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On the Angle Between Two Fourier Subspaces

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Examining an approximation inspired by equalization theory, we consider the minimum angle Ω_N between the subspaces of Hilbert space generated by the sequences $\{e^{ik\omega}\}^N_{-N}$ and $\{e^{ik\omega}\}_{|k|>N}$. Here $\omega \in [-\pi,\pi]$ and the inner product for the Hilbert space involve a positive, bounded weight function $r(\omega)$. The finite Toeplitz matrices R and Γ generated by $r(\omega)$ and $1/r(\omega)$, respectively, play a crucial role, and, in fact, $\sin^2\Omega_N$ is the reciprocal of the largest eigenvalue of $R\Gamma$. In general, $\sin^2\Omega_N$ is shown to be bounded away from unity as N becomes large. The geometry of the problem enables us to give some results concerning the product matrix $R\Gamma$, which, out of the present context, may seem surprising.

I. INTRODUCTION AND SUMMARY

Let H be the Hilbert space of square-integrable functions on $[-\pi,\pi]$ with an inner product† given by

$$(f,g)_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(\omega)g(\omega)r(\omega)d\omega, \tag{1}$$

where the weight function $r(\omega)$ is bounded and strictly positive; i.e.,

$$0 < r \leqq r(\omega) \leqq R. \tag{2}$$

We call a Fourier subspace of H any subspace generated by a finite or infinite collection of functions of the form $e^{in\omega}$, n an integer. In particular, we shall be interested in the Fourier subspaces [relative to the metric $r(\omega)$]:

$$F_N = \left\{\sum_{-N}^{N} f_n e^{in\omega}\right\}_r$$
$$G_N = \left\{\sum_{|n|>N} g_n e^{in\omega}\right\}_r \tag{3}$$

† The subscript r, as in (1), will be used when we wish to emphasize that the weight function $r(\omega)$ is being used. No subscript will refer to an arbitrary inner product, while the case $r(\omega) = 2\pi$ will be called the "usual" metric. The usual inner product will be written without a subscript as well.

for $N \geqq 0$.† If $r(\omega)$ is constant, then the subspaces F_N and G_N are orthogonal. We will be concerned with the minimum angle between F_N and G_N for a general weight function satisfying (2), and with the limiting behavior of this angle as $N \to \infty$. (The concept of the angle between subspaces is not new to the engineering literature. See for example Ref. 1.)

The main results of our investigation are stated in terms of two finite Toeplitz matrices, R and Γ, which are generated by the weight functions $r(\omega)$ and $g(\omega) = 1/r(\omega)$, respectively [see eqs. (30) and (33) for precise definitions]. We also need the Fourier coefficients r_n and g_n of $r(\omega)$ and $g(\omega)$. Then, we show

$$(i)\quad \sin^2\Omega_N = \frac{1}{\text{largest eigenvalue of } R\Gamma}.$$

$$(ii)\quad \lim_{N\to\infty} \sin^2\Omega_N \leqq 2\left[1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\omega)d\omega \times \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\omega}{r(\omega)}\right]^{-1} < 1.$$

(iii) All eigenvalues of $R\Gamma$ are $\geqq 1$.

$$(iv)\quad \sum_{-\infty}^{\infty} |n| r_n g_n^* < 0.$$

This is not a trivial inequality in the sense that $\sum_{-\infty}^{\infty} r_n g_n^* = 1$ is.

The case $r(\omega) = 1 + a \cos \omega$ is solved exactly in Section V, showing that the obvious bound $\sin^2\Omega_N \geqq r_{\min}/r_{\max}$ is often loose. A better bound, still involving only this ratio, is given in (68).

In somewhat general terms, this problem arose in the mean-square equalization theory of data transmission, where the question is one of bounding the effect of replacing tap weight values by certain Fourier coefficients. To be specific, let us ignore the effects of noise and note that the job of the equalizer is to invert the Nyquist equivalent channel. That is, if we had an infinite number of taps at our disposal, we would take the transfer function of the equalizer $C_\infty(\omega)$ to be

$$C_\infty(\omega) = \frac{1}{X(\omega)} = \sum_{-\infty}^{\infty} \epsilon_n e^{in\omega}.$$

The equalizer transfer function $C_N(\ )$ when only the usual $(2N + 1)$ taps are available can always be written as

$$C_N(\omega) = \frac{1}{X(\omega)} - \sum_{-N}^{N} \delta_n e^{in\omega} - \sum_{|n|>N} \epsilon_n e^{in\omega}.$$

In the above expression δ_n, $|n| < N$ are "corrections" to the Fourier coefficients ϵ_n, $|n| < N$. The mean-square error resulting from the

