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On the Phase of the Modulation Transfer Function of a Multimode Optical-Fiber Guide

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We consider the range of validity of a Hilbert-transform approach in which the measured magnitude of the modulation-transfer-function of an optical fiber is used to compute the fiber's impulse response. It is argued that a key "minimum-phase assumption" can fail to be satisfied in important cases, and a few closely related experimental and analytical results are presented.

I. INTRODUCTION

Pulse dispersion in an optical fiber transmission line limits its information-carrying capacity by limiting the temporal spacing of input pulses that can be resolved at the output. The impulse response $g(t)$, by which we mean the output power of a fiber excited by a unit impulse of optical power, provides the necessary information concerning the distortion of the pulse by modal and material dispersion. For a strictly monochromatic pulse source, only modal dispersion contributes to the distortion. However, with regard to the corresponding measurement problems, it is difficult to obtain sufficiently short (<0.5 nsec) and monochromatic (< 10 Å) input pulses to accurately study fibers with very low modal dispersion. Another difficulty is the lack of availability of suitable sources that are tunable over a wide range of wavelengths, including wavelengths longer than 1 μm which are of interest for practical fiber systems.

One can obtain $g(t)$ from the modulation frequency transfer function (MTF) $G(\omega)$, which is the envelope response of the fiber to an incoherent optical signal sinusoidally modulated in amplitude at angular frequency ω. Personick[1] has shown that to the extent that certain reasonable approximations hold, $g(t)$ and $G(\omega)$ are a Fourier transform pair. In principle, the MTF can be determined experimentally. The method employed in Refs. 2 and 3 uses a xenon lamp and monochromator as a tunable source that can be sinusoidally modulated to high frequency (>1 GHz) by an electrooptic modulator. It is straightforward to measure the magnitude $|G(\omega)|$ of the transfer function

$$G(\omega) = |G(\omega)|e^{i\theta(\omega)} \tag{1}$$

using available components, including an RF spectrum analyzer. However, because the fibers must be long (~1 km) in order to obtain good measurement precision, the measurement of $\theta(\omega)$ appears to be formidable. As $f = \omega/2\pi$ varies from zero to 1 GHz, $\theta(\omega)$ varies nearly linearly with f from zero to $10^4\pi$ radians for a 1 km long fiber. However, the contribution of $\theta(\omega)$ to pulse dispersion is due to a nonlinear deviation $\Delta\theta(\omega)$, on the order of 2π, from the much larger linear phase shift $\theta_0(\omega)$; i.e.,

$$\theta(\omega) = \theta_0(\omega) + \Delta\theta(\omega) \tag{2}$$

$$\theta_0(\omega) = \omega L/v \tag{3}$$

where L is the fiber length and v is an effective envelope velocity taken to be independent of ω. Hence, direct measurement of phase distortion in the presence of the large frequency-dependent $\theta_0(\omega)$ could be subject to large error as v varies with temperature or other environmental factors. One is therefore led[2,3] to consider methods of mathematical computation of $\theta(\omega)$ from $|G(\omega)|$ using Hilbert-transform theory.

The main purpose of this note is to report on results which indicate that unfortunately the Hilbert-transform approach described in Refs. 2 and 3 is, in general, not a helpful one for the particular problem at issue, even though early experimental results suggested otherwise.†

Roughly speaking, it is known that by using a Hilbert-transform relation the phase can be obtained from the magnitude of a transfer function provided that the transfer function is "minimum phase." A standard condition (which is by no means sufficient) for "minimum-phase behavior" is that the Laplace transform of the impulse response that corresponds to the transfer function have no zeros in the closed right half-plane. This is in accord with the observation that for an ideal waveguide with constant positive delay τ_0 and transfer function $G_0 = e^{-i\omega\tau_0}$, the phase cannot be determined from a knowledge of the function $|G_0(\omega)|$ alone, but a transfer function that has the same magnitude as that of the waveguide is $G_1(\omega) = 1$ for which the phase is zero for all ω.††

† The statement on page 1518 of Ref. 2 concerning the possible lack of "approximate minimum phase behavior" was motivated by the results of the joint work described here.

†† The mathematical reason that G_0 is not a "minimum-phase" function (even though $e^{-z\tau_0}$ has no zeros for $\mathrm{Re}(z) \geq 0$) is that $\ln|e^{-z\tau_0}|$ fails to satisfy a sufficiently strong growth condition in the half-plane $\mathrm{Re}(z) \geq 0$. See Sections 5.1 and 5.3 for related material.

