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Elastic State of Stress in a Stalpeth Cable Jacket Subjected to Pure Bending

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The stresses in the plastic jacket of a slightly bent telephone cable are analyzed within the linear theory of elasticity. The jacket is considered to be bonded to the underlying corrugated steel by a flooding compound. The constraining effect of the steel results in a three-dimensional state of stress that differs substantially from the predictions of elementary beam theory. For the thin jackets typically used on telephone cables it is found that the stress state is essentially biaxial, the axial and circumferential normal stresses being at least an order of magnitude larger than the others. On the tensile side, the stresses are closely approximated (at any given point) by those in the well-known biaxial strip experiment, in which the principal stresses are in proportion by the Poisson's ratio of the plastic. The compressive side is likewise in biaxial compression, and there the flooding compound is subjected to tensile stresses even before the onset of any jacket buckling. The results confirm the validity of previous approaches to the effects introduced by imperfections and indicate further that the probability of spontaneous cracking is increased by the adherence of the jacket to the soldered steel layer.

I. INTRODUCTION

The selection and development of plastic jacketing compounds for multipair cables depend to a large extent on cable behavior during bending. Cable jackets are expected to be relatively flexible for ease of handling and installation while at the same time surviving large strains (up to 15 percent) without cracking, splitting, or severe wrinkling. Temperature extremes encountered in the field render these criteria even more stringent.

Recently completed analyses have led to easily performed laboratory tests for the screening of candidate compounds with regard to some of these requirements. Now, for example, the relative influence of various sheath-grade plastics on the bending stiffness of cables can be evaluated by conducting ordinary tensile tests.[1] In addition, the relative sensitivity of compounds to low temperature and high strain-rate cracking can be determined through impact tests on notched specimens.[1] Still, there remains the observation of slow crack growth at high temperatures during bending and the occurrence of wrinkles in cable jackets during duct installation at low temperatures. None of these phenomena nor how they are affected by cable parameters such as jacket moduli and thickness, flooding compound tackiness, and the depth of the corrugations in the underlying steel, is presently understood.

This paper is devoted to a study of the state of stress in the jacket of a Stalpeth cable subjected to classical pure bending. The problem is treated within the framework of the linear theory of elasticity, which supposes small strains and rotations and an elastic material. Although the bending strains in telephone cable are frequently large, our analysis is intended to provide insight into the circumstances at incipient cracking, buckling, or yielding of the cable jacket. Standard techniques from linear viscoelasticity theory[2,3] can be applied to the elastic results given here to account for the time dependence inherent to plastics.

The primary emphasis in the present investigation is directed toward estimating the effect on the stress field in the jacket resulting from the constraint imposed on its inner surface by the underlying cable structure (see Fig. 1 for a detail of the Stalpeth construction). For slight bends, the influence of the soldered steel shield dominates that of the wire core, and we shall, therefore, consider the pure bending of a plastic jacket bonded to a corrugated metal shell. The analysis is further simplified by the realization that the corrugation wave length used in Stalpeth cable is one to two times larger than the jacket thickness, while the valleys imprinted on the jacket's inner surface are reasonably shallow because of the presence of the flooding compound. We are thus afforded the privilege of averaging field variables over a corrugation wave length to arrive at a boundary value problem involving a uniform cylindrical geometry. The amplification of jacket stresses created by the corrugation imprints can be deduced from the results obtained here together with published concentration factors in the usual way.

We begin in the next section by formulating the relevant three-dimensional field equations and show that they can be reduced through a change in dependent variables to plane strain equations. The approach taken is reminiscent of the scheme used in elementary elasticity for the St. Venant-bending of cylinders with irregularly shaped cross sections, but differs in that here the strain field, rather than the stress field, is supposed to conform to the elementary Bernoulli-Euler theory.

Next, the boundary conditions at the jacket-steel interface are con-

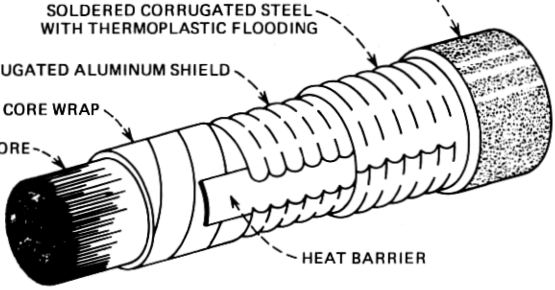

Fig. 1—Stalpeth cable.

sidered in detail. The steel is assumed to have negligible stiffness in the corrugation direction and to otherwise obey the usual hypotheses on the deformation of thin shells. Integration over a corrugation wave length then permits the stresses and displacements of the jacket to be related to those of the steel shield.

The appropriate (plane strain) boundary value problem for the stress state in the sheath having been set up, we find that it admits an elementary solution in closed form through the introduction of an Airy stress function. The full three-dimensional stress and displacement fields are then calculated.

When these results are applied to Stalpeth cables as presently manufactured, it is found that, on the tensile side of the cable, the jacket is essentially in a state of biaxial tension. That is, the ratios of the axial and circumferential normal stresses to the shear and transverse (thickness-direction) stresses are of the order of the diameter-to-thickness ratio. Furthermore, if E denotes the Young's modulus of the plastic jacket, D the cable diameter and ρ the bend radius, then the longitudinal and hoop stresses are shown to be in approximate proportion to the bending stresses $ED/2\rho$ from elementary beam theory. Of particular interest is the conclusion that these constants of proportionality vary appreciably only with the Poisson's ratio of the jacket, at least for the ranges of cable size, corrugation geometry and jacket moduli encountered in Stalpeth applications. With Poisson's ratio chosen in the typical range for plastics (0.3–0.5), the longitudinal stress varies from about 110 to 130 percent of the beam theory stress, while the hoop stress factor ranges from 30 to 45 percent. The stress state on the tensile side of the cable is thus closely approximated by the biaxial strip experiment discussed at the end of the paper. The importance of biaxiality in bending has been recognized previously.[4]

All of the results outlined above for the tensile side of the cable apply as well on the compressive side, except that the stress state is, of course, one of biaxial compression. It is also observed that the flooding compound is subjected to a tension on the compressive side that exceeds that which currently used compounds are likely to support, even for very small bend radii.

