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Minimum Impulse Response in Graded-Index Fibers

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A straightforward analysis slightly extending the work of Kawakami and Nishizawa and of Gloge and Marcatili provides some insight about the extent to which the mode differential delay in a graded-index fiber can be minimized. We show that an ideal fiber (no mode mixing) with uniform mode excitation and loss and uniform material dispersion can theoretically have an rms pulse broadening due to mode differential delay as small as about $0.02L\Delta^2 n_0/c$. *We suggest that further improvement can result through recognition of differential mode loss and by accurate control of the (non-zero) rate of change of dispersion with fiber index.*

I. INTRODUCTION

It is known through simple first-order analysis that differential delay between the propagational modes in multimode optical fibers can be greatly reduced by grading the optical index of the core so that the index

$$n = n_0(1 - \Delta R^2), \tag{1}$$

where R is the fiber radius normalized to unity at the core-cladding boundary. It is also known that even less differential delay can be realized theoretically by slightly perturbing the gradient from this parabolic shape.

We have taken some direct steps based on existing analyses to determine how much further improvement might be realized if the optimum gradient could be realized in an "ideal" fiber, where geometry is invariant over its length (no mode mixing) and where material dispersion is invariant with radius. The approach is very simple and will be so stated, but the algebra is tedious and what little has been included will be found in the appendix.

In their very nice analysis, Kawakami and Nishizawa[1] showed that an improvement in fiberguide impulse response could be obtained by perturbing the parabolic profile through the addition of a small fourth-order variation in the index gradient. They suggested that the minimum pulse width would be obtained when the fourth-order coefficient, δ, lies between the values $2/3$, where all meridional modes are synchronous, and 1, where circular spiral modes are synchronous. Minimum total pulse width, τ, in fact occurs when $\delta = 2/3$, and minimum rms width, σ, occurs when $\delta = 5/7$. Their expression for n can be written:

$$n = n_0[1 - 2\Delta R^2 + \delta(2\Delta)^2 R^4]^{1/2}. \tag{2}$$

It will be seen presently that when the index gradient is near optimum it takes some care to keep track of which propagating modes are the fastest and which are the slowest. The overall pulse width, τ, is simply the difference in arrival time between the slowest and fastest modes. If we plot τ vs δ [found by solving (5) for all mode numbers at each δ and taking the difference between the extremes], the curve is continuous but its derivative is not (Fig. 3 below). Personick[2] has pointed out that the rms pulse width, σ, (the second moment of the received pulse) is more useful for fiberguide system analysis than is τ.

$$\sigma^2 = \frac{\int_\tau t^2 p(t)dt}{\int_\tau p(t)dt} - \left[\frac{\int_\tau t p(t)dt}{\int_\tau p(t)dt}\right]^2, \tag{3}$$

where $p(t)$ is the power arriving at a given point at time, t. Note that $d\sigma/d\delta$ is continuous, hence minimum σ can be found by direct computation.

Gloge and Marcatili have shown in their analysis[3] that minimum total pulse width occurs when

$$n = n_0[1 - 2\Delta R^{2(1-\Delta)}]^{1/2}. \tag{4}$$

For convenience, we introduce a multiplier, ρ, in the exponent of R in (4), namely,

$$n = n_0[1 - 2\Delta R^{2(1-\rho\Delta)}]^{1/2}. \tag{4$'$}$$

By definition, minimum total pulse width occurs for $\rho = 1$; minimum rms width, σ, however, occurs when $\rho = 1.2$.

These results are found by determining in each case the time of arrival of energy propagating in the μ, ν mode (radial, azimuthal mode number) with respect to the arrival time of the zero-order mode:

$$t(\delta) = \frac{L\Delta^2 n_0}{2cM^2}[(1 - 3\delta/2)(2\mu + \nu + 1)^2 + \delta/2(\nu - 1)^2] \tag{5}$$