For our purposes, the difference between $G_0(\omega)$ and $G_1(\omega)$ is unimportant, because we are willing to ignore a constant time delay that can easily be estimated. More generally, it is reasonable to ask if $e^{i\omega\tau_0}G(\omega)$ is a minimum-phase function, where τ_0 denotes the linear part of the delay. An early impulse response measurement on an actual fiber suggested[2,3] that this might indeed be the case. However, the further analytical and experimental study reported on here shows that nonminimum-phase behavior is likely to arise, and can arise, in important actual cases. Some additional closely related material is also presented.

Methods for circumventing the difficulties described in this note are under study, and it is expected that they will be described in a later paper.

II. SOME ANALYTICAL PROPERTIES OF THE MULTIMODE TRANSFER FUNCTION

In the general case, the transfer function of a fiber can be written in the form

$$G(\omega) = \int_{T_a}^{T_b} e^{-i\omega\tau} da(\tau) \tag{4}$$

in which the integrator $a(\tau)$ is a real-valued monotonically nondecreasing function of τ,† and T_a and T_b, which are fixed by the refractive indices of core and cladding, are the smallest and largest modal delays, respectively.†† Often $a(\tau)$ is normalized so that

$$\int_{T_a}^{T_b} da(\tau) = 1$$

For a fiber that can propagate n discrete modes without mode mixing, (4) becomes

$$G(\omega) = \sum_{j=1}^{n} d_j e^{-i\omega\tau_j} \tag{5}$$

in which each d_j is a positive constant that represents the initial excitation of the jth mode. Typically, $n > 100$. Most of our discussion is concerned with the important particular case in which (5) holds, and, in order to avoid a lack of continuity of the presentation, proofs of the results discussed are given in a separate section. We assume that the τ_j are ordered so that $\tau_1 < \tau_2 < \ldots < \tau_n$.

In (5), $G(\omega)$ is the generalized Fourier transform of a finite train of

† Thus, roughly speaking, $da(\tau)$ in (4) can be replaced with $b(\tau)d\tau$ in which the function $b(\tau)$ is nonnegative and may contain impulses corresponding to discrete modes. See Refs. 1 and 2 for the relevant background material.

††We are of course assuming that material dispersion can be neglected.

not-necessarily-equally-spaced impulses. Let $H(z)$ denote the corresponding Laplace transform. That is, let $H(z)$ denote

$$\sum_{j=1}^{n} d_j e^{-z\tau_j}$$

for all complex z. Of course $G(\omega) = H(i\omega)$ for all ω.

A standard Hilbert-transform method[4,2,3] for determining the phase $\hat{\theta}(\omega)$ of a transfer function $\hat{G}(\omega)$, from the function $|\hat{G}(\omega)|$, when it is possible to do so, is to use the formula†

$$\hat{\theta}(\omega) = \frac{2\omega}{\pi} \int_0^{\infty} \frac{\ln|\hat{G}(y)|}{y^2 - \omega^2} dy \tag{6}$$

which, roughly speaking, amounts to a direct application of the Hilbert transform

$$\mathrm{Im}[f(i\omega)] = \frac{2\omega}{\pi} \int_0^{\infty} \frac{\mathrm{Re}[f(iy)]}{y^2 - \omega^2} dy \tag{7}$$

in which $f(z)$ is any complex-valued function of z that satisfies certain conditions (such as those described in Section 4.1††) which include the condition that $f(z)$ is analytic for $\mathrm{Re}(z) \geq 0$.

The way in which (7) is used to obtain (6) is of course to let $f(z)$ be a suitably defined single-valued variant of $\ln[\hat{H}(z)]$, in which $\hat{H}(z)$ is the Laplace transform of the time function whose Fourier transform is $\hat{G}(\omega)$. That is what gives rise to the well-known requirement that $\hat{H}(z)$ be zero-free in the closed right half-plane.

2.1 The zeros of H(z), and related material

With regard to the location of the zeros of $H(z)$, according to Proposition 2 of Section IV: $H(z) \neq 0$ for $\mathrm{Re}(z) \geq 0$ if

$$d_1 > \sum_{j=2}^{n} d_j \tag{8}$$

and if (8) is not satisfied, then given an arbitrarily small positive number ϵ, and *any* set of n numbers $t_1 < t_2 < \ldots < t_n$, we have $H(z) = 0$ for some z in the closed right half-plane for a choice of the τ_j such that $|\tau_j - t_j| \leq \epsilon$ for all j. In particular, and roughly speaking, unless (8) is satisfied, given any set of n delays, there is a set of delays arbitrarily close to those

† The integrals in (6) and (7) are to be interpreted as Cauchy principal values.