The implications of these results on the cracking to which we alluded earlier are discussed qualitatively in the final section of the paper. For the time being, we remark only that the biaxiality tends to increase the likelihood of spontaneous cracking. Finally, it should be mentioned that the jacket stresses generated in the bending of cables other than Stalpeth can differ drastically from those obtained here. Indeed, the biaxial state of stress produced by the constraining effect of the soldered steel layer would not be present in those designs that allow relative motion between the jacket and underlying metallic layers.

II. REDUCTION OF THE PROBLEM TO ONE OF PLANE STRAIN

Consider a cylindrical shell of inner radius r_i and outer radius r_o. Referring to Fig. 2, choose cylindrical coordinates (r,θ,z) in the obvious way, and let the shell be subjected to pure bending in the y-z plane. The center line of the bent shell is then a circle of prescribed radius ρ.

Recall the field equations of linear elasticity in cylindrical coordinates:[5]

Strain-displacement relations

$$\epsilon_r = \frac{\partial u_r}{\partial r}, \ \epsilon_\theta = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), \ \epsilon_z = \frac{\partial u_z}{\partial z} \tag{1}$$

$$2\epsilon_{r\theta} = \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}, \ 2\epsilon_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \ 2\epsilon_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}$$

Equations of equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{2a}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0 \tag{2b}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{2c}$$

Hooke's law

$$2\mu(1+\nu)\epsilon_r = \sigma_r - \nu(\sigma_\theta + \sigma_z) \tag{3a}$$

$$2\mu(1+\nu)\epsilon_\theta = \sigma_\theta - \nu(\sigma_r + \sigma_z) \tag{3b}$$

$$2\mu(1+\nu)\epsilon_z = \sigma_z - \nu(\sigma_r + \sigma_\theta) \tag{3c}$$

$$2\mu\epsilon_{r\theta} = \sigma_{r\theta}, \ 2\mu\epsilon_{rz} = \sigma_{rz}, \ 2\mu\epsilon_{\theta z} = \sigma_{\theta z} \tag{3d}$$

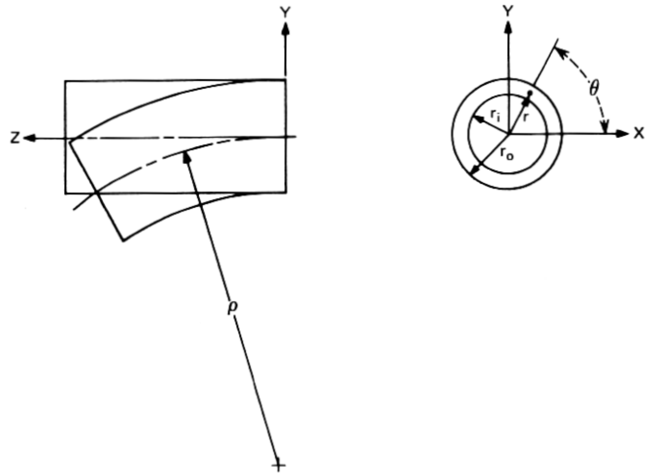

Fig. 2—Choice of cylindrical and Cartesian coordinates.

Here, the symbols u, ϵ, and σ, suitably subscripted, stand for components of displacement, strain, and stress, respectively. The constants μ and ν represent the shear modulus and Poisson's ratio for the cable jacket, while the coefficient

$$E = 2\mu(1 + \nu) \tag{4}$$

of ϵ_r, ϵ_θ, and ϵ_z in the first three equations of (3) is Young's modulus.

Next, make the assumption that the annular cross sections remain plane and normal to the shell's axis during bending.* It then follows from a familiar geometric argument[7] that

$$\epsilon_{rz} = \epsilon_{\theta z} = 0, \; \epsilon_z = \frac{r \sin \theta}{\rho} \tag{5}$$

With no consequent loss of generality, we assume further that the cross section at $z = 0$ remains in the x-y plane and undergoes no rotation about the z-axis nor overall rigid translation in the x-y plane, so that

$$u_z(r,\theta,0) = 0 \tag{6a}$$

$$u_\theta(r_o,0,0) = u_\theta\left(r_o, \frac{\pi}{2}, 0\right) = u_\theta(r_o,\pi,0) = 0 \tag{6b}$$

We show next that the formulae (5) and (6a) require the in-plane stresses σ_r, σ_θ, and $\sigma_{r\theta}$ to be associated with a state of plane strain. That is, we establish the existence of plane displacements $\hat{u}_r$ and $\hat{u}_\theta$, independent of z, that generate strains $\hat{\epsilon}_r$, $\hat{\epsilon}_\theta$, and $\hat{\epsilon}_{r\theta}$ obeying the plane strain

* The viewpoint here is similar to that of the traditional St. Venant "semi-inverse" method.[6] The solution thus obtained is justified by exhibiting stress distributions on the ends of the cylinder that support the assumed deformation.

form of Hooke's law:[5]

$$2\mu\hat{\epsilon}_r = (1-\nu)\sigma_r - \nu\sigma_\theta \tag{7a}$$

$$2\mu\hat{\epsilon}_\theta = (1-\nu)\sigma_\theta - \nu\sigma_r \tag{7b}$$

$$2\mu\hat{\epsilon}_{r\theta} = \sigma_{r\theta} \tag{7c}$$

We first confirm (7) and then prove the existence of the requisite displacements.

