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A Generalization of Takagi's Theorem on Optimal Channel Graphs

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A channel graph, also called a linear graph,[5] *is a multistage graph with the properties that (i) each of the first and the last stages consists of a single vertex (denoted by I and O respectively); (ii) for any vertex* $v \neq I$ *or* O, v *is adjacent to at least one vertex from the preceding stage and at least one vertex from the following stage. In a switching network, the union of all paths connecting a fixed input terminal to a fixed output terminal can usually be studied as a channel graph by taking each switch as a vertex. In comparing the blocking probabilities of two channel graphs with the same number of stages, we say one is* superior *to another if its blocking probability is less than or equal to that of the other under any link occupancies. Takagi proved a basic theorem in showing one type of channel graph is superior to another. In this note we present a more powerful result which includes Takagi's theorem as a special case.*

I. INTRODUCTION

A graph is called a *multistage graph* if its vertex set can be partitioned into subsets $V_1, \ldots, V_s$, for some number s, and its edge set into subsets $E_1, \ldots, E_{s-1}$ such that E_i connects V_i with V_{i+1}. A *channel graph,* also called a linear graph,[5] is a multistage graph with the properties that (i) each of the first and the last stages consists of a single vertex (denoted by I and O respectively); (ii) for any vertex $\neq I$ or O, v is adjacent to at least one vertex from the preceding stage and at least one vertex from the following stage. In a switching network, the union of all paths connecting a fixed input switch to a fixed output switch can usually be studied as a channel graph by taking each switch as a vertex and each link as an edge.

In a switching network, a link can be in either one of two states, *busy* or *idle,* depending on whether it is part of a connection carrying a call. A path from a fixed input switch to a fixed output switch is *blocked* if

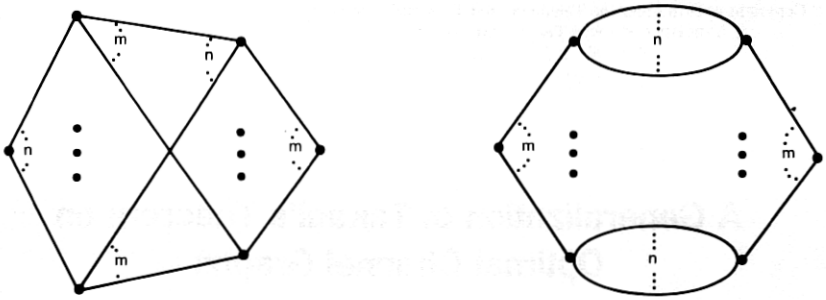


Fig. 1—Channel graphs in Takagi's theorem.

it contains a busy link. A pair of switches is blocked if every path between them is blocked. The same notion of "blocking" applies to the study of channel graphs and therefore we can talk about the blocking probability of a channel graph.

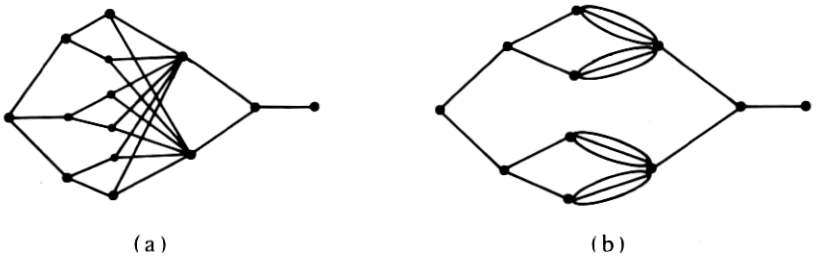
A *series-parallel* channel graph is a channel graph which is either a series combination or a parallel combination of two smaller series-parallel channel graphs with an edge being the smallest such graph. Channel graphs which are not series-parallel are often called spider-web channel graphs. Recent studies have shown, either by analysis or by simulation (see Refs. 3 and 9, for example), that spider-web channel graphs can sometimes significantly reduce blocking probabilities over series-parallel channel graphs for given switching network hardware. In particular, Takagi[8,9] gives a useful theorem which compares the blocking probability of the spider-web channel graph in Fig. 1a and the series-parallel channel graph in Fig. 1b.