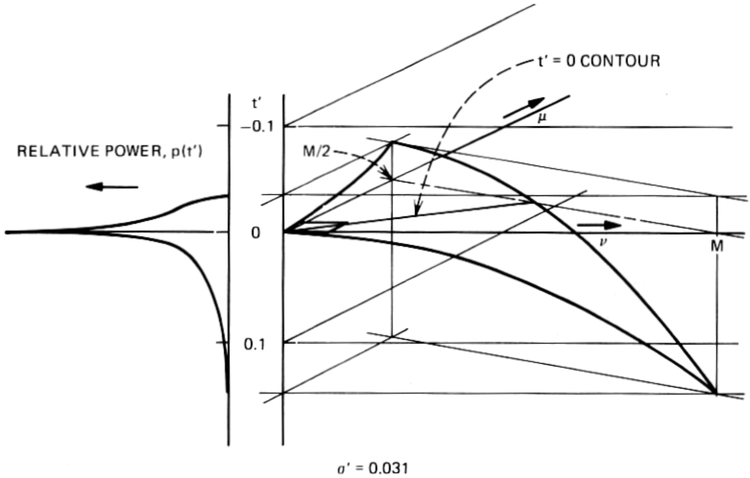

Fig. 1—Normalized time of arrival vs mode number (right) and mode count vs time of arrival (left) for $n = n_0[1 - 2\Delta R^2 + (5/7)(2\Delta)^2R^4]^{1/2}$; i.e., $\delta = 5/7$.

$$t(\rho) = \frac{L\Delta^2 n_0}{2cM^2}\left[(2\mu+\nu+1)^2 - \rho M(2\mu+\nu+1)\right], \tag{6}$$

where L is the length of the fiber, c is free-space light velocity, and M is the largest guided-mode number.

In both cases,

$$M = (2\mu+\nu+1)_{\max} \approx \pi n_0(a/\lambda_0)\sqrt{2\Delta}\,, \tag{7}$$

where a is the core radius and λ_0 is the free-space wavelength of light. Also, since we must include both E and H waves and two polarizations, a fiber with near-parabolic core gradient carries a total of about M^2 modes.

Equation (3) can be solved easily if we assume all modes are equally excited $[p(\mu, \nu) = 1]$ and integrate over all modes. So

$$\sigma^2 = \frac{\int_0^M \int_0^{(M-\nu)/2} t^2(\mu,\nu)\,d\mu\,d\nu}{M^2/4} - \left[\frac{\int_0^M \int_0^{(M-\nu)/2} t(\mu,\nu)\,d\mu\,d\nu}{M^2/4}\right]^2. \tag{8}$$

Substitution of (5) and (6) into (8) (assuming $M \gg 1$ to simplify the algebra) and minimization with respect to δ and ρ, respectively, produce the results already stated.

We can substitute these minimum values back into (5) and (6) and plot arrival time as a function of μ and ν, as shown in Figs. 1 and 2. Also shown in these figures (on the left) is a plot of relative power vs time of arrival, again assuming uniform excitation of the modes.

Figures 3 and 4 show τ' and σ' vs δ and ρ, respectively. The prime denotes normalization with respect to $L\Delta^2 n_0/c$.

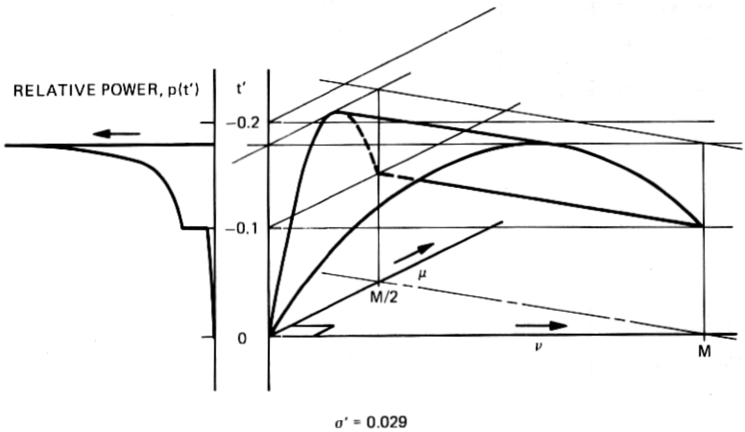

Fig. 2—Normalized time of arrival vs mode number (right) and mode count vs time of arrival (left) for $n = n_0[1 - 2\Delta R^{2(1-6\Delta/5)}]^{1/2}$; i.e., $\rho = 6/5$.