† Any function of the form (3) is in H if and only if the associated coefficient sequence is square-summable.

equalizer $C_N(\omega)$ can then be shown to be given by

$$\frac{1}{2\pi}\sum_{-\pi}^{\pi}|X(\omega)C_N(\omega)-1|^2d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{-N}^{N}\delta_k e^{ik\omega}+\sum_{|k|>N}\epsilon_k e^{ik\omega}\right|^2|X(\omega)|^2d\omega.$$

The minimum mean-square error $E^2{}_{\min}$ is the minimum of the above expression over the δ_k. Now we can imagine taking the (fixed) vector

$$\sum_{|k|>N}\epsilon_k e^{ik\omega}$$

and decomposing into a vector in the space F_N and one perpendicular to it. The part in F_N can be "subtracted off" by the choice of δ's, leaving the remainder. The fraction subtracted off can never be greater than $\cos^2\Omega_N$, where Ω_N is the angle between the two subspaces F_N and G_N when $X_{\text{eq}}(\omega)^2$ is used as the weight function $r(\omega)$ for inner products. Thus,

$$\sin^2\Omega_N\times\|\epsilon_N\|_r^2\leqq E_{\min}^2\leqq\|\epsilon_N\|_r^2, \tag{4}$$

where

$$\|\epsilon_N\|_r^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{|k|>N}\epsilon_k e^{ik\omega}\right|^2|X_{eq}(\omega)|^22d\omega.$$

The point of replacing the exact tap values of the finite equalizer by Fourier coefficients is not to replace one calculation by another. Rather, it is to supplement calculation by insight, since much is known about properties of Fourier coefficients. This will be done for a specific equalization problem involving timing recovery for finite equalizers in a later work.

II. WARMUP EXERCISES

The angle θ between two fixed vectors, f and g, is defined by

$$\theta=\cos^{-1}\frac{Re(f,g)}{\|f\|\ \|g\|},\quad \theta\in[0,\pi]. \tag{5}$$

If f and g are restricted to be in subspaces F and G, respectively, the infimum of (5) (call it Ω) over all f and g (so restricted) is called the angle between the two subspaces. We easily see that

$$\|f-g\|^2\geqq\|f\|^2+\|g\|^2-2\|f\|\times\|g\|\cos\Omega, \tag{6}$$

and thus by minimizing the right member of (6) with respect to the norm of f, we have

$$\inf_f\|f-g\|^2\geqq\sin^2\Omega\|g\|^2. \tag{7}$$

In fact, we can also calculate $\sin^2\Omega$ *via* the formula

$$\sin^2\Omega = \inf_g \inf_f \frac{\|f - g\|^2}{\|g\|^2}. \tag{8}$$

When (2) holds, the infimum angle between our subspaces F_N and G_N [given by (3)] is actually attained and its value is strictly positive. In fact, it follows from an application of a theorem by Paley and Wiener[2] that the two sequences (usual metric)

$$\begin{aligned} \{\phi_n\} &= \left\{\sqrt{\frac{r(\omega)}{2\pi}}\, e^{in\omega}\right\} \\ \{\psi_n\} &= \left\{\frac{1}{\sqrt{2\pi r(\omega)}}\, e^{in\omega}\right\} \end{aligned} \tag{9}$$

form a complete biorthogonal pair, i.e.,

$$(\phi_n,\psi_m) = \delta_{nm} \tag{10}$$

and

$$h = \sum_{-\infty}^{\infty} (\phi_n,h)\psi_n = \sum_{-\infty}^{\infty} (\psi_n,h)\phi_n \tag{11}$$

for any h in L_2. Thus, either sequence in (9) forms a basis for L_2, or, equivalently, $\{e^{in\omega}\}_r$ forms a basis for H [with weight function $r(\omega)$]. Now if $f \in F_N$, $g \in G_N$,

$$\inf_g \|f - g\|_r$$

is attained when g is the orthogonal projection of f on G_N, and is a continuous function of the finite dimensional f. Therefore,

$$\inf_f \left[\inf_g \frac{\|f - g\|_r^2}{\|f\|_r^2}\right]$$

is attained, since we may restrict $\|f\|_r = 1$ and thus are minimizing over a compact set. The basis property of $\{e^{in\omega}\}_r$ in H assures that the minimum is not zero.

There are several ways to get at the minimum angle Ω_N between F_N and G_N. We shall begin by using (8) and the calculus of variations.