Since ϵ_z is known from (5), we have from (3c) that

$$\sigma_z = 2\mu(1+\nu)\epsilon_z + \nu(\sigma_r + \sigma_\theta) \tag{8}$$

Thus, σ_z may be eliminated from eqs. (3a), (3b) for ϵ_r, ϵ_θ to obtain

$$\epsilon_r = \hat{\epsilon}_r - \nu\epsilon_z,\ \epsilon_\theta = \hat{\epsilon}_\theta - \nu\epsilon_z \tag{9}$$

where $\hat{\epsilon}_r, \hat{\epsilon}_\theta$ are given by (7a), (7b). The last of (7) is, of course, satisfied by taking

$$\hat{\epsilon}_{r\theta} \equiv \epsilon_{r\theta} \tag{10}$$

To see that the strain field $\hat{\epsilon}_r$, $\hat{\epsilon}_\theta$, and $\hat{\epsilon}_{r\theta}$ is indeed generated from (7) by in-plane displacements $\hat{u}_r$ and $\hat{u}_\theta$, independent of z, observe that the displacement field*

$$\overline{u}_r = -\frac{(\nu r^2 + z^2)\sin\theta}{2\rho},\ \overline{u}_\theta = \frac{(\nu r^2 - z^2)\cos\theta}{2\rho},\ \overline{u}_z = \frac{rz\sin\theta}{\rho} \tag{11}$$

has an associated strain field

$$\overline{\epsilon}_r = \overline{\epsilon}_\theta = -\nu\epsilon_z,\ \overline{\epsilon}_z = \epsilon_z,\ \overline{\epsilon}_{r\theta} = \overline{\epsilon}_{rz} = \overline{\epsilon}_{\theta z} = 0 \tag{12}$$

Thus, the defining equations

$$u_r = \overline{u}_r + \hat{u}_r,\ u_\theta = \overline{u}_\theta + \hat{u}_\theta,\ u_z = \overline{u}_z + \hat{u}_z \tag{13}$$

for $\hat{u}_r$, $\hat{u}_\theta$, and $\hat{u}_z$, together with (1), (9), and (12), reveal that the displacements $\hat{u}_r, \hat{u}_\theta$ satisfy the strain-displacement relations for $\hat{\epsilon}_r, \hat{\epsilon}_\theta, \hat{\epsilon}_{r\theta}$. Moreover, when (13) is combined with (7c), (5), and (12), there results

$$\frac{\partial \hat{u}_r}{\partial z} + \frac{\partial \hat{u}_z}{\partial r} = 0,\ \frac{\partial \hat{u}_\theta}{\partial z} + \frac{1}{r}\frac{\partial \hat{u}_z}{\partial \theta} = 0,\ \frac{\partial \hat{u}_z}{\partial z} = 0 \tag{14}$$

Integration of (14) subject to the constraint (6a) yields

$$\hat{u}_z = \frac{\partial \hat{u}_r}{\partial z} = \frac{\partial \hat{u}_\theta}{\partial z} = 0$$

and the desired result is established.

* This displacement field (11) is that which the shell would exhibit were it hollow (see Sokolnikoff,[6] Article 32, for example).

The three-dimensional problem is now reduced to determining the plane-strain elastic state with displacements $(\hat{u}_r,\hat{u}_\theta)$, strains $(\hat{\epsilon}_r,\hat{\epsilon}_\theta,\hat{\epsilon}_{r\theta})$, and stresses $(\sigma_r,\sigma_\theta,\sigma_{r\theta})$, subject to boundary conditions on the cylindrical surfaces $r = r_i$ and $r = r_o$. The remaining stresses $\sigma_z,\sigma_{rz},\sigma_{\theta z}$ are then provided by (8) and

$$\sigma_{rz} = \sigma_{\theta z} = 0 \tag{15}$$

which follows from (5) and (3c).

III. THE BOUNDARY CONDITIONS

As previously mentioned, the intent of this paper is to account for the influence of the soldered steel shield on the plastic jacket. Since the longitudinal (z direction in Fig. 3) stiffness of the corrugated steel is small for small longitudinal extensions,[8] the constraint imposed on the inner surface of a Stalpeth jacket is essentially confined to the x-y plane, *provided the jacket field variables are interpreted as averages over a corrugation wave-length.*

Moreover, since the corrugation depth H (see Fig. 3) is small compared to the radius r_i of the cable, the shield will deform in approximate accordance with Kirchoff's hypothesis of classical shell theory.[9] In particular, denoting by u and v the circumferential and radial displacements of the midsurface of the shield (see Fig. 3), one has the relations[9]

$$u_r^s(\xi,\theta) = v \tag{16a}$$

$$u_\theta^s(\xi,\theta) = u + \frac{2\xi}{2r_i - H}\left(u - \frac{dv}{d\theta}\right) \tag{16b}$$

for the radial and circumferential displacements of a particle at a distance ξ from the midsurface (again, see Fig. 3). Requiring the displacements of the steel to conform to those of the jacket at $\xi = H/2$ results in

$$\begin{aligned} v(\theta) &= u_r(r_i,\theta) \\ \left(\frac{2r_i}{2r_i - H}\right) u(\theta) &= u_\theta(r_i,\theta) + \frac{H}{2r_i - H}\frac{\partial u_r}{\partial\theta}(r_i,\theta) \end{aligned} \tag{17}$$