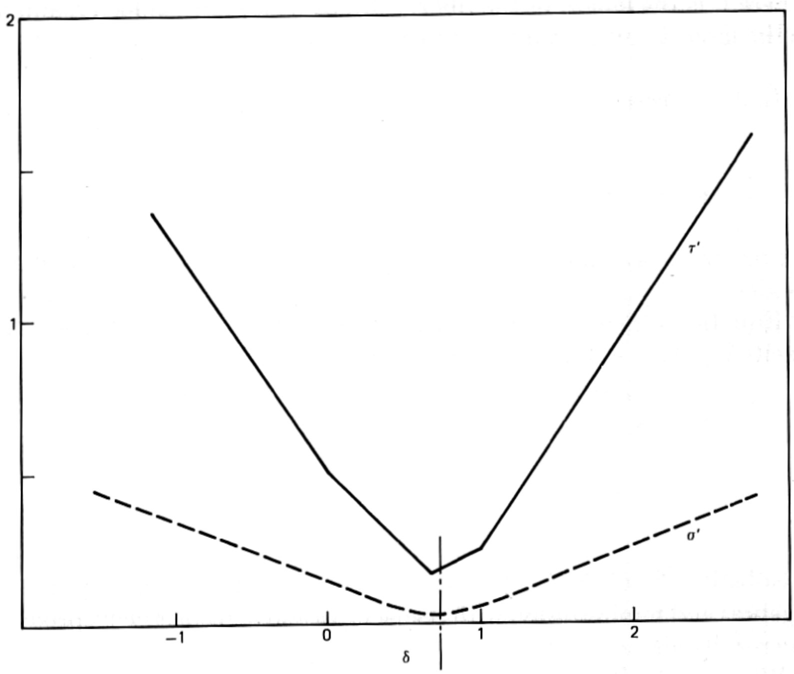

Fig. 3—Normalized total pulse width, τ', and rms pulse width σ' vs δ for $n = n_0[1 - 2\Delta R^2 + \delta(2\Delta)^2 R^4]^{1/2}$.

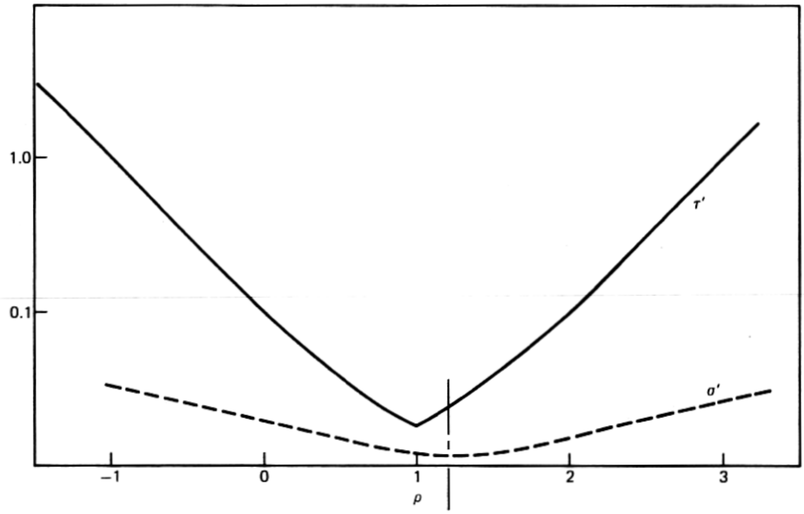

Fig. 4—Normalized total pulse width, τ', and rms pulse width σ' vs ρ for $n = n_0[1 - 2\Delta(R)^{2(1-\rho\Delta)}]^{1/2}$.

It is natural to question whether we can improve on these independent functional perturbations from the parabola by an appropriate combination thereof. Combination of (5) and (6) yields:

$$t(\delta, \rho) = \frac{L\Delta^2 n_0}{2cM^2}\left[\left(1 - \frac{3\delta}{2}\right)(2\mu + \nu + 1)^2 + \frac{\delta}{2}(\nu - 1)^2 - \rho M(2\mu + \nu + 1)\right]. \tag{9}$$

Substituting (9) into (8), as before, and minimizing with respect to both δ and ρ produces: $\delta = 1/3$, $\rho = 2/3$. Substituting these into (9) and neglecting the 1s (large mode count) yields:

$$t \approx \frac{L\Delta^2 n_0}{cM^2}\left[\mu^2 + \mu\nu + \left(\frac{1}{3}\right)\nu^2 - \left(\frac{1}{3}\right)M(2\mu + \nu)\right]. \tag{10}$$

This is plotted in Fig. 5.

