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An Approximation for the Variance of the UPCO Offered Load Estimate

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This paper develops a generalization of some available approximations for the variance of the estimate for offered load to a trunk or server group operating in a blocked-calls-cleared mode, using measurements of usage, offered attempts (peg count), and overflow. The analysis takes into account the peakedness of the offered traffic stream, the level of blocking on the group, the duration of the measurement interval, and switch count errors due to sampling usage. The resulting approximation is quite accurate over a wide range of conditions, is easily computable, and clearly displays the role of the basic factors that control the precision of the estimator. The variance approximation is useful in studies of the relationship between traffic measurement errors and the performance of the provisioning and administration processes.

I. INTRODUCTION

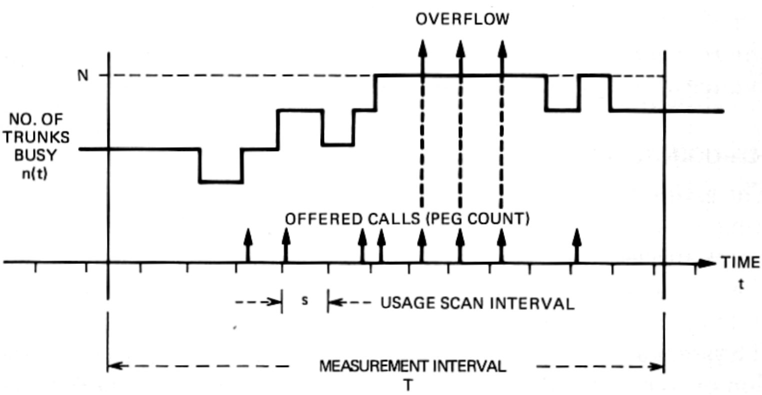
The estimation of loads offered to a trunk group or server group operating in a blocked-calls-cleared mode plays an important role in many network-provisioning processes. The preferred measurement combination for developing such load estimates consists of usage, offered attempts (peg count), and overflow attempts (usually referred to in the Bell System as UPCO measurements). This paper develops a generalization of some available approximations for the variance of the UPCO offered load estimate for a single measurement interval. The analysis considers the peakedness of the offered traffic stream, the level of blocking or call congestion for the group, the duration of the measurement interval, and switch count errors due to the sampling of usage at discrete points in time. The resulting approximation is quite accurate over a wide range of conditions, is easily computable, and clearly displays the role of the basic factors that control the precision of the estimator.

Variance approximations are useful in designing measurements and in studying relationships between traffic measurement errors and the performance of the provisioning and administration processes. For example, the relationship of actual traffic measurement accuracies (which can be further corrupted by wiring, data base, and recording errors) to the quality of the trunk provisioning process was studied in Ref. 1. The variance approximation developed here was useful in quantifying the background accuracy of the process.

This paper is organized as follows. The basic approximation is presented and discussed in Section II. The development of the approximation is given in Section III; supporting analysis of switch count error is developed in the appendix. Concluding remarks are given in Section IV.

II. THE BASIC APPROXIMATION

Figure 1 illustrates UPCO measurements for a measurement interval of length T, with usage scan interval s. The UPCO estimate for the offered load during this measurement interval is given by



PEG COUNT P = Σ (OFFERED CALLS IN MEASUREMENT INTERVAL)
OVERFLOW O = Σ (OVERFLOW CALLS IN MEASUREMENT INTERVAL)

USAGE $U = \dfrac{1}{\left(\text{NO. SCANS IN MEASUREMENT INTERVAL}\right)} \sum_{\text{SCANS}} (\text{NO. TRUNKS BUSY})$

Fig. 1—UPCO measurements.

$$\hat{a} = \frac{\text{average measured usage}}{\text{1-measured blocking}}, \tag{1}$$

where the measured blocking is the ratio of overflow to offered attempts. It is well known that (under reasonable conditions subsequently discussed) this is an unbiased estimate for the true offered load a during this interval.

Early work on analyzing offered load estimators was carried out, among others, by R. I. Wilkinson,[2] who addressed the reliability of holding time estimates. In a 1952 paper,[3] W. S. Hayward, Jr., drawing on some of Wilkinson's analysis, addressed the variance of offered load estimates based on sampled usage. Hayward's model assumed Poisson arrivals, exponential holding times, and no blocking, yielding the result

$$\text{var}(\hat{a}) = \frac{\bar{h}a}{T}(2 + q), \tag{2}$$

where a is the offered load in erlangs, $\bar{h}$ is the average holding time, and T is the length of the measurement interval. The parameter q is given by

$$q = v\frac{1 + e^{-v}}{1 - e^{-v}} - 2, \tag{3}$$

where $v = s/\bar{h}$, and s is the usage scan interval; q determines the variance contribution due to switch count (sampling) error, e.g., $q = 0$ for $s = 0$, the continuous scan case.