In Fig. 1a, the connection between the two middle stages can be viewed as a complete bipartite graph on m and n vertices. In Fig. 1b, the connection between the two middle stages can be viewed as a matching of m pairs, each pair joined by n multiple edges.

The above theorem is the basis of Takagi's work[8,9] on optimal channel graphs which has been widely quoted in the literature (see Refs. 1, 2, 4, 6, 7, 10, 11, and 12, for example). In this note we present a more powerful result which deals with a much larger class of channel graphs and includes Takagi's theorem as a special case.

II. TAKAGI'S THEOREM

Takagi's comparison of two 4-stage channel graphs as shown in Fig. 1 actually has broader applications than it appears. The extension is made possible by interpreting each edge in the 4-stage channel graphs as a reduction of a multistage graph. The only requirement is that the



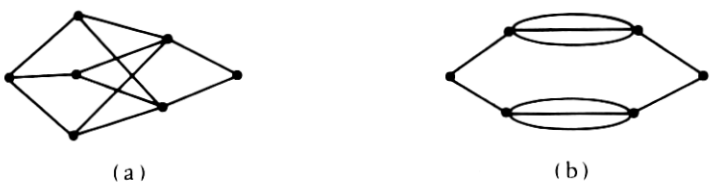
(a) (b)

Fig. 2—Two 6-stage channel graphs.

multistage graphs represented by the edges in the same set E_i are isomorphic. For instance, the two 6-stage channel graphs shown in Fig. 2 can be reduced to the two 4-stage graphs shown in Fig. 3.

Note that a vertex in the 4-stage graphs can represent a group of vertices (from the same stage) in the 6-stage channel graphs. Furthermore, two disjoint edges in the 4-stage channel graphs can come from two nondisjoint subgraphs of the 6-stage channel graphs. Finally, an edge in the 4-stage channel graphs can have more than one state where a state is basically a distinct subset of nonblocking paths in the corresponding multistage graph.

By associating a probability distribution to the joint states of the edges, the blocking probability of a channel graph can be computed. Let D be a collection of probability distributions on the joint state of the edges. Then an s-stage channel graph G is said to be *superior* to another s-stage channel graph G' with respect to D if given any member of D, the blocking probability of G never exceeds that of G'. Let $P(X_i)$, $i = 1$,



(a) (b)

Fig. 3—Two 4-stage channel graphs.

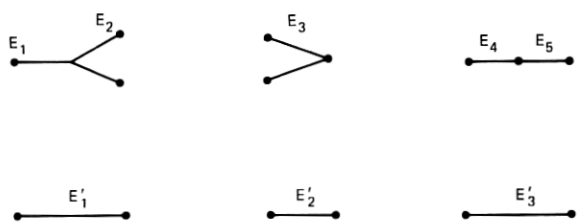
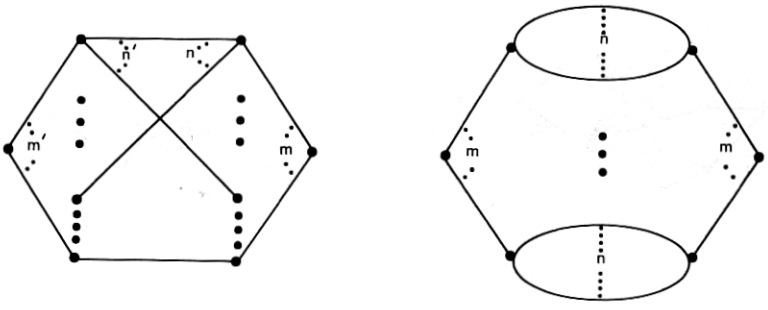


Fig. 4—The mapping from Fig. 2 to Fig. 3.



(a) (b)

Fig. 5—Channel graphs in main theorem.

$\ldots,x$, denote the probability that an edge of E_1 is in state X_i, let $P(Z_j)$, $j = 1, \ldots, y$, denote the probability that an edge of E_3 is in state Z_j, and let $Y(i,j)$ denote the blocking probability of a path from I to O which contains an edge of E_1 in state X_i and an edge of E_3 in state Z_j. Finally, let S be a joint state of the edges in E_3. Then Takagi proves:

Takagi's theorem. The channel graph of Fig. 1a is superior to the channel graph of Fig. 1b for arbitrarily given S, $P(X_i)$, and $Y(i,j)$ under the following assumptions:

(*i*) The states of the edges are independent.
(*ii*) $n \geq m$.