Now combine (2) and (4′) and let $\delta = 1/3$ and $\rho = 2/3$ to find

$$n = n_0[1 - 2\Delta R^{2(1-2\Delta/3)} + (1/3)(2\Delta)^2 R^4]^{1/2} \tag{11}$$

as the near-optimum index gradient. Equation (11) can also be written

$$n \approx n_0[1 - \Delta R^2 - \Delta^2\epsilon(R)], \tag{12}$$

which may be more convenient if we are seeking the excursion of the

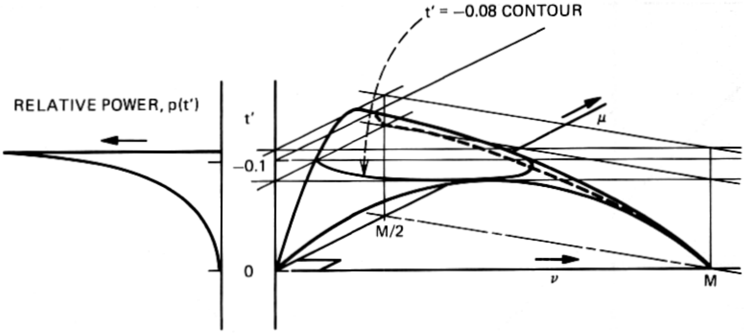

Fig. 5—Normalized time of arrival vs mode number (right), and mode count vs time of arrival (left), for $n = n_0[1 - 2\Delta R^{2(1-2\Delta/3)} + (1/3)(2\Delta)^2 R^4]^{1/2}$; i.e., $\rho = 2/3$, $\delta = 1/3$.

improved gradient from the inverse parabola of (1):

$$\epsilon(R) \approx \left(\frac{2}{3}\right) R^2 \ln(1/R^2) - \frac{R^4}{6}. \tag{13}$$

This is plotted in Fig. 6.

II CONCLUSION

The rms pulse width in an ideal fiber with parabolic gradient is

$$\sigma \approx 0.145 \frac{L\Delta^2 n_0}{c}. \tag{14}$$

For the profile determined in this paper,

$$\sigma = 0.0216 \frac{L\Delta^2 n_0}{c}. \tag{15}$$

Hence, nearly an order-of-magnitude improvement in fiberguide impulse response over that produced by an inverse parabolic gradient could be realizable in an ideal multimode fiber.

Uniform excitation of the modes was assumed for this analysis. If the low-order modes tend to carry more power,[2] even smaller σ could be realized and Fig. 1 would suggest that $\rho \to 0$ and $\delta \to 5/7$ might be more nearly optimum.

Olshansky and Keck[4] have shown that when material dispersion varies with radius, the optimum gradient is significantly different from parabolic and from those discussed here. On the other hand, it can be shown that procedures similar to those indicated here can be carried out; and if the dependence of the dispersion on optical index could be controlled to a predetermined practical value, even less differential delay than that