††Section 4.1 contains an outline of a proof that (7) holds under certain specific conditions. The derivation given, for instance in Ref. 4, lacks rigor in that, for example, the point s in Ref. 4 is initially assumed to be a point internal to a certain contour, while subsequently an expression based on that assumption is evaluated for s *on* the contour. The main reason for including the basically tutorial material of Section 4.1 is that it is used to prove another result described in this note.

delays such that $H(z)$ is not "minimum phase." Notice that it is not claimed that $H(z)$ has a zero in the closed right half-plane whenever (8) is violated.† However, Proposition 2 does imply that whenever (8) is not satisfied it is incorrect to assert that $H(z) \neq 0$ for $\text{Re}(z) \geq 0$ when the τ_j are known only to within some positive tolerance ϵ, no matter how small ϵ is. Therefore, $H(z)$ is zero-free in the closed right half-plane, and that property is structurally stable in the sense indicated, if and only if (8) is met. This result suggests that it would not be surprising to encounter "nonminimum-phase" behavior with fibers for which the total power in a sufficient number of the modes corresponding to the delays $\tau_2, \tau_3, \ldots, \tau_n$ is considerable.

An idealized example in which a somewhat analogous conclusion is reached is as follows. Consider an n-mode fiber without mode mixing for which the modal delays are equally spaced by δ sec, so that $\tau_j = \tau'_0 + j\delta$ for each positive integer j. Assume that the fractional power into the jth mode is given by $d_j = ce^{\gamma j}$ for all j, in which γ and c are constants with $c > 0$. Then

$$H(z) = ce^{-z\tau'_0}\frac{e^{(\gamma - z\delta)n} - 1}{1 - e^{-(\gamma - z\delta)}}$$

and the condition that $H(z) \neq 0$ for $\text{Re}(z) \geq 0$ will be satisfied if and only if $\gamma < 0$.†† Of course $\gamma < 0$ means that modes with larger delays have smaller excitation. A similar conclusion is reached for the continuum mode-mixing case‡ in which the integrator $a(\tau)$ of (4) has a continuous derivative that is proportional to $e^{\gamma\tau}$.

While the discussion in the preceding paragraphs suggests‡‡ that there are important cases in which (6) cannot be used, it certainly does not rule out the possibility that there is some other method for determining $\theta(\omega)$ from $|G(\omega)|$ (which, for example, might possibly exploit the fact that the d_j are positive).

In this connection, consider (5). In order to avoid the necessity of introducing a function equal to $e^{i\omega\tau_1}G(\omega)$, assume throughout the remainder of this section that $\tau_1 = 0$ (which of course simply provides a normalization†‡).

Suppose, for example, that $n = 2$ and that $d_2 = (1 - d_1)$ with $0 < d_1 < 1$ and $d_1 \neq 1/2$. Then $|G(\omega)| = d_1{}^2 + (1 - d_1)^2 + 2d_1(1 - d_1)\cos(\omega\tau_2)$

† In fact, we show that that claim would be false.

†† Since (8) is violated when γ is negative and sufficiently close to zero, we see that a given specific $H(z)$ can be zero-free in the closed right half-plane when (8) is violated.

‡ This example was suggested by H. E. Rowe.

‡‡ Little information is available concerning how to accurately specify $G(\omega)$ for an actual fiber using purely analytical methods. Very small geometrical perturbations can have significant effects on the impulse response of graded-index fibers.[5]

†‡ It is easy to see that without some such normalization, it is is not possible to determine $\theta(\omega)$ from $|G(\omega)|$.