However, before we begin to work on this, let us review an old problem of linear prediction (really, linear interpolation) theory. We are required to find the minimum value of

$$E^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|1 - \sum_{m\neq 0} a_m e^{im\omega}\right|^2 r(\omega)d\omega \tag{12}$$

over all l_2 sequences $\{a_m\}$, under assumption (2) for $r(\omega)$. We let

$$a(\omega) = \sum_{m\neq 0} a_m e^{im\omega}$$

be any element of $L_2(-\pi,\pi)$ which has zero for its zeroth Fourier coefficient. We then have

$$E^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |1 - a(\omega)|^2 r(\omega) d\omega. \tag{13}$$

The calculus of variations yields

$$\int \delta a^*[1 - a(\omega)]r(\omega)d\omega = 0 \tag{14}$$

or

$$\int e^{in\omega}[1 - a(\omega)]r(\omega)d\omega = 0, \quad n \neq 0. \tag{15}$$

Thus,

$$[1 - a(\omega)]r(\omega) = \text{const} = k. \tag{16}$$

From (13) and (16), on the one hand, we have

$$E_{\min}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{k^2}{r^2(\omega)} r(\omega)d\omega = \frac{k^2}{2\pi}\int_{-\pi}^{\pi} \frac{d\omega}{r(\omega)} \tag{17}$$

and, on the other,

$$E_{\min}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega(1 - a^*)[(1 - a)r] = \frac{k}{2\pi}\int_{-\pi}^{\pi} (1 - a^*)d\omega = k, \tag{18}$$

since a^* has no $m = 0$ term. Equating the results of (17) and (18) enables us to solve for k [note that $k = 0$ must be excluded under (2)], yielding

$$E_{\min}^2 = \frac{1}{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \frac{d\omega}{r(\omega)}} = k. \tag{19}$$

If $r(\omega)$ has a zero somewhere in such a manner that $\int 1/r = \infty$, then (19) says $E_{\min}^2 = 0$. This turns out to be the correct conclusion for the infimum, but this infimum is never attained. As is well known, the calculus of variations can only be applied if the infimum is attained. In fact, in the present problem, if we set $k = 0$ in (16), we would conclude that there is an l_2 sequence $\{a_m\}$ such that

$$1 - \sum_{m\neq 0} a_m e^{im\omega} = 0 \quad \text{a.e.}, \tag{20}$$

which obviously cannot be.

Another way to do this problem is to use the biorthogonal sequences (9). We note that all vectors of the type

$$\sum_{m\neq 0} a_m \phi_m \tag{21}$$

form the subspace orthogonal to the vector ψ_0. Hence, $E^2_{\min}$ must simply be the squared norm of the projection of ϕ_0 onto ψ_0. This is (in the usual norm)

$$\|\phi_0\|^2\cos^2(\phi_0,\psi_0) = \|\phi_0\|^2 \frac{|(\phi_0,\psi_0)|^2}{\|\phi_0\|^2\|\psi_0\|^2}$$

$$= \frac{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{r(\omega)} \times \frac{1}{\sqrt{r(\omega)}}\,d\omega\right)^2}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\omega}{r(\omega)}} = \frac{1}{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\omega}{r(\omega)}}. \qquad (22)$$

III. FINDING THE MINIMUM ANGLE

We proceed with the calculus-of-variations approach to finding $\sin^2\Omega_N$ via (8). Let

$$f(\omega) = \sum_{-N}^{N} f_k e^{ik\omega} \in F_N$$

$$g(\omega) \in G_N. \qquad (23)$$

Then, if we vary g^* in $\|f - g\|_r^2$, we obtain

$$\int \delta g^*(w)[f(\omega) - g(\omega)]r(\omega)d\omega = 0 \qquad (24)$$

for all allowed variations. Thus, (24) means $[f - g]r \in F_N$, or, in other words,

$$[f(\omega) - g(\omega)]r(\omega) = \sum_{-N}^{N} b_k e^{ik\omega} \equiv b(\omega) \qquad (25)$$

for some numbers b_k. As in Section II, (25) permits us to write two expressions for $\min\|f - g\|_r^2$. They are

$$\min_g \|f - g\|_r^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|b(\omega)|^2}{r(\omega)}d\omega \qquad (26)$$

and

$$\min_g \|f - g\|_r^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega f^*(\omega)b(\omega) = \mathbf{f}^*\cdot\mathbf{b}. \qquad (27)$$

The vector notation in (27) refers to a row of $(2N + 1)$ numbers. Letting $\mathbf{b} = \mathrm{k}(\mathbf{f} + \mathbf{b}_\perp)$, $\mathbf{f}^*\cdot\mathbf{b}_\perp = 0$, we may equate (26) and (27), solve for k, and obtain

$$\min_g \|f - g\|_r^2 = \frac{\left(\sum_{-N}^{N}|f_k|^2\right)^2}{\min\limits_{\substack{\gamma \\ \mathbf{f}\cdot\gamma = 0}} \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|\sum_{-N}^{N} f_k e^{ik\omega} + \sum_{-N}^{N}\gamma_k e^{ik\omega}\right|^2}{r(\omega)}d\omega}. \qquad (28)$$