In the next section, we give a generalization of Takagi's theorem.

III. THE MAIN THEOREM

Consider the two channel graphs in Fig. 5.

In Fig. 5a, each vertex in stage 2 has n' edges connected to n' distinct vertices in stage 3, and each vertex in stage 3 has n edges connected to n distinct vertices in stage 2. Furthermore $mn = m'n'$, $n' \leq m$ and $n \leq m'$. Figure 5b is the same as Fig. 1b.

Using the same notation as in Section II, we now prove:

Main theorem. The channel graph in Fig. 5a is superior to the channel graph in Fig. 5b for arbitrarily given S, $P(X_i)$ and $Y(i,j)$ under the following assumptions:

(*i*) The states of the edges are independent.
(*ii*) $m' \geq m$.

Before we prove the theorem we state a lemma proved by Takagi[8] which is a generalized version of Hölder's inequality.

Lemma. If $a_{ij} \geq 0$, and

$$\sum_{j=1}^{m} \frac{1}{b_j} = 1$$

for $b_j > 1$, and $\lambda_i \geq 0$, then the following inequality holds:

$$\sum_{i=1}^{n} \left(\lambda_i \prod_{j=1}^{m} a_{ij} \right) \leq \prod_{j=1}^{m} \left(\sum_{i=1}^{n} \lambda_i a_{ij}^{b_j} \right)^{1/b_j} \tag{1}$$

Proof of the theorem. Let S be an arbitrary state on the set of edges between stage 3 and stage 4 in Fig. 5. Suppose under S, z_j edges between stage 3 and stage 4 are in the state Z_j, $j = 1, \ldots, y$. In Fig. 5a, let w_{kj} be the number of edges joining the kth vertex in stage 2 to a vertex v in stage 3 such that the edge between v and O is in the state Z_j. Then it can be easily checked:

$$\sum_{k=1}^{m'} w_{kj} = nz_j \tag{2}$$

$$\sum_{j=1}^{y} w_{kj} = n' \tag{3}$$

and

$$\sum_{j=1}^{y} z_j = m \tag{4}$$

Let $Y(i,j)$ denote the blocking probability of a path from I to O which contains an edge in the state X_i between stage 1 and stage 2 and an edge in the state Z_j between stage 3 and stage 4. Furthermore, let $P(X_i)$ denote the probability of X_i and let x be the number of possible states X_i. Then the blocking probabilities of the channel graphs in Fig. 5a and 5b for the given state S, denoted by B_a and B_b, are

$$B_a = \prod_{k=1}^{m'} \left[\sum_{i=1}^{x} P(X_i) \prod_{j=1}^{y} Y(i,j)^{w_{kj}} \right]$$

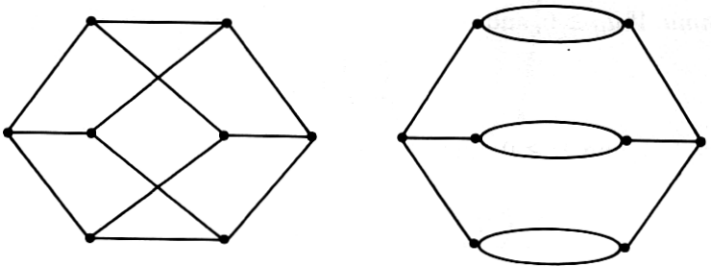
and

$$B_b = \prod_{j=1}^{y} \left[\sum_{i=1}^{x} P(X_i) Y(i,j)^n \right]^{z_j}$$

Using (2), (3), and (4), we have

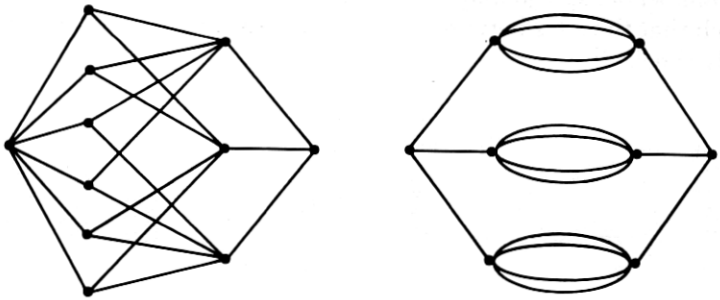
$$B_b = \prod_{j=1}^{y} \left[\sum_{i=1}^{x} P(X_i) Y(i,j)^n \right]^{\sum_{k=1}^{m'} w_{kj}/n}$$