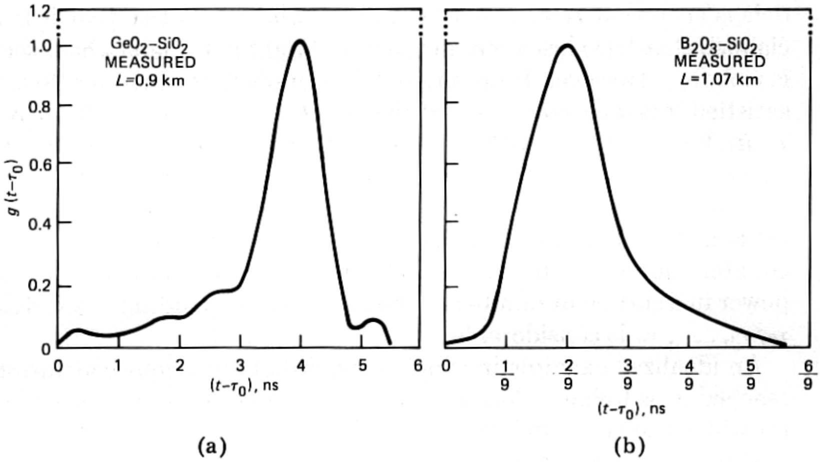

Fig. 1—Measured (deconvolved) impulse responses for (a) a GeO_2–SiO_2 fiber with $\alpha < \alpha_{opt}$; (b) a B_2O_3–SiO_2 fiber with $\alpha \approx \alpha_{opt}$.

which clearly is unchanged if d_1 is replaced with $(1 - d_1)$. Hence given only $|G(\omega)|$ for all ω, and that (5) with $\tau_1 = 0$ holds, it is not possible to find a unique $\theta(\omega)$.†

With regard to results in the opposite direction, when (5) holds and (8) is satisfied, it is true that the phase function $\theta(\omega)$ can be obtained from $|G(\omega)|$ by using (6). This is proved in Section 5.3.

III. EXPERIMENT

For each of two fibers A and B the impulse response was measured by injecting 0.4 nsec pulses (2σ width) from a GaAs laser ($\lambda = 0.9$ μm) and observing the pulse distortions after propagation through the fibers. Fiber A was 1 km long and had a graded index GeO_2–SiO_2 core with $\alpha \approx 1.9$ and $\alpha_{opt}(0.9\ \mu m) \approx 2.0$. Fiber B was 1 km long and had a graded index B_2O_3–SiO_2 core with $\alpha \approx \alpha_{opt}(0.9\ \mu m) \approx 1.8$. The measured impulse responses are shown in Figs. 1a and b. These impulse-response functions were Fourier transformed to obtain $|G(\omega)|$ for each case. Then each $|G(\omega)|$ together with its phase *calculated* from piecewise-linear formulas[4] based on (6) was used to calculate a corresponding impulse response. The plots are shown in Figs. 2a and b for fibers A and B, respectively. It may be seen that the calculated and measured responses agree quite well for fiber B with a suitable normalization and translation to bring them into

† Analogous examples can be given which hold for all $n \geq 2$. For instance, let $H(z, \gamma)$ denote the expression for $H(z)$ for the idealized exponential-excitation case mentioned above, with c chosen to depend on γ so that $H(0, \gamma) = 1$. It is not difficult to verify that we have $|H(i\omega, \gamma)| = |H(i\omega, -\gamma)|$ for all ω for each γ.

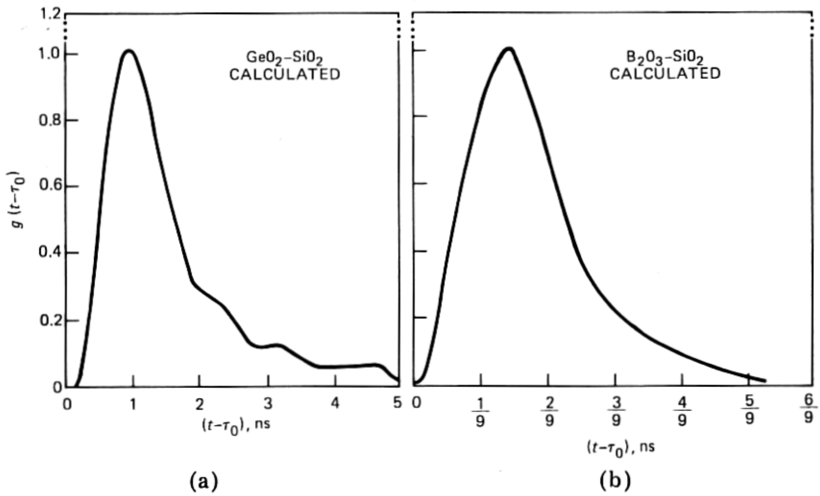

Fig. 2—Impulse responses calculated from the magnitudes of the fourier transforms of the measured impulse responses in Figs. 1(a) and (b).

coincidence. On the other hand, the measured and calculated responses for fiber A do not agree.†

The fact that the functions of Figs. 1a and 2a appear to be approximate mirror images of one another suggests that all of the dominant zeros associated with the transfer function of fiber A might lie in the *right* half-plane. In this connection, we note that a general function $H(z)$ of the form defined in Section II can have zeros in *both* half-planes. For example, with $H(z, \gamma)$ as defined in a preceding footnote, the product $H(z, 1)H(z, -1)$, which can be written in the same form as $H(z)$, has zeros in both half planes.