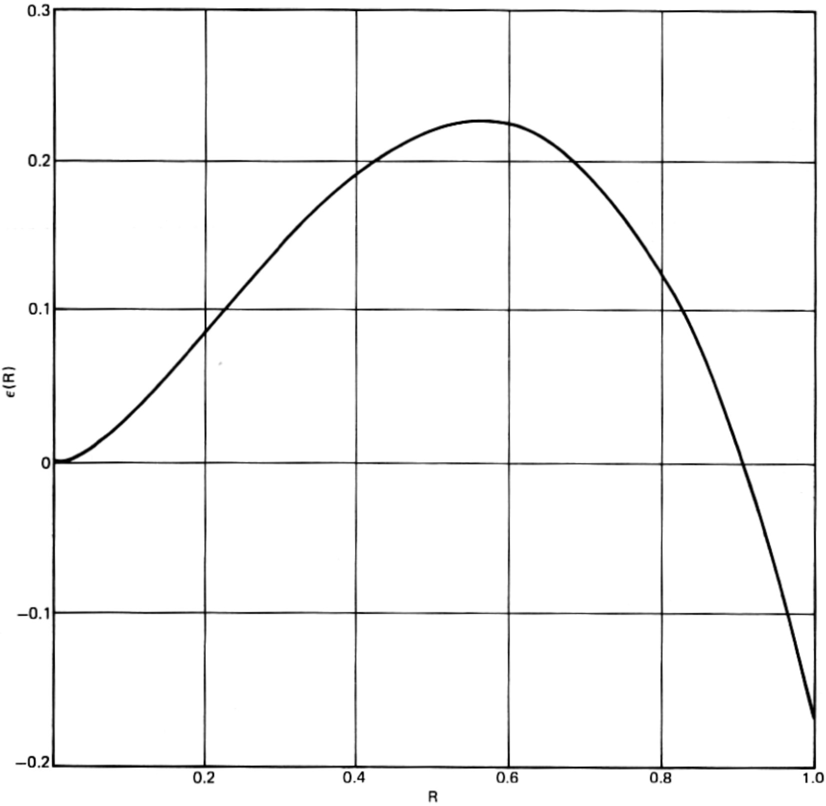

Fig. 6—Index perturbation from inverse parabola for minimum rms pulse width [see eqs. (12), (13)].

indicated here could be realized. This does not necessarily contradict the results of the calculation by Arnaud and Fleming[5] based on an analysis[6] which includes tunneling modes at equal weight with the more clearly guided modes. They showed that if we assume a very specific non-zero dependence of dispersion on index, namely, that resulting from the inclusion of germania in silica, the differential delay of the optimum graded-index fiber is considerably degraded. We suggest, however, that if we carefully choose materials to enhance the index (a propitious mix of germania and phosphorous oxide, for example), we might provide a dispersion vs index dependence that would improve rather than degrade the mode differential delay, at least at a particular light wavelength.

This goes well beyond our present ability to make measurements and control materials, however, and only leads us to conclude that improvement in technology can potentially bring significant improvements in the information-carrying capacity of low-loss graded-index fibers.

APPENDIX

It is our purpose first to justify (5) and (6). Equation (5) comes from Ref. 1, eq. (29), which translates directly to

$$\frac{\omega^2 n_0^2}{c^2} - \beta^2 = \frac{2\omega n_0}{c}(2\mu + \nu + 1)\frac{\sqrt{2\Delta}}{a} - \delta[6\mu^2 + 6\mu(\nu + 1) + (\nu + 1)(\nu + 2)]\frac{2\Delta}{a^2}, \quad (16)$$

where a is the core radius.

This can be solved for β to find

$$\beta \approx \frac{\omega n_0}{c} - (2\mu + \nu + 1)\frac{\sqrt{2\Delta}}{a} - \frac{c\Delta}{\omega n_0 a^2}\left[\left(1 - \frac{3\delta}{2}\right)(2\mu + \nu + 1)^2 + \frac{\delta}{2}(\nu^2 - 1)\right]. \quad (17)$$

The largest propagating mode numbers can be found by recognizing that the phase constant, β, can be no lower than the phase constant, k_a, of an unbound wave in the cladding, where

$$n_a \approx n_0[1 - 2\Delta]^{1/2} \approx n_0(1 - \Delta). \quad (18)$$

So

$$\beta_{\min} \approx \frac{\omega n_0}{c} - (2\mu + \nu + 1)\frac{\sqrt{2\Delta}}{a} = k_a \approx \frac{\omega n_0}{c}(1 - \Delta). \quad (19)$$

Eq. (19) yields

$$(2\mu + \nu + 1)_{\max} \equiv M = \frac{\omega n_0 a\sqrt{\Delta}}{c\sqrt{2}} \quad (20)$$

$$M^2 = \frac{\omega^2 n_0^2 a^2 \Delta}{2c^2}. \quad (21)$$

Substitution of (20) into (17) shows that each term on the right-hand side thereof is of order Δ smaller than the previous.

The time of arrival of energy carried in any mode through length, L, of fiber relative to the arrival of energy in the zero-order mode is

$$t = L\left[\frac{\partial\beta}{\partial\omega} - \frac{\partial\beta_0}{\partial\omega}\right]. \quad (22)$$

Differentiation of (17) and substitution of that [utilizing (20)] into (22) produces eq. (5).