Next, take the shield to have an (in-plane) modulus E_s and use (16), (17) to compute the tension T and moment M in the shield averaged over a corrugation wave length ℓ (refer again to Fig. 3). There results

$$\begin{aligned} T(\theta) &= \frac{E_s A}{\ell}\,\epsilon_\theta(r_i,\theta) - \frac{AH}{I} M(\theta) \\ M(\theta) &= \frac{E_s I}{r_i{}^2\ell}\left[u_r(r_i,\theta) + \frac{\partial^2 u_r}{\partial\theta^2}(r_i,\theta)\right] \end{aligned} \tag{18}$$

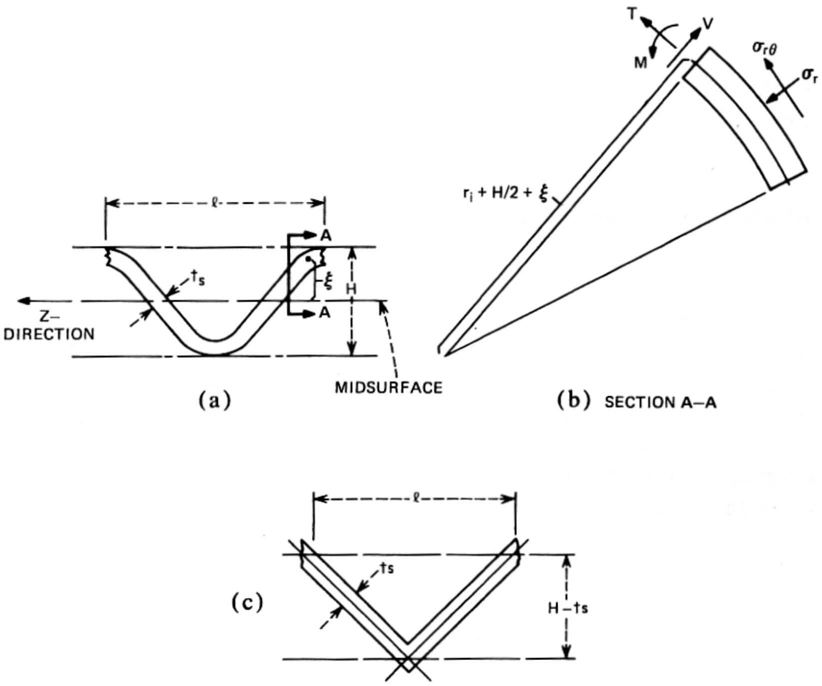

Fig. 3—Corrugation details.

where A and I are respectively the area and area moment of inertia about the midsurface of a corrugation wave length (see Fig. 3a), and ϵ_θ is the circumferential strain in the jacket.

Finally, recall that for the shield to be in equilibrium under the interfacial loads $\sigma_r, \sigma_{r\theta}$ exerted by the jacket, T and M must satisfy[10]

$$\left(r_i - \frac{H}{2}\right) \sigma_r = T - \frac{1}{r_i - H/2} \frac{d^2M}{d\theta^2} \qquad \text{at } r = r_i$$
$$\left(r_i - \frac{H}{2}\right) \sigma_{r\theta} = -\frac{dT}{d\theta} - \frac{1}{r_i - H/2} \frac{dM}{d\theta} \qquad \text{at } r = r_i \tag{19}$$

The sought-after boundary conditions at the inner surface of the jacket can now be extracted from (18) and (19) with the aid of (5), (9), (11), and (13). Neglecting terms in $H/2r_i$ compared to unity, one has

$$-r_i\sigma_r + \frac{E_sA}{\ell}\hat{\epsilon}_\theta - \left(\frac{AH}{I} M + \frac{1}{r_i}\frac{d^2M}{d\theta^2}\right) = \frac{E_sA}{\ell}\frac{\nu r_i}{\rho}\sin\theta \tag{20a}$$

$$r_i\sigma_{r\theta} + \frac{E_sA}{\ell}\frac{\partial\hat{\epsilon}_\theta}{\partial\theta} - \left(\frac{AH}{I} - \frac{1}{r_i}\right)\frac{dM}{d\theta} = \frac{E_sA}{\ell}\frac{\nu r_i}{\rho}\cos\theta \tag{20b}$$

where

$$M = -\frac{E_s I}{r_i{}^2 \ell}\left(\hat{u}_r + \frac{\partial^2 \hat{u}_r}{\partial \theta^2}\right) \tag{20c}$$

The remaining boundary conditions

$$\sigma_r = \sigma_{r\theta} = 0 \qquad \text{at } r = r_o \tag{21}$$

insure the absence of load on the exterior of the cable.

