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A Probability Inequality and Its Application to Switching Networks

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The use of channel graphs to study the blocking probabilities of multistage switching networks was first proposed by Lee and has gained popularity ever since. A channel graph between an input terminal and an output terminal is the union of all paths connecting them in the network. Usually, the assumption is made that links connecting the same two stages, say the i*th stage and the* $(i + 1)$st *stage, have constant and identical probability* p_i *of being busy. Let* $G(s,\lambda)$ *denote the class of channel graphs with* s *stages and* λ *paths. We show that for every channel graph in* $G(s,\lambda)$ *with multiple links, there exists a channel graph in the same class without multiple links that has smaller or equal blocking probabilities for all* $\{p_i\}$*. We obtain this result by first proving a probability inequality of a more general nature.*

I. INTRODUCTION

In this paper we consider a switching network as a directed graph. A vertex is called a *switch* if its in-degree and out-degree are both positive, an *input terminal* if it has in-degree zero and out-degree one, and an *output terminal* if it has in-degree one and out-degree zero. The edges between the switches are called *links*. A switch is said to be of size $n \times m$ if it has in-degree n and out-degree m. Every switch in our network is assumed to be *two-sided nonblocking* in the sense that when the network is in actual use, traffic can be routed from every input link to every output link in a switch, provided the two links involved are not carrying other traffic, and regardless of the traffic carried by other links.

In a multistage switching network, the switches are partitioned into a sequence of stages with the following properties.

(i) The sizes of switches in a given stage are identical.

(*ii*) All input terminals are connected to the switches of the first stage; all output terminals are connected to the switches of the last stage.

(*iii*) Links exist only between two switches in adjacent stages. [We call links between the ith stage and the $(i + 1)$st stage the ith-stage links.] The direction of an ith-stage link is from the ith stage to the $(i + 1)$st stage.

A *channel graph* between a given input terminal and a given output terminal is the smallest subgraph containing all paths connecting the two terminals. Since a link in a path is also shared by other paths connecting possibly other pairs of terminals, the actual routing of a path will fail if any link involved has already been used to route some other path. In that case, we say that the path is blocked. The blocking probability of a channel graph is the probability that every path in it is blocked.

Lee[5] first suggested the use of channel graphs to study the blocking performances of switching networks. Usually, the assumption that each ith-stage link has the constant and independent probability p_i of being *busy* (meaning the link is used in routing some other path) is made to simplify the computations of blocking probabilities. Lee's method has gained popularity both in theory and in practice since its proposal.

A class of multistage switching networks that has been widely used but only recently has come under systematic study is the class of *balanced networks*[4]. Balanced networks are characterized by the property that the channel graphs for all pairs of input terminals and output terminals are isomorphic. Thus, the blocking performance of a balanced network can be studied by analyzing just one channel graph.

Let $G(s,\lambda)$ denote the class of channel graphs with s stages and λ paths. Comparisons of channel graphs in a given $G(s,\lambda)$ have been made in Refs. 1 and 3. This paper is a continuation of this study. We are particularly interested in channel graphs with multiple links. Networks with multiple links between a pair of switches have recently been studied by Fontenot.[2] In this paper, we show that for every such channel graph, there is a channel graph in the same class but without multiple links with an equal or smaller blocking probability for any arbitrarily given set $\{p_i\}$. In some cases, a switching network constructed using such a channel graph has a larger number of crosspoints than the corresponding multiple-link network. In other cases, however, our construction produces a network with the same number of crosspoints (and therefore cost), but lower blocking probability. This is illustrated by a simple example at the end of our paper.

II. A PROBABILITY INEQUALITY

We prove a probability inequality which is itself of some interest and has application to our study of channel graphs.

Theorem 1:

$$\prod_{i=1}^{k} (1 - p_i^{c_i}) \leqq 1 - \left\{1 - \prod_{i=1}^{k} (1 - p_i)\right\}^{\Pi_{i=1}^{k} c_i},$$

where p_i and c_i are real numbers satisfying $1 \geqq p_i \geqq 0$ and $c_i \geqq 1$ for $i = 1, \ldots, k$.

Proof: Proof is by induction on k. Theorem 1 is trivially true for $k = 1$. For general k, assume Theorem 1 is true for all $k' = 1, \cdots, k - 1$.

Let $b = \Pi_{i=1}^{k-1} c_i$, $y = \Pi_{i=1}^{k-1} (1 - p_i)$, $p_k = p$ and $c_k = c$. Then $b \geqq 1$ and $1 \geqq y \geqq 0$. By induction,

$$\prod_{i=1}^{k} (1 - p_i^{c_i}) \leqq (1 - p^c) - (1 - p^c)(1 - y)^b. \tag{1}$$