Equation (6) is found from substitution of GM(17) [i.e., eq. (17) of Gloge and Marcatili in Ref. 2] into GM(19), recognizing that the α of Gloge and Marcatili is our $2(1 - \rho\delta)$. First, we must identify m/M of

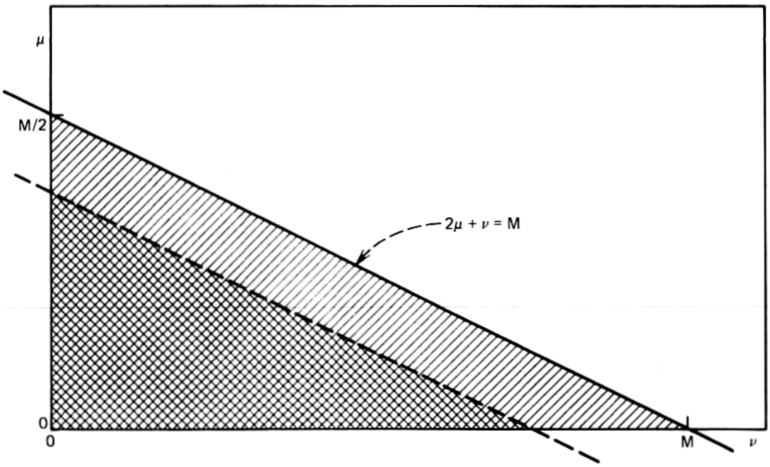

Fig. 7—All modes that lie under the line intercepting the μ axis at $M/2$, the ν axis at M, are propagating modes.

GM(17). The symbols are unfortunate, but we find this equates to our $(2\mu + \nu + 1)^2/M^2$.

This is easier to justify than derive. All modes are associated with positive mode numbers that lie in the μ, ν plane, as shown in Fig. 7. We can plot the identity of (20) on the plane to bound the crosshatched area in Fig. 7 within which lie all propagating mode numbers. The total number of propagating modes is four times the total mode-number combinations in that area, which can be seen by inspection to be M^2. Gloge and Marcatili[2] showed that for their assumed functional index variation, the phase constant depends on the total mode number independent of the ratio of μ to ν. If, then, we draw a dotted line parallel to the limit line, as in Fig. 7, the sum of all modes with phase constant greater than that represented by those on the line are identified by numbers that lie within the double crosshatched area. This Gloge and Marcatili call m, and we call $(2\mu + \nu + 1)^2$. Their M is the total number of propagating modes, which is our M^2. [We have neglected the extra (1) since we have assumed $M \gg 1$. We would have to be more precise if only a small number of modes were involved.]

We can now translate GM(17) to be

$$\frac{(2\mu + \nu + 1)^2}{M^2} \approx (\delta_{\mathrm{GM}}/\Delta)^2$$

or

$$\delta_{\mathrm{GM}} = \frac{\Delta(2\mu + \nu + 1)}{M}. \tag{23}$$

This in GM(19) results in (6).

Performing the integrations for (8) is simple but tedious. A helpful formula, however, is

$$\int_0^{(M-\nu)/2} (2\mu + \nu)^n d\mu = \frac{M^{n+1} - \nu^{n+1}}{2(n+1)}. \tag{24}$$

The evaluations of (8) for the two cases of (5) and (6) are

$$\sigma(\delta) = \frac{L\Delta^2 n_0}{c} \frac{1}{2\sqrt{3}} \left[\frac{1}{4} - \frac{2\delta}{3} + \frac{7\delta^2}{15}\right]^{1/2} \tag{25}$$

$$\sigma(\rho) = \frac{L\Delta^2 n_0}{c} \frac{1}{2\sqrt{3}} \left[\frac{1}{4} - \frac{2\rho}{5} + \frac{\rho^2}{6}\right]^{1/2}. \tag{26}$$

For the combined case,

$$\sigma(\delta, \rho) = \frac{L\Delta^2 n_0}{c} \frac{1}{2\sqrt{3}} \left[\frac{1}{4} - \frac{2\delta}{3} + \frac{7\delta^2}{15} - \frac{2\rho}{5} + \frac{\rho^2}{6} + \frac{8\delta\rho}{15}\right]^{1/2}. \tag{27}$$

Minimizing these is straightforward.

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