IV. SOLUTION OF THE PROBLEM

According to the plane theory of elasticity, there exists an Airy stress function ϕ generating the stresses σ_r, σ_θ, and $\sigma_{r\theta}$ through the relations[5]

$$\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} \\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) \end{aligned} \tag{22}$$

In addition, ϕ must satisfy the biharmonic equation

$$\nabla^4 \phi = 0 \tag{23}$$

Michell[11] has given a general form for Airy's function for annular domains subject to the requirement that the stresses and displacements be single-valued.* Examining Michell's solution (see also Fung,[5] p. 246) and the boundary conditions (2), it seems worth trying ϕ in the form

$$\phi = (ar^3 + b/r)\sin\theta \tag{24}$$

To determine the constants a and b in (24), substitute first into (22) and impose the boundary conditions (21) to conclude that

$$b = ar_o{}^4$$

and hence

$$\begin{aligned} \sigma_r = -2af(r)\sin\theta,\ \sigma_\theta = 2ag(r)\sin\theta,\ \sigma_{r\theta} = 2af(r)\cos\theta \\ f(r) = r(r_o{}^4/r^4 - 1),\ g(r) = r(r_o{}^4/r^4 + 3) \end{aligned} \tag{25}$$

Before enforcing (20), we compute the displacements and strains gen-

* Although Michell's argument for the generality of his solution is sketchy, his result is nevertheless correct. This can be established directly with the aid of Muskhelishvili's[12] complex valued stress functions.

erated by the stress state in (25). Toward this end, substitute (25) into the plane strain form of Hooke's law to arrive at

$$\hat{\epsilon}_r = -\frac{a}{\mu}\hat{\mathscr{E}}_r(r)\sin\theta,\ \hat{\epsilon}_\theta = \frac{a}{\mu}\hat{\mathscr{E}}_\theta(r)\sin\theta,\ \hat{\epsilon}_{r\theta} = -\frac{a}{\mu}f(r)\cos\theta$$
$$\hat{\mathscr{E}}_r = r\left(\frac{{r_o}^4}{r^4} + 4\nu - 1\right),\ \hat{\mathscr{E}}_\theta = r\left(\frac{{r_o}^4}{r^4} + 3 - 4\nu\right) \tag{26}$$

If eqs. (26) are now incorporated into the strain displacement relations (1) and an elementary integration is performed, it is found that

$$\hat{u}_r = \frac{a}{2\mu}\left[\frac{{r_o}^4}{r^2} - (4\nu - 1)r^2 + \hat{A}\right]\sin\theta + \hat{B}\cos\theta$$
$$\hat{u}_\theta = -\frac{a}{2\mu}\left[\frac{{r_o}^4}{r^2} + (5 - 4\nu)r^2 + \hat{A}\right]\cos\theta - \hat{B}\sin\theta + \hat{C}r \tag{27}$$

where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are constants fixed by the condition (6b). In fact, (27), (13), (11), and (6b) give

$$\hat{B} = \hat{C} = 0,\ \hat{A} = \left(\frac{\mu\nu}{\rho a} + 2(2\nu - 3)\right){r_o}^2 \tag{28}$$

We are now in a position to determine the unknown constant a. Observe first that (27) and (20) require that the moment M in the steel vanish identically. Thus, the second of (20) follows from the first provided

$$\frac{\partial\sigma_{r\theta}}{\partial\theta} = \sigma_r \qquad \text{at } r = r_i$$

a condition that is met for all r by the stress field in (25). Equations (25), (26), and the first of (20), therefore, reveal that the boundary relations (20) are all satisfied provided

$$a = \frac{\mu\nu r_i/\rho}{\hat{\mathscr{E}}_\theta(r_i) + Sf(r_i)},\ S = \frac{2\mu\ell r_i}{E_s A} \tag{29}$$

The result in (29) may be rewritten in the form

$$a = \frac{\mu\nu}{\lambda\rho},$$
$$\lambda = 3 - 4\nu + \frac{{r_o}^4}{{r_i}^4} + S\left[\frac{{r_o}^4}{{r_i}^4} - 1\right] \tag{30}$$

with the aid of (25) and (26). The stress field is now completely determined since by (8) and (25),

$$\sigma_z = \left(1 + \nu + \frac{4a\rho\nu}{\mu}\right)\frac{2\mu r}{\rho}\sin\theta$$

or, using (30) and (4),

$$\sigma_z = \left[1 + \frac{4\nu^2}{\lambda(1+\nu)}\right]\frac{Er}{\rho}\sin\theta \tag{31}$$

By the same token, the displacements are given by (11), (13), (27), (28), and (30) in the form

$$\begin{aligned} u_r &= \frac{a{r_o}^2}{2\mu}\left[\frac{{r_o}^2}{r^2} - (\lambda + 4(\nu - 1))\frac{r^2}{{r_o}^2} - \frac{\lambda}{\nu}\frac{z^2}{{r_o}^2} + \lambda + 2(2\nu - 3)\right]\sin\theta \\ u_\theta &= -\frac{a{r_o}^2}{2\mu}\left[\frac{{r_o}^2}{r^2} + (5 - 4\nu - \lambda)\frac{r^2}{{r_o}^2}\right. \\ &\qquad\qquad \left. + \frac{\lambda}{\nu}\frac{z^2}{{r_o}^2} + \lambda + 2(2\nu - 3)\right]\cos\theta \end{aligned} \tag{32}$$

The solution to the original three-dimensional problem has thus been found. Although there is no available uniqueness theorem encompassing the present problem (the boundary conditions (20) are nonstandard), it can be shown from Michell's[11] representation for the Airy function that the solution obtained here is unique except for certain peculiar choices of the elastic and geometric parameters of the jacket and shield. The physical significance of these instances of non-uniqueness is related to the important sheath-buckling phenomenon that has been observed in telephone cables. Since the present analysis is insufficient to adequately describe the crucial z-dependence of the ripples, we shall explore this issue no further in this paper.

V. DISCUSSION OF THE STRESSES IN THE JACKET

Observe first [eq. (31)] that the axial stress σ_z varies linearly with the distance $y = r\sin\theta$ from the neutral plane ($y = 0$), just as in the elementary Bernoulli-Euler theory. In addition, (25) reveals that σ_r and σ_θ are likewise antisymmetric about the neutral plane, σ_r being compressive and σ_θ tensile when σ_z is tensile and vice versa. The shear stress $\sigma_{r\theta}$, as expected, is symmetric about the neutral plane.