In more recent work, Hill and Neal[4] addressed the question of the variance of $\hat{a}$ for peaked traffic,* but did not consider congestion or switch count error. Through the application of an asymptotic result for the variance of the renewals for a peaked traffic stream, they obtained the expression

$$\text{var}(\hat{a}) \cong \frac{2\bar{h}az}{T}, \tag{4}$$

where z is the peakedness factor for the stream.

In this paper, we combine elements of both of these previous analyses

* Peaked traffic refers to overflow traffic, or to streams containing some overflow traffic. The peakedness factor $z(\mu)$ (or z if μ is understood) is the equilibrium variance-to-mean ratio of busy servers when this traffic is offered to an infinitely large group of exponential servers with service rate μ. The peakedness factor is one for Poisson traffic and is larger than one for overflow traffic.

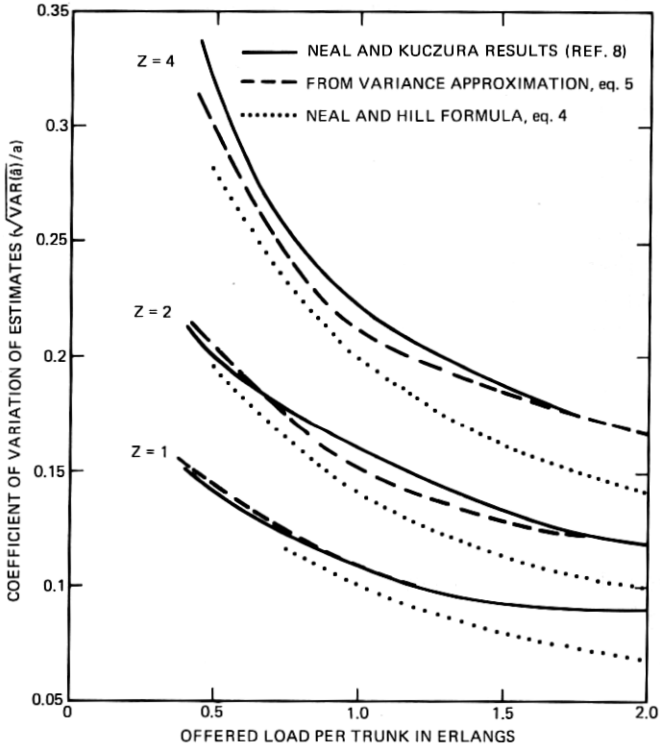


Fig. 2—Comparison of variance approximations for $N = 10$ servers ($\bar{h} = 180$ s, $T = 3600$ s, $s = 100$ s).

and explicitly consider the effect of blocking on the group, to obtain the generalization

$$\mathrm{var}(\hat{a}) \cong \frac{\bar{h}a}{T}\left(2z + \frac{B+q}{1-B}\right), \tag{5}$$

where B is the equilibrium call congestion,* i.e., the fraction of attempts blocked. Thus, congestion basically adds a term to the previous various approximations.

Figures 2 and 3 show comparisons of the variance approximation (5) with the reference approximations obtained via the error theory devel-

* The blocking B is defined in theory as the probability that an arbitrary attempt is blocked. In practice, when the load parameters a,z are given, the blocking or call congestion B is assumed to be defined by the equivalent random method (Ref. 5), so that $B = f(N,a,z)$ where N is the number of trunks in the group. Otherwise, as shown by Holtzman (Ref. 6), the blocking B is not uniquely defined by N,a,z, but may take on a range of values, depending on higher order characteristics of the traffic stream. The actual value of $f(N,a,z)$ may be obtained from traffic tables normally used in administering trunking networks. It may also be estimated by Hayward's approximation, $f(N,a,z) \cong B(N/z,a/z)$ (Ref. 7), thus allowing Erlang $B(.,.)$ tables or formulas to be used.

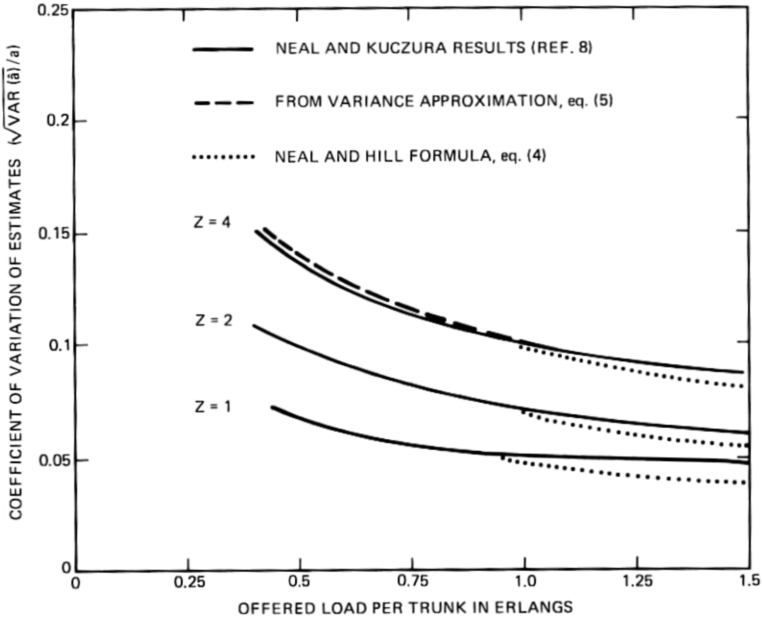


Fig. 3—Comparison of variance approximations for $N = 40$ servers ($\bar{h} = 180$ s, $T = 3600$ s, $s = 100$ s).