Thus, from (8) and (28)

$$\sin^2\Omega_N = \min_{\substack{\mathbf{u} \\ \sum_{-N}^{N} |u_i|^2 = 1}} \frac{1}{\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{-N}^{N} u_k e^{ik\omega}\right|^2 r(\omega)d\omega \times \min_{\substack{\mathbf{w} \\ \mathbf{w}\cdot\mathbf{u}=0}} \frac{1}{2\pi}\int_{-\pi}^{\pi} d\omega \frac{\left|\sum_{-N}^{N}(u_k + w_k)e^{ik\omega}\right|^2}{r(\omega)}\right]}. \tag{29}$$

If we introduce the Toeplitz matrix

$$\Gamma_{nm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\omega}\frac{1}{r(\omega)}d\omega, \quad |n|, \quad |m| \leqq N, \tag{30}$$

the second term in the denominator of (29) has the form

$$u^+\Gamma u + w^+\Gamma w + 2u^+\Gamma w. \tag{31}$$

Expression (31) may be minimized over the appropriate $\mathbf{w}$ using a Lagrange multiplier, yielding

$$\frac{1}{u^+\Gamma^{-1}u}$$

Thus,

$$\sin^2\Omega_N = \min_{\mathbf{u}} \frac{u^+\Gamma^{-1}u}{u^+Ru}, \tag{32}$$

where R is the Toeplitz matrix corresponding to $r(\omega)$, i.e.,

$$R_{nm} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\omega}\, r(\omega)d\omega, \quad |n|, \quad |m| \leqq N. \tag{33}$$

Finally, the minimization of (32) yields

Theorem I. Let the matrices R and Γ be defined as in (30) and (33). Then the minimum angle Ω_N between the Fourier subspaces F_N and G_N, defined in (3), is

$$\sin^2\Omega_N = \frac{1}{\text{largest eigenvalue of } R\Gamma} \tag{34}$$

This theorem implies that Ω_N is invariant under the replacement $r(\omega) \to 1/r(\omega)$.

Since similarity transformations preserve eigenvalues, we note that the eigenvalues of $R\Gamma$ are the same as those of $\sqrt{\Gamma}\,R\sqrt{\Gamma}$, which is, by (2), (30), and (33), a strictly positive definite Hermitian matrix.

Before exploring consequences of (34), we shall rederive it from a geometric point of view. Let $\{\phi_i\}_1^\infty$ and $\{\psi_i\}_1^\infty$ be complete biorthogonal sequences of vectors for a Hilbert space. Let

$$V = \{\phi_i\}_1^N, \quad W = \{\phi_i\}_{N+1}^\infty \qquad T = \{\psi_i\}_1^N \tag{35}$$

be subspaces generated by the indicated vectors. Note that the orthogonal complement of W, $W_\perp$, is given by $W_\perp = T$. Also note that our problem is equivalent to that of finding the minimum angle between V and W. If $v \in V$, and α is the angle between v and W (i.e., the angle between v and its projection on W), and β is the angle between v and $W_\perp$, we have†

$$\alpha + \beta = \frac{\pi}{2} \cdot \tag{36}$$

Thus, the minimum angle between V and W (call it Ω) is the complement of the maximum angle between V and T, called θ_M. Thus, we have

$$\sin^2\Omega = \cos^2\theta_M. \tag{37}$$

The spaces V and T both have dimension N here.

Let P represent the orthogonal projection operator onto T, and Q the orthogonal projection operator onto V. It can be shown that if $V \in V$ is a vector in V which attains the minimum or maximum angle between V and T, we must have ‡

$$QPv = \lambda v, \tag{38}$$

where $\lambda \geqq 0$ is the square of the cosine of the indicated angle. We shall see shortly that (34) is a form of (38) when we represent P and Q by matrices that are representations of the restrictions of P to V and Q to T.

We begin by deriving these matrix representations. For general biorthogonal sequences $\{\phi_i\}$, $\{\psi_i\}$, let

$$R_{nk} = (\phi_n, \phi_k) \qquad \Gamma_{nk} = (\psi_n, \psi_k) \qquad n,k = 1,2,\cdots,N. \tag{39}$$

† If we call the projection of v on W by v_1, then $v - v_1$ is the projection of v on $W_\perp$. Since v, v_1, and $v - v_1$ all lie in a plane, (36) follows immediately from a simple diagram depicting these three vectors.