The variation in σ_r (and hence $\sigma_{r\theta}$) and σ_θ across the thickness of the jacket is illustrated in Fig. 4, where the dimensionless functions

$$\text{FBAR} = -f/4r_o,\ \text{GBAR} = g/4r_o \tag{33a}$$

[Eq. (25)] are plotted against the dimensionless radius

$$\text{RBAR} \equiv \bar{r} \equiv r/r_o \tag{33b}$$

for $0.5 \le \bar{r} \le 1$. It is seen that as long as the jacket thickness is less than 10 percent of the cable radius ($0.9 < \bar{r} < 1$), the transverse stress σ_r is less than 10 percent of the hoop stress σ_θ, which is itself essentially constant across the jacket.

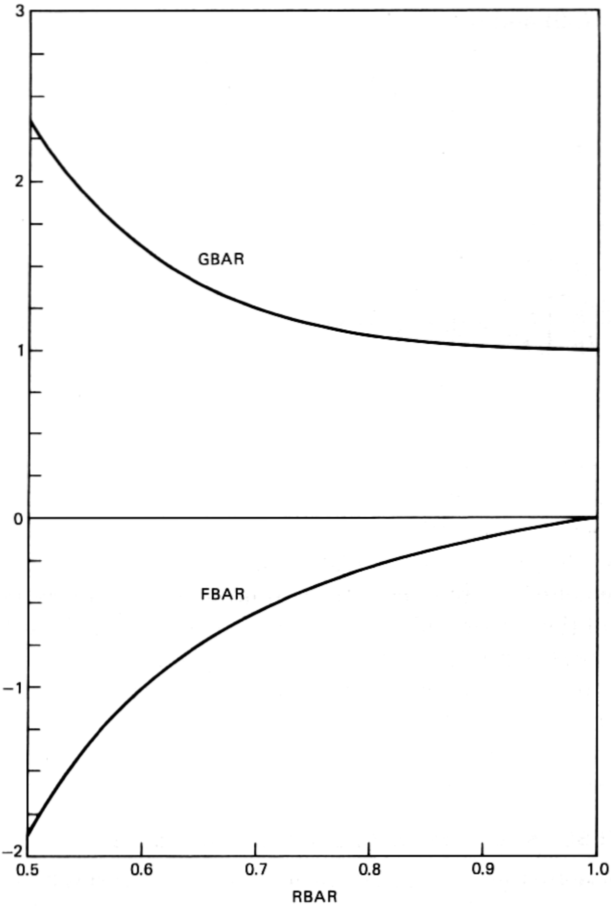

Fig. 4—Radial dependence of σ_r (FBAR) and σ_θ (GBAR).

With a view toward examining the influence of the properties of the jacket on the stress state, let $\sigma_z{}^m$ and $\sigma_\theta{}^m$ stand for the maximum values of the axial and hoop stresses, respectively. Then (25) and (31) together with (30) and (4) give

$$\begin{aligned} \sigma_\theta{}^m &\equiv \sigma_\theta(r_i,\pi/2,z) = \frac{E\nu}{(1+\nu)\lambda\rho} g(r_i) \\ \sigma_z{}^m &\equiv \sigma_z(r_o,\pi/2,z) = \left[1 + \frac{4\nu^2}{(1+\nu)\lambda}\right]\frac{Er_o}{\rho} \end{aligned} \tag{34}$$

The ratio of the maximum bending stress in the cable to the prediction of elementary beam theory is thus

$$\text{SIGMAX} \equiv \frac{\sigma_z^m}{Er_o/\rho} = 1 + \frac{4\nu^2}{(1+\nu)\lambda} \tag{35}$$

Further, the degree of biaxiality created in the jacket by its constrained inner surface is described by the parameter

$$\text{ALPHA} \equiv \frac{\sigma_\theta{}^m}{\sigma_z{}^m} = \frac{\nu g(r_i)/r_o}{(1+\nu)\lambda + 4\nu^2} \tag{36}$$

In Fig. 5, the dimensionless stresses SIGMAX and ALPHA are plotted versus Poisson's ratio ν for a 3-inch diameter Stalpeth cable conforming to the current Bell System design. In computing the parameter S that enters the formula (30) for λ, the area A of a half-corrugation wave length has been approximated by

$$A = \sqrt{m^2+1}\,\ell t_S,\; m = \frac{H - t_S}{\ell} \tag{37}$$

which is the area of the parallelogram shown in Fig. 3c. Also, the modulus E_S was assumed to be that of steel and the jacket modulus E was supposed to be 45,000 psi, representative of low-density polyethylene at room temperature.[13]

The curves shown are quite insensitive to variations in jacket modulus, SIGMAX changing by less than ½ percent and ALPHA by less than 2 percent when E is increased to 90,000 psi. The results indicate that for a Poisson's ratio $\nu = 0.35$, the bending stress in the jacket is about 12 percent higher than the prediction of the elementary theory while the hoop stress is roughly ⅓ of the axial stress.

The formulae (35) and (36) can also be used to investigate the effect of cable size on the state of stress in the jacket. Once the cable diameter is chosen, all other geometric parameters (e.g., jacket thickness, corrugation height) are fixed. This information was incorporated into (35) and (36) resulting in equations relating SIGMAX and ALPHA to cable diameter and the elastic constants of the jacket. For the range of Stalpeth cable currently manufactured, SIGMAX and ALPHA are independent of cable diameter.