IV. CONCLUSIONS

With regard to the overall problem of determining the impulse response $g(t)$, direct measurement of the phase appears to be difficult and the general use of the "minimum phase" assumption to calculate the phase does not appear to be justified.

Methods for circumventing the difficulties described are under study, and it is expected that they will be described in a later paper.

V. APPENDIX: PROOFS

5.1 Outline of a derivation of a well-known formula

Let z be a complex variable with real and imaginary parts x and y, respectively, and let z^* denote the complex conjugate of z. Let f be a

† An early computational error led to a reversal of the sign of the time scale for the phase calculations as reported in Refs. 2 and 3. Thus, it was erroneously reported that $g(t)$ for measured and calculated responses matched for fiber A.

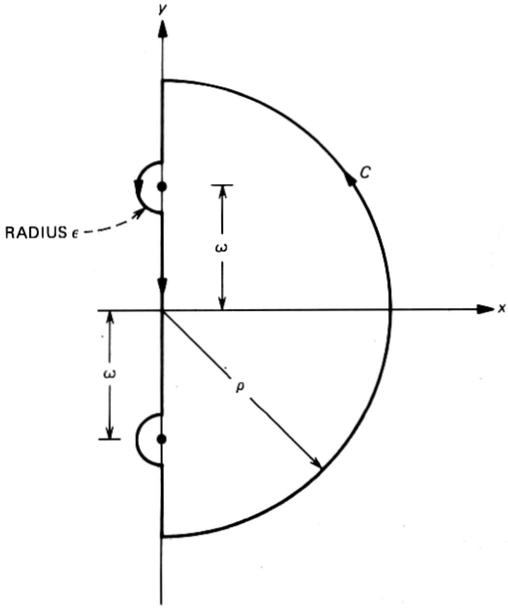

Fig. 3—Contour in the (x, y) plane.

complex-valued function of z defined throughout an open subset S of the (x, y)-plane that contains the half-plane $x \geq 0$ such that: $f(z^*) = f(z)^*$, $\mathrm{Re}[f(iy)]$ is an even function of y, and $\mathrm{Im}[f(iy)]$ is an odd function of y.

Proposition 1. Suppose that f is analytic on S, and that $|f(z)/z| \to 0$ as $|z| \to \infty$ in the half-plane $x \geq 0$. Then

$$\mathrm{Im}[f(i\omega)] = \frac{2\omega}{\pi} P \int_0^\infty \frac{\mathrm{Re}[f(iy)]}{y^2 - \omega^2}\, dy$$

for each real $\omega > 0$, in which P denotes a Cauchy principal value.

Outline of a Proof. Assume that the hypotheses of Proposition 1 are satisfied. Let $\omega > 0$ be given, and choose $\epsilon > 0$ such that $\epsilon < \omega$ and f is analytic for $|z - i\omega| < 2\epsilon$. With ρ any positive number such that $\rho > \omega + \epsilon$, let C denote the contour shown in Fig. 3 which consists of a semicircular arc of radius ρ, two semicircular arcs of radius ϵ, and a portion of the y-axis.

By Cauchy's integral theorem,

$$f(i\omega) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - i\omega}\, dz$$

Therefore,

$$\mathrm{Im}[f(i\omega)] = \frac{\omega}{2\pi i} \int_C \frac{f(z)}{(z - i\omega)(z + i\omega)} dz \tag{10}$$

Consider separately the following contributions to the right side of (10): I_1 the integral along the y-axis from $z = i\rho$ to $z = -i\rho$ excluding the two gaps due to the semicircular ϵ arcs, the sum I_2 of the integrals along the ϵ arcs, and the integral I_3 along the remaining arc of radius ρ.

We find at once that

$$I_1 = \frac{\omega}{\pi}\left[\int_0^{\omega-\epsilon} + \int_{\omega+\epsilon}^{\rho}\right] \frac{\mathrm{Re}[f(iy)]}{y^2 - \omega^2} dy$$

Using the fact that the integral of $(z - i\omega)^{-1}$ over the upper ϵ arc is $i\pi$, it follows that $I_2 = \frac{1}{2}\,\mathrm{Im}[f(i\omega)] + \delta(\epsilon)$ in which $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Also, $I_3 = \beta(\rho)$ in which $\beta(\rho) \to 0$ as $\rho \to \infty$.