‡ Let α be any vector in the space such that $Q\alpha \neq 0$. Then if θ is the angle between $Q\alpha \in V$ and W, we have $\cos^2\theta = \|PQ\alpha\|^2/\|Q\alpha\|^2 = (\alpha,QPQ\alpha)/(\alpha,Q\alpha)$. Vectors α which yield stationary values of this ratio of quadratic forms can be obtained by differentiating $(\alpha,QPQ\alpha)$ holding $(\alpha,Q\alpha)$ constant (via a Lagrange multiplier λ). This procedure yields (38) upon setting $Q\alpha = v$.

Any vector x can be written uniquely as a vector in V plus a vector in the orthogonal complement of V, i.e.,

$$x = \sum_{1}^{N} a_i \phi_i + \sum_{N+1}^{\infty} b_i \psi_i. \tag{40}$$

If we form the inner product of (40) with $\phi_j, j = 1,2, \cdots, N$, we can calculate

$$a_k = \sum_{l=1}^{N} (R^{-1})_{kl} (\phi_l, x). \tag{41}$$

Thus, given any x, its projection onto V is simply

$$\sum_{1}^{N} a_i \phi_i \tag{42}$$

with a_i given by (41). Similarly, the projection of x onto T is

$$\sum_{1}^{N} b_i \psi_i \tag{43}$$

with

$$b_k = \sum_{l=1}^{N} (\Gamma^{-1})_{kl} (\psi_l, x) \tag{44}$$

Hence, if we start with any vector $v \in V$,

$$v = \sum_{1}^{N} v_i \phi_i, \tag{45}$$

the result of projecting it onto T and then projecting this vector back to V is another vector $v'' \in V$ with components v''_m given by

$$v''_m = \sum_{i=1}^{N} (R^{-1}\Gamma^{-1})_{mi} v_i. \tag{46}$$

Hence, the operator equation (38) becomes the $N \times N$ matrix equation

$$(\Gamma R)^{-1} v = \lambda v. \tag{47}$$

Equation (34) is thus rederived, after an appropriate relabeling of indices.

Since the reciprocals of the largest and smallest eigenvalues of $R\Gamma$ have interpretations as squares of cosines of angles, we have

Theorem II. Let matrices R and Γ be defined as in (30), (33). Then all eigenvalues of $R\Gamma$ are ≥ 1.

IV. IMPLICATIONS CONCERNING $\mathrm{SIN}^2\Omega_N$

From (30) and (33), we see that the matrix elements of $r\Gamma$ are given in terms of the Fourier coefficients r_j and g_j of $r(\omega)$ and $g(\omega) = 1/r(\omega)$.

Thus, (33) reads $R_{nm} = r_{m-n}$. We see that, in this notation

$$(R\Gamma)_{nk} = \sum_{m=-N}^{N} g_{k-m} r_{m-n}, \; |n|, |k| \leqq N. \tag{48}$$

This can also be written

$$(R\Gamma)_{nk} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} e^{i(n\omega - k\omega')} \frac{r(\omega)}{r(\omega')} \frac{\sin \frac{2N+1}{2}(\omega - \omega')}{\sin \frac{1}{2}(\omega - \omega')} d\omega d\omega'. \tag{49}$$

Equation (49) follows easily from (36), (33), and the identity

$$\sum_{m=-N}^{N} e^{im\psi} = \frac{\sin \frac{2N+1}{2} \psi}{\sin \frac{\psi}{2}}.$$

If the largest eigenvalue of $R\Gamma$ approaches 1 as N becomes large, then $\sin^2 \Omega_N \to 1$, and the subspaces F_N and G_N described in (3) eventually become orthogonal. Equivalently, we have seen that the question becomes the following: Does the largest angle between the subspaces (with the usual metric)

$$F_N = \left\{ e^{in\omega} \sqrt{\frac{r(\omega)}{2\pi}} \right\}_{-N}^{N}$$

and

$$G_N = \left\{ e^{in\omega} \frac{1}{\sqrt{2\pi r(\omega)}} \right\}_{-N}^{N}$$

approach zero? If we *set* $N = \infty$, the two generated spaces are identical† (all of L_2), so from this point of view it comes as a surprise that the limiting angle between F_N and G_N is bounded away from zero.

Let us assume that $r(\omega)$ has only a finite number of Fourier coefficients; that is, assume $r_j = 0$ if $|j| > k$. For this case, the reader may verify, using either (48) or (49), that the $(2N + 1) \times (2N + 1)$ matrix $R\Gamma$ has the form (once $N \geqq k$)‡

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† F_N is never G_N for finite N unless $r(\omega) = \text{const}$. This follows from using (11) to show that you cannot expand each ϕ_n, $n \leqq |N|$ in terms of the ψ_k, $|k| < N$.