The conclusions already reached as well as others of interest may be inferred more easily from the approximations of the previous equations arising from assuming the jacket to be thin. Toward this end, let

$$t = r_o - r_i \tag{38}$$

so that, for any n

$$\left(\frac{r_i}{r_o}\right)^n = \left(1 - \frac{t}{r_o}\right)^n = 1 - n\frac{t}{r_o} + 0\left[\left(\frac{t}{r_o}\right)^2\right] \quad \text{as } t/r_o \to 0 \tag{39}$$

If the approximation in (39) is applied to the formula (30) for λ, one gets

$$\lambda = 4(1-\nu) + 4t/r_o + 4St/r_o + 0[(t/r_o)^2]$$

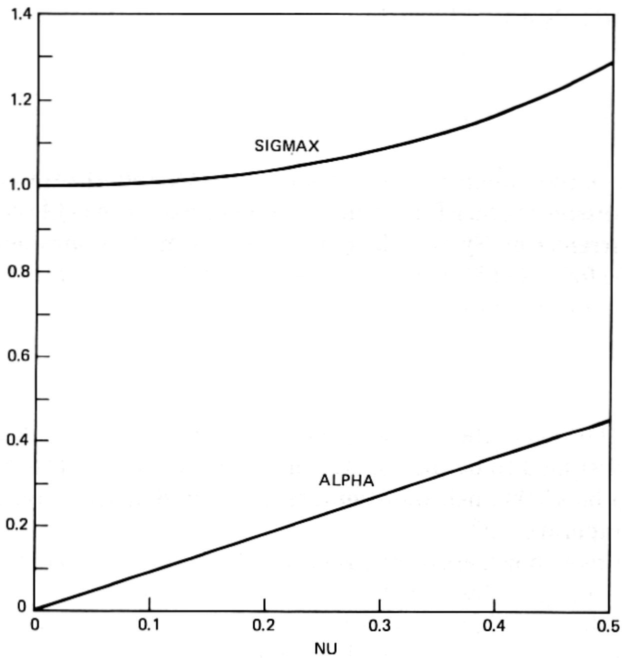

Fig. 5—Dependence of bending and hoop stresses on Poisson's ratio (cable diameter = 3 inches, E = 45,000 psi).

But from (30), (37), and (39)

$$\frac{St}{r_o} = \frac{2\mu}{E_S\sqrt{m^2+1}}\frac{t}{t_S}\frac{r_i}{r_o} = \frac{2\mu t}{E_S t_S\sqrt{m^2+1}}\left(1-\frac{t}{r_o}\right) + 0\left[\left(\frac{t}{r_o}\right)^2\right]$$

whence

$$\begin{aligned}\lambda &= 4\left(1-\nu+\frac{S_o}{1+\nu}\right) + 0(t/r_o)\\ S_o &= \frac{2\mu(1+\nu)t}{\sqrt{m^2+1}\,E_S t_S} = \frac{Et}{\sqrt{m^2+1}\,E_S t_S}\end{aligned} \tag{40}$$

Equations (40) and (35) give

$$\text{SIGMAX} = 1 + \frac{\nu^2}{1-\nu^2+S_o} + 0(t/r_o) \tag{41}$$

Similarly,

$$\frac{g(r_i)}{r_o} = 4 + 0[(t/r_o)^2]$$

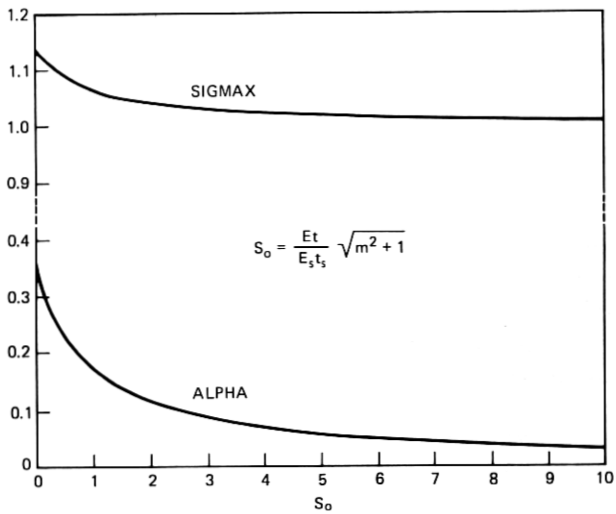

Fig. 6—Dependence of bending and hoop stresses on the modulus parameter S_o (ν = 0.35).

so that, using (40) as well, ALPHA may be approximated from (36) by

$$\text{ALPHA} = \frac{\nu}{1 + S_o} + 0(t/r_o) \tag{42}$$

The dependence of the nondimensionalized stresses on Poisson's ratio is thus seen from (41) and (42) to have the form indicated in Fig. 5. Moreover, (41) and (42) exhibit no explicit dependence on cable radius, though there is a slight implicit dependence entering through S_o, since t and m are governed by cable diameter.

The approximate relations (41) and (42) have been used to generate the curves in Fig. 6. The parameter S_0 for present cable designs is very small (10^{-2} to 10^{-3}) which accounts for the insensitivity of the curves in Fig. 5 to variations in jacket modulus in contrast to those in Fig. 6. In fact, the ratio $Et/E_s t_s$ is so small for all conceivable cable applications that S_o may as well be neglected. If this is done, then (41), (42) can be used in conjunction with (25), (31), (35), and (36) to provide

$$\sigma_z \approx \frac{E}{1 - \nu^2} \epsilon, \ \sigma_\theta \approx \nu \sigma_z$$
$$\epsilon = \frac{r_o \sin \theta}{\rho} \tag{43}$$
All other stresses zero

as a good approximation to the jacket stresses.