Since $\mathrm{Im}[f(i\omega)] = I_1 + I_2 + I_3$, we have

$$\mathrm{Im}[f(i\omega)] = 2I_1 + 2\delta(\epsilon) + 2\beta(\rho)$$

from which it is clear that the limit

$$\lim_{\rho\to\infty} \lim_{\epsilon\to 0} 2I_1$$

exists and is equal to $\mathrm{Im}[f(i\omega)]$. This completes the outline of the proof.†

5.2 Result concerning the zeros of H(z)

Let $H(z)$ denote††

$$\sum_{j=1}^{n} d_j e^{-z\tau_j}$$

for all complex z, in which $n \geq 2$, $d_j > 0$ for all j, and the τ_j are real numbers such that $\tau_1 < \tau_2 < \ldots < \tau_n$.

Let $t_1, t_2 \ldots, t_n$ denote *any* set of n real numbers with the property that $t_1 < t_2 < \ldots < t_n$.

Consider the condition

$$d_1 > \sum_{j=2}^{n} d_j \tag{11}$$

Proposition 2. We have $H(z) \neq 0$ for $\mathrm{Re}(z) \geq 0$ if (11) holds. If (11) is not satisfied, then for any positive ϵ there is a choice of the τ_j such that $|\tau_j - t_j| \leq \epsilon$ for all j and $H(z) = 0$ for some z with $\mathrm{Re}(z) \geq 0$.

† For further results concerning Hilbert transforms, see, for example, Ref. 6.
†† For the reader's convenience the definition of $H(z)$ is repeated here.

Proof. If (11) holds and $z = x + iy$ with $x \geq 0$,

$$|H(z)| = \left| e^{-z\tau_1} \left[d_1 + \sum_{j=2}^{n} d_j e^{-z(\tau_j - \tau_1)} \right] \right| \geq e^{-x\tau_1} \left(d_1 - \sum_{j=2}^{n} d_j \right) > 0$$

Suppose now that

$$d_1 \leq \sum_{j=2}^{n} d_j \tag{12}$$

and let ϵ be given. Let $z = x + iy$ with $y = \pi/\epsilon$. Choose $\tau_1 = t_1 - \epsilon$, and for each $j = 2, 3, \ldots, n$, choose τ_j such that $e^{-iy(\tau_j - \tau_1)} = -1$ and $|\tau_j - t_j| \leq \epsilon$. We have

$$H(z) = e^{-z\tau_1} \left[d_1 - \sum_{j=2}^{n} d_j e^{-x(\tau_j - \tau_1)} \right] \tag{13}$$

in which $\tau_j - \tau_1 > 0$ for $j \geq 2$. Using (12) and (13), it is clear that there is an $x \geq 0$ such that $H(z) = 0$, which completes the proof.

5.3 A corollary of proposition 1

Concerning the function H defined in Section 5.2 consider the following hypothesis.

Hypothesis: $\tau_1 = 0$ and (11) is satisfied.

We notice that if the hypothesis holds, then, for each real ω, $\text{Re}[H(i\omega)] > 0$ and we have $H(i\omega) = |H(i\omega)|e^{i\phi(i\omega)}$, in which $\phi(i\omega)$ denotes the principal value of $\tan^{-1}\{\text{Im}[H(i\omega)]/\text{Re}[H(i\omega)]\}$.

Proposition 3. If the hypothesis holds, then

$$\phi(i\omega) = \frac{2\omega}{\pi} P \int_0^\infty \frac{\ln|H(i\omega)|}{y^2 - \omega^2} dy$$

for $\omega > 0$ in which P denotes a Cauchy principal value.

Proof: Assume that the hypothesis is satisifed. We shall show that Proposition 1 can be used.

Let x_0 be a negative number such that

$$d_1 > \sum_{j=2}^{n} d_j e^{-x_0\tau_j}$$

We see that there is a positive constant δ such that $\text{Re}[H(z)] > \delta$ for $\text{Re}(z) > x_0$. Let $\phi(z)$ denote the principal value of $\tan^{-1}\{\text{Im}[H(z)]/\text{Re}[H(z)]\}$ for $\text{Re}(z) > x_0$.

For each complex z such that $z \neq 0$, let $p(z)$ denote the principal value of $\ln z$. Thus, $p[H(z)] = \ln|H(z)| + i\phi(z)$ for $\text{Re}(z) > x_0$, and $p[H(z)]$ is analytic in z for $\text{Re}(z) > x_0$.

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