‡ To see this, let us evaluate $(R\Gamma)_{ab}$ from (48). If $|a| < (N - k)$, the summation in (48) may be extended from $N = -\infty$ to $N = +\infty$, since $r_j = 0$ if $j > k$. Using the duality between l_2 and L_2, the resulting sum is then $1/2\pi \int_{-\pi}^{\pi} r(\omega) exp(ia\omega)][g(\omega) exp(ib\omega)]^* \, d\omega = \delta_{ab}$, since $g(\omega) = 1/r(\omega)$.

$$
\begin{bmatrix}
A & X & X & X & \cdot & \cdot & \cdot & X & B \\
 & 1 & & & & & & & \\
 & & 1 & & & & \bigcirc & & \\
 & \bigcirc & & 1 & & & & & \\
 & & & & \cdot & & & & \\
 & & & & & \cdot & & & \\
 & & & & & & \cdot & & \\
 & & & & & & & 1 & \\
C & X & X & X & & & & X & D
\end{bmatrix}
\tag{50}
$$

That is, the first k rows and the last k rows are nonvanishing. The remaining $2(N-k)+1$ diagonal elements are exactly unity, while all other matrix elements vanish. The four $k \times k$ matrices in the corners are singled out for special attention and are labeled A, B, C, D. The capital X's in the first and last rows are inserted only to indicate that elements in the first and last k rows are not vanishing, in general.

As an example, we write the elements of A explicitly, labeling the elements of **A** by a_{rs}, $r, s = 0,1, \cdots, k-1$; i.e., $a_{00} = (R\Gamma)_{-N,-N}$, etc. Then

$$
a_{rs} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sum_{l=-r}^{k} f_l e^{il\omega} e^{-is\omega}}{f(\omega)} \, d\omega. \tag{51}
$$

The elements of D can be determined from A using the property

$$
(R\Gamma)^*_{m,l} = (R\Gamma)_{-m,-l}. \tag{52}
$$

It is important to note that (for $N \geqq k$) the elements of A and D do not depend on N. However those of B and C do. For example, the upper right corner of B is the element $(R\Gamma)_{-N,N}$, given from (48) as

$$
(R\Gamma)_{-N,N} = \sum_{l=0}^{k} r_l g_{2N-l}. \tag{53}
$$

Not only does (53) depend on N, but, by the Riemann-Lebesgue lemma, $g_{2N-l} \to 0$ as N increases for l bounded. As N increases, all elements of B and C similarly vanish.

We now look further at the problem of calculating the eigenvalues of (50). If one is not an eigenvalue then the matrix $R\Gamma - \lambda I$ has $2(n-k)+1$ diagonal elements $(1-\lambda)$. By multiplying a row in which such an element occurs by the appropriate constant and adding the result to one of the first or last rows, all the elements indicated by "X" in (50) can be made to vanish. Clearly then, the eigenvalues that are not unity are given by the nonunity eigenvalues of the $2k \times 2k$ matrix

$$
K \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{54}
$$

where B and C depend on N. All other eigenvalues of (49) are one. Since B and C vanish in the limit of large N, we have†

$$\lim_{N\to\infty} \lambda_{\max}(N) = \text{largest eigenvalue of } A. \tag{55}$$

The simple fact that F_N is never G_N for finite N implies that the largest eigenvalue of $R\Gamma$, and hence K, is strictly greater than one. Also the fact that all eigenvalues of $R\Gamma$ are greater than or equal to 1 implies $tr\, K = tr\, A + tr\, D > 2k$. But A and D do not depend on N and have the same trace. Hence, $tr\, A > k$, and A has an eigenvalue strictly greater than unity. Thus, from (34) and (55), $\lim \sin^2\Omega_N < 1$.

The above discussion implies the following:‡

Theorem III. If $r(\omega)$ has only a finite number of nonvanishing Fourier components and is not constant, then $\lim_{N\to\infty} \sin^2\Omega_N < 1$.

Theorem IV. Let $1 \geqq r(\omega) > 0$ on $[-\pi,\pi]$ have only a finite number of nonvanishing Fourier coefficients, r_n. Set $g(\omega) = 1/r(\omega)$ and call its Fourier coefficients g_n. Then

$$\sum_{n=-\infty}^{\infty} |n| r_n g_n^* < 0,$$

unless $r(\omega)$ is constant.