VI. IMPLICATIONS ON CABLE PERFORMANCE

One important result that emerges from the present investigation is that (43) (for fixed θ, $0 < \theta < \pi$) is the stress state that arises in the so-called "biaxial strip" experiment (Fig. 7). When a strain $\epsilon > 0$ is imposed in the y direction, the strip is in plane stress ($\sigma_z \approx 0$) throughout and plane strain ($\epsilon_y \approx 0$) in a region near the center. The circumferential and longitudinal directions in the cable thus correspond, respectively, to the x and y directions in Fig. 7.

The approximate plane strain condition in the bent Stalpeth cable is created by the adherence of the jacket to the steel. Should this constraint remain intact during continued bending of the cable, the analogy with the strip experiment also continues. In that event, the elongation at break in the biaxial strip experiment would be an important material parameter. Another biaxial experiment[4] (equal principal stresses) indicates that some low-density polyethylenes exhibit an ultimate elongation biaxially that is substantially reduced from the uniaxial value, say to 20 percent or perhaps even less at lower temperatures or higher rates. Since jacket strains typically reach 15 percent during duct installation, failure due simply to biaxiality is indeed a concern. The strip experiment should, therefore, be instituted as a materials screening test.

Even if the material exhibits a high elongation in the biaxial strip experiment, however, the cable jacket may fail because of the localization of deformation in the neighborhood of imperfections, such as corrugation imprints or surface scratches. In these cases, failure occurs at bend radii for which the present analysis is applicable to points in the jacket away from the imperfection. Thus, if the geometry and orientation of the imperfection are known, concentration factors may be used to determine under what conditions the transition to highly localized deformation occurs. The critical parameters at which this transition occurs are currently found from impact tests on relatively narrow notched bars.[1] The

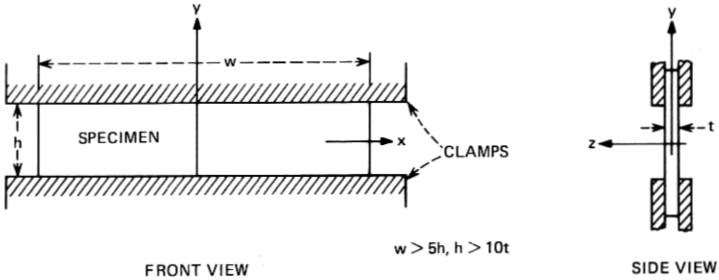

Fig. 7—Biaxial strip experiment (clamps pulled in y direction).

biaxiality evidenced here should have little effect on the conclusions drawn from such a procedure, since the notches create a complex state of stress which is largely independent of width.[14]

In addition, the growth of sharp internal or surface flaws under the influence of the far-field stresses calculated here can be analyzed with the aid of a recently developed viscoelastic theory[15] coupled with stress intensity factors available from the literature.[16] Theoretical failure times would then be compared against the duration of loading to give a probability of cracking failure for a given flaw population. Here, we remark only that the probability of premature fracture is obviously enhanced by the constraining effect of the steel, since the severity of loading for those flaws not oriented in the θ direction is increased by the biaxiality.

In addition to the effect on fracture performance, biaxiality can also have a substantial effect on the yielding behavior of plastics.[4,17] But in view of the fact that cable jackets, including Stalpeth jacket, are imprinted with corrugation valleys, *localized* yielding behavior is more relevant.

The failure mechanism on which the present analysis sheds the most light is that of sheath buckling. Firstly, on the compressive side of the cable, (25) indicates that the jacket exerts tensile loads on the flooding compound. Estimates from the preceeding formulae reveal the magnitude of this stress to be as high as 200 psi during installation. It would therefore seem that only an exceptional flooding compound would successfully restrain the jacket at low temperatures and suggests a need for the mechanical characterization of these compounds. The situation is further aggravated by the biaxiality and the fact that the stresses vary in proportion to the modulus. This accounts for the occurrence of buckling at low installation temperatures. The temperature at which buckling will occur for a given duration of loading can be estimated from this analysis and elementary viscoelasticity theory. This buckling analysis is presently being pursued.

VII. CONCLUSIONS

We have shown that:

(*i*) The state of stress in the jacket of a bent telephone cable is essentially biaxial and constant across the thickness.

(*ii*) At any fixed point in the jacket of a Stalpeth cable, the stress state is essentially that in the biaxial strip:

$$\sigma_z \approx \frac{E}{1-\nu^2}\epsilon, \ \sigma_\theta \approx \nu\sigma_z, \ \epsilon \equiv \frac{r_o \sin\theta}{\rho}$$

with all other stresses and strains negligible. Thus, the maximum bending stress is 110–130 percent of the elementary beam theory prediction while the maximum hoop stress is 30–45 percent of the bending stress.

(*iii*) The flooding compound on the compressive side of the cable is subjected to tensile and shear stresses as high as 200 psi.

(*iv*) Practical variations in cable design parameters such as corrugation depth and frequency, steel thickness and jacket thickness have no significant effect on the global stresses due to bending.

Having the results of this as well as previous investigations in mind, we have concluded that:

(*i*) The currently used techniques[1] for evaluating the notch sensitivity of plastics are applicable to the bending conditions encountered in cable installation.

(*ii*) The failure strength of flooding compounds can play an important role in preventing jacket buckling. For this reason, any anticipated change in flooding material should be thoroughly examined with respect to tensile strength and adhesive strength.

(*iii*) The probability of spontaneous cracking is increased by the adherence of the jacket to the soldered steel layer.

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