoam PS—Low-Loss Adjustable Dielectric Constant Foam," Technical Bulletin No. 6-2-4, Canton, Massachusetts, June 15, 1970.