Proof. Using (49) for the product $R\Gamma$ and the identity immediately following it, we calculate

$$tr\, R\Gamma - (2N+1) = (2N+1)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{r(\omega)}{r(\omega')}K_N(\omega,\omega')d\omega d\omega' - 1\right], \tag{56}$$

where $K_N(\omega,\omega')$ is the well-known Fejer kernel[3]

$$K_N(\omega,\omega') = \frac{1}{2\pi(2N+1)}\,\frac{\sin^2\frac{2N+1}{2}(\omega-\omega')}{\sin^2\frac{1}{2}(\omega-\omega')}. \tag{57}$$

It has the following property. Let $X(\omega) \in L_2(-\pi,\pi)$, and let

$$\tilde{X}(\omega) = \int_{-\pi}^{\pi} K_N(\omega,\omega')X(\omega')d\omega'. \tag{58}$$

† The $k \times k$ matrices A and D have the same eigenvalues.
‡ The restrictions in Theorem III and Theorem IV to only a finite number of nonvanishing components of $r(\omega)$ is removed in Section V.

Call the Fourier coefficients of $\tilde{X}(\omega)$ and $X(\omega)$, $\tilde{X}_n$ and X_n, respectively. Then

$$\tilde{X}_n = X_n \left[1 - \frac{|n|}{2N+1}\right], \quad |n| \leqq 2N+1 \tag{59}$$

$$= 0 \text{ otherwise.}$$

Thus, if $r(\omega) = \sum_{-k}^{k} r_n e^{in\omega}$,

$$\int_{-\pi}^{\pi} K_N(\omega,\omega') r(\omega) d\omega = r(\omega') - \frac{1}{2N+1} \sum_{-k}^{k} r_n |n| e^{in\omega'} \tag{60}$$

if only $(2N+1) \geqq k$. Substituting (60) into the right member of (56), we obtain

$$\sum_{-k}^{k} r_n |n| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{in\omega'}}{r(\omega')} d\omega' = \sum_{-k}^{k} r_n |n| g_n^{*}. \tag{61}$$

Noting that we have already established that the left-hand side of (56) is strictly positive $[r(\omega) \neq \text{const}]$, the theorem follows.

V. EXAMPLE AND FURTHER COMMENTS

A particular example is provided by choosing

$$r(\omega) = 1 + a \cos \omega, \; |a| < 1, \tag{62}$$

and, thus, $k = 1$. We calculate

$$A = D = \frac{1}{2}\left[1 + \frac{1}{\sqrt{1-a^2}}\right]$$

$$B(N) = C(N) = \frac{\rho^{2N-1}}{\sqrt{1-a^2}}\left[\frac{a}{2} + \rho\right] > 0, \quad N > 0, \tag{63}$$

$$\rho = \frac{-1 + \sqrt{1-a^2}}{a}.$$

When $N = 0$, we have, from (49),

$$\lambda_{\max}(N=0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\omega) d\omega \times \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\omega}{r(\omega)} = \frac{1}{\sqrt{1-a^2}},$$

and otherwise

$$\lambda_{\max}(N) = A + B(N). \tag{64}$$

From (29) it follows that

$$\sin^2\Omega_N \geqq \frac{r_{\min}}{r_{\max}}, \tag{65}$$

where $r_{\min}$ and $r_{\max}$ denote the minimum and maximum of $r(\omega)$, and it is interesting to compare numerically (64) with (65). Set $a = 0.6$, so that $r_{\min}/r_{\max} = 0.25$. Then $\Omega_N \geqq 30$ degrees from (65). On the other hand $\lambda_{\max}(N) = 1.125 + 0.125/9^N$, $N \geqq 0$, which means Ω_N starts at about 63 degrees and increases to 70 degrees. Equivalently, while (65) allows a factor of four between the upper and lower bounds of (4) for this example, the more exact evaluation has them differing by only 12 to 25 percent depending on N.†

Exact solutions, as we have just found, may be useful for estimating Ω_N for some particular $r(\omega)$. If we already know $\tilde{\Omega}_N$ for some other $\tilde{r}(\omega)$ and if it is true that there are constants $\mu, \mu' \geqq 0$ so that

$$\frac{\tilde{r}(\omega)}{1+\mu} \leqq r(\omega) \leqq (1+\mu')\tilde{r}(\omega), \tag{66}$$

then, in a similar way to which (65) was derived, we can show that

$$\frac{\sin^2\tilde{\Omega}_N}{(1+\mu)(1+\mu')} \leqq \sin^2\Omega_N \leqq (1+\mu)\,(1+\mu')\sin^2\tilde{\Omega}_N. \tag{67}$$

Equation (67) could be useful when $r(\omega)$ has a large or infinite number of Fourier coefficients.