It is sufficient to prove that

$$(1 - p^c) - (1 - p^c)(1 - y)^b \leqq 1 - \{1 - (1 - p)y\}^{bc},$$

or equivalently,

$$\{1 - (1 - p)y\}^{bc} \leqq p^c + (1 - p^c)(1 - y)^b. \tag{2}$$

Let $z = p^c$. Then, $1 \geqq z \geqq 0$. Ineq. (2) can be written as

$$\{1 - (1 - z^{1/c})y\}^{bc} \leqq z + (1 - z)(1 - y)^b. \tag{3}$$

We first show that

$$f(z,y) = \{1 - (1 - z^{1/c})y\}^c \leqq 1 - (1 - z)y = g(z,y). \tag{4}$$

Clearly $f(1,y) = g(1,y)$. Furthermore,

$$\frac{\partial}{\partial z} f(z,y) = c\{1 - (1 - z^{1/c})y\}^{c-1} y \frac{1}{c} z^{1/c-1}$$

$$\geqq y = \frac{\partial}{\partial z} g(z,y),$$

since

$$\{1 - (1 - z^{1/c})y\}^{c-1} z^{1/c-1} \geqq \{1 - (1 - z^{1/c})\}^{c-1} z^{1/c-1}\, 1 = 1.$$

Therefore, Ineq. (4) is true. Consequently, to prove Ineq. (3), it suffices to prove

$$h(z,y) = g(z,y)^b = \{1 - (1 - z)y\}^b$$
$$\leqq z + (1 - z)(1 - y)^b = u(z,y). \tag{5}$$

We have $h(z,0) = u(z,0)$. Furthermore

$$\frac{\partial}{\partial y} h(z,y) = -b\{1 - (1 - z)y\}^{b-1}(1 - z)$$

$$\leqq -(1 - z)b(1 - y)^{b-1} = \frac{\partial}{\partial y} u(z,y),$$

since

$$1 \geqq 1 - z \geqq 0$$

and

$$1 - (1 - z)y \geqq 1 - y.$$

Therefore, Ineq. (5) is true. The proof is completed.

It is easy to construct counter-examples of Theorem 1 if the conditions $c_i \geqq 1$ for $i = 1, \ldots, k$ are violated.

III. A THEOREM ON CHANNEL GRAPHS

Consider a channel graph which contains the subgraph of Fig. 1,

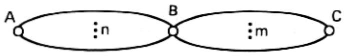

Fig. 1—Graph with multiple links.

where A is an ith-stage switch, B an $(i + 1)$st-stage switch, C an $(i + 2)$nd-stage switch and $\text{Max}\{m,n\} > 1$. Since B is a two-sided nonblocking switch, there are nm paths from A to C. Let p_i, $i = 1, \cdots, s$ be the probability that an ith-stage link is busy. We show that if we replace Fig. 1 with the subgraph of Fig. 2,

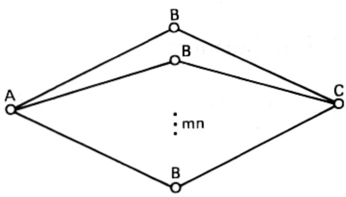

Fig. 2—Graph without multiple links.

then the new channel graph, clearly in the same class $G(s,\lambda)$, will have an equal or smaller blocking probability. It suffices to show that the blocking probability of the graph in Fig. 1 is equal or larger than that of the graph in Fig. 2. Routing from A to C can be realized in Fig. 1 if at least one link from each of the n and m links is available. The probability of this event is

$$(1 - p_i^n)(1 - p_{i+1}^m).$$

The same routing can be realized in Fig. 2 if at least one of the nm two-link paths (see Fig. 3)

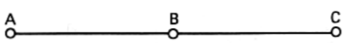

Fig. 3—Two-link path.

is nonblocking. The probability of this event is

$$1 - \{1 - (1 - p_i)(1 - p_{i+1})\}^{nm}.$$

That the first probability is equal or smaller than the second probability is an immediate consequence of the method of Theorem 1 by setting $k = 2$. Therefore we have proved the following theorem.

Theorem 2: For every channel graph with multiple links, there is a channel graph in the same $G(s,\lambda)$ class without multiple links that has equal or smaller blocking probability.

Example. Let us compare the blocking probabilities of the two five-stage balanced networks in Fig. 4 and Fig. 5.

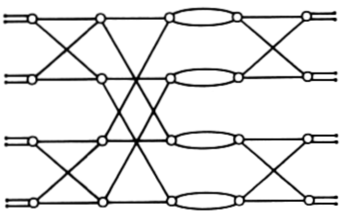

Fig. 4—Network with multiple links.

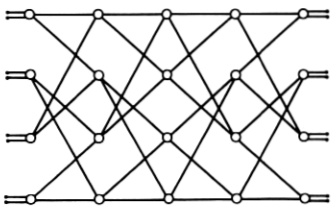

Fig. 5—Network without multiple links.

The two channel graphs are shown in Fig. 6 and Fig. 7, respectively.

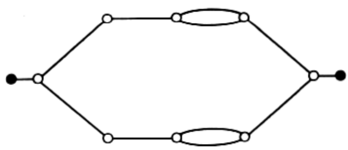

Fig. 6—Channel graph of network with multiple links.

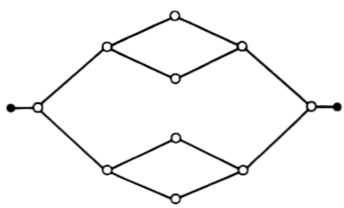

Fig. 7—Channel graph of network without multiple links.

By Theorem 1, the blocking probability of the channel graph in Fig. 6 is equal to or greater than that of the channel graph in Fig. 7. Therefore, we conclude that the network in Fig. 5 has smaller blocking probabilities than the one in Fig. 4. Note that the two networks are identical except for the way the switches are linked.

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