$$= \prod_{j=1}^{y} \left\{ \prod_{k=1}^{m'} \left[\sum_{i=1}^{x} P(X_i) Y(i,j)^n \right]^{w_{kj}/n} \right\} = \prod_{k=1}^{m'} \left\{ \prod_{j=1}^{y} \left[\sum_{i=1}^{x} P(X_i) Y(i,j)^n \right]^{w_{kj}/n} \right\}$$



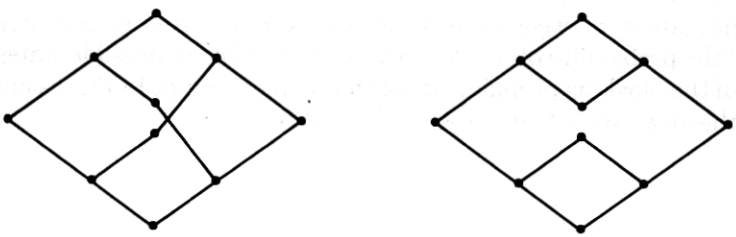
(a) (b)

Fig. 6—Example 1.



(a) (b)

Fig. 7—Example 2.



(a) (b)

Fig. 8—A counterexample.

(a) (b)

Fig. 9—A reduction of Fig. 8.

If $m' = m$, then $n' = n$. By substituting $\lambda_i = P(X_i)$, $a_{ij} = Y(i,j)^{w_{kj}}$ and $b_j = n/w_{kj}$ in (1), we obtain

$$B_b \geq \prod_{k=1}^{m'} \left\{ \sum_{i=1}^{x} P(X_i) \prod_{j=1}^{y} Y(i,j)^{w_{kj}} \right\} = B_a$$

If $m' > m$, then $n > n'$. Define

$$Y(i,y+1) = 1 \quad \text{and} \quad b_{y+1} = \frac{n}{n-n'}$$

Then $b_{y+1} > 1$ and

$$\sum_{j=1}^{y+1} \frac{1}{b_j} = \sum_{j=1}^{y} \frac{w_{kj}}{n} + 1 - \frac{n'}{n} = 1$$

Now

$$B_b = \prod_{k=1}^{m'} \left\{ \prod_{j=1}^{y+1} \left[\sum_{i=1}^{x} P(X_i) Y(i,j)^n \right]^{b_j} \right\}$$

$$\geq \prod_{k=1}^{m'} \left\{ \sum_{i=1}^{x} P(X_i) \prod_{j=1}^{y+1} Y(i,j)^{n/b_j} \right\}$$

$$= \prod_{k=1}^{m'} \left\{ \sum_{i=1}^{x} P(X_i) \prod_{j=1}^{y} Y(i,j)^{w_{kj}} \right\} = B_a$$

where the inequality is obtained by making a similar substitution as before. The proof is complete.

We note that by setting $m' = n$ (hence $n' = m$) in the given theorem, we obtain Takagi's theorem immediately.

IV. DISCUSSION

Two examples to which Takagi's theorem does not apply while our theorem does are shown in Figs. 6 and 7. In both figures, the channel graph (a) is superior to that of (b). The comparison in Fig. 6 is especially useful since the degrees of corresponding vertices are exactly the same.

Next we ask can we generalize our theorem in the direction that the states of the edges between stage 1 and stage 2 can also be dependent. We conjecture the theorem will still be true but no proof is known yet. Certainly we cannot hope to imitate the proof given here. A ready counterexample to this approach is illustrated in Fig. 8.

If the two states we assume are such that the first edge between the first two stages and the first edge between the last two stages are blocked, then the two channel graphs can be reduced to the channel graphs in Figs. 9a and b, respectively. Clearly, the channel graph in Fig. 9b is superior. But from Takagi's theorem or our theorem, we know that the channel graph in Fig. 9a is superior to that in Fig. 9b.

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