Proceeding further along the direction of bounds, we note that, using only $r_{\min}$ and $r_{\max}$, (65) can be considerably sharpened. One can show, in fact, letting $E = r_{\min}/r_{\max}$, that

$$\sin^2\Omega_N \geqq \left[\frac{1}{2} + \frac{1}{4}\left(E + \frac{1}{E}\right)\right]^{-1}. \tag{68}$$

The two basic ingredients are that [see (32) through (34)]

$$\lambda_{\max}(N) = \max_{\psi} \frac{\psi^+ R\psi}{\psi^+\Gamma^{-1}\psi} \tag{69}$$

and

$$\frac{\|\psi\|^4}{\psi^+ A\psi} \leqq \psi^+ A^{-1}\psi \tag{70}$$

for any positive definite Hermitian matrix.[4] Combining (69) and (70) yields

$$\lambda_{\max}(N) \leqq \sup_{\psi} \frac{(\psi^+ R\psi)(\psi^+\Gamma\psi)}{\|\psi\|^4}. \tag{71}$$

The right member of (71) is further upper bounded by

$$\max_{u(\omega)} \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(\omega)|^2 r(\omega)d\omega \times \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(\omega)|^2 \frac{1}{r(\omega)}\,d\omega, \tag{72}$$

† Another bound involving only the ratio $r_{\min}/r_{\max}$ is given in (68). For the present example it yields $\Omega_N \geqq 53$ degrees, a considerable improvement over (65).

where

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|u(\omega)|^2 d\omega = 1, \tag{73}$$

$u(\omega)$ being any L_2 function, not just one having Fourier components u_n restricted to $|n| \leqq N$.

To maximize (72), consider maximizing

$$Q = (\sum p_i a_i)\left(\sum p_i \frac{1}{a_i}\right) \tag{74}$$

with $\sum p_i = 1$, $p_i \geqq 0$, $a_i > 0$ and distinct. Introducing a Lagrange multiplier λ and differentiating, we obtain

$$a_l \sum \frac{p_j}{a_j} + \frac{1}{a_l} \sum p_j a_j = \lambda \tag{75}$$

for all nonvanishing p_l. Whatever values the optimum p_i's take, we may regard the sums in (75) as fixed numbers, independent of the index l. The resulting quadratic equation in a_l can be satisfied by at most two values of a_l and, hence, only two p_l are nonvanishing, and are easily seen to correspond to the maximum and minimum a_l if we are to maximize Q. Also the two p_l have equal values. Thus, (for a maximum Q)

$$Q_{\max} = \frac{1}{4}(a_{\max} + a_{\min})\left(\frac{1}{a_{\max}} + \frac{1}{a_{\min}}\right) \tag{76}$$

and (68) follows.

Our next theorem says something about the limiting behavior of $\lambda_{\max}(N)$, and in fact bounds the latter away from unity in the general case.

Theorem V. If $r(\omega) \neq$ *const.*

$$\lim_{N\to\infty} \lambda_{\max}(N) \geqq \frac{1}{2}\left[1 + \frac{1}{2\pi}\int_{-\pi}^{\pi} r(\omega)d\omega \times \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\omega}{r(\omega)}\right] > 1. \tag{77}$$

This immediately removes the restriction in Theorem III. Also, the left side of (56) is now, in general, bounded away from zero, and, by simple limiting procedures, Theorem IV also follows without restricting $r(\omega)$ as was previously required. Of course, "$-\infty$" is included in the statement "<0."

Proof. We begin with a modified form of (69) (let $\psi = \Gamma\phi$) which states

$$\lambda_{\max}(N) = \max_{\phi} \frac{\phi^+ \Gamma R \Gamma \phi}{\phi^+ \Gamma \phi}. \tag{78}$$

Thus, any particular choice for ϕ provides a lower bound. We choose for the components ϕ_k of ϕ, $|k| \leqq N$, $\phi_k = \delta_{-N,k}$. Inserting this choice into (78) yields

$$\lambda_{\max}(N) \geqq \frac{(\Gamma R \Gamma)_{-N,-N}}{\Gamma_{-N,-N}} = \frac{1}{g_0} \sum_{n,m=-N}^{N} g_{n+N} g_{-N-m} r_{m-n}$$
$$= \frac{1}{g_0} \sum_{s,t=0}^{2N} g_s^* g_t r_{s-t} = \frac{1}{2g_0} \left[\sum_{s,t=-2N}^{2N} g_s^* g_t r_{s-t} + g_0^2 r_0 \right]. \tag{79}$$

Since† $\lim \sum_{-N}^{N} g_s^* g_t r_{s-t} = g_0$, the theorem follows.

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† Namely, $\sum g_e r_{s-t}$ becomes, in the limit, the sth component of $1/r(\omega) \times r(\omega)$ which is, of course, δ_{s0}.