oped by Neal and Kuczura.[8]* These results assume that $\bar{h} = 180$ s, $T = 3600$ s (i.e., $\bar{h}/T = 0.05$), and $s = 100$ s. For a wide range of congestion and peakedness conditions, the agreement between eq. (5) and the reference results is very good. Neal and Kuczura also determined by numerical comparisons that switch count error was a small contributor to var($\hat{a}$). Since q is small for typical scan-interval-to-holding-time ratios (e.g., $q \cong 0.05$ for $s = 100$ s and $\bar{h} = 180$ s, which are typical scan intervals and holding times for Bell System trunks), this conclusion is also evident from eq. (5).

Figures 2 and 3 also show the behavior of the Neal and Hill result, eq. (4). As the load per trunk increases, it is clear that the contribution of the congestion term in eq. (5) is increasingly important. These higher levels of congestion occur quite commonly on high usage groups, where a substantial fraction of the busy hour loads may be overflowed to an alternate route. As the load is increased to very large values, the coefficient of variation using eq. (4) goes to 0, whereas Figs. 2 and 3 suggest that the coefficient of variation has a positive limit as $a \rightarrow \infty$. It can be shown that (for any z) as the attempt rate $\lambda \rightarrow \infty$,

$$\lim_{\lambda \rightarrow \infty} \text{var}(\hat{a})/a^2 = \bar{h}/TN, \tag{6}$$

* This error theory is applicable to general functions of the e UPCO measurements. The approximation developed for the UPCO offered load estimate is computationally much more complex, as well as less transparent, than eq. (5). The Neal and Kuczura approximation agreed well with simulation results, and hence is a suitable reference for comparing eq. (5).

where N is the number of servers in the group.* Equation (6) has a simple interpretation. The UPCO offered load estimate may be viewed as the product of essentially independent estimators for the attempt rate λ and for the mean holding time $\bar{h}$. As $\lambda \to \infty$, the coefficient of variation for the first estimator goes to 0. Equation (6) represents the squared coefficient of variation for the second estimator, i.e., the positive limit results from having only a finite number of carried attempts from which to estimate mean holding time. For Figs. 2 and 3, the asymptotic limits for the coefficient of variation are 0.071 and 0.035, respectively.

If a is assumed to have a mean a_o and variance $\sigma_a{}^2$, one is often interested in estimating a_o. The results of this section can be applied to obtain $\text{var}(\hat{a}_o)$ for a single measurement period by interpreting them as conditional results, i.e., $\text{var}(\hat{a}\,|\,a)$, in the expression

$$\text{var}(\hat{a}_o) = \sigma_a{}^2 + E_a \,\text{var}(\hat{a}\,|\,a). \tag{7}$$

In many cases, the $\sigma_a{}^2$ term can be a significant contributor. For example, in trunk engineering $\sigma_a{}^2$ may represent a day-to-day variance under an i.i.d. model for busy-hour loads (in this case, a_o is usually estimated from 5 to 20 busy-hour loads) and can be quite large in relation to the other sources of variability.

III. DEVELOPMENT OF THE APPROXIMATION

Consider a full access group of N servers operated in a blocked-calls-cleared mode. The offered traffic process is assumed to be a (nonlattice) renewal process with rate parameter λ, and server holding times are assumed to be exponential with hang-up rate μ. We define the mean and peakedness of the offered load by $a = \lambda/\mu$, $z = \text{var}(n(t))/E(n(t))$, where $n(t)$ is the equilibrium occupancy when the renewal process is offered to an infinitely large group of exponential servers with rate μ. The parameters (a,z) are conventionally used in traffic engineering, and hence it is useful to relate the variance approximation to these parameters.

For a measurement period of length T, let u,p,o denote average measured usage, offered attempts, and overflow attempts, as illustrated by Fig. 1. The average measured usage is defined by $u = 1/m \; \Sigma_{j=1}^{m=[T/s]} n(js)$ if $s > 0$, and by $u = 1/T \int_0^T n(t)dt$ if $s = 0$, where $n(t)$ is the number of busy servers at time t. It is assumed that equilibrium conditions apply at the beginning of the measurement interval, both for the occupancy on the servers and for the renewal processes corresponding to arrivals and overflows.

* This result is not the same as the limit obtained from eq. (5) as $a \to \infty$, which gives $(1 + q)\bar{h}/TN$. The discrepancy arises because the model for switch count error used in the development of eq. (5) breaks down as $\lambda \to \infty$. For this unrealistic limiting case, the servers are occupied 100 percent of the time, and *no* error is introduced by scanning. The correct result is thus obtained by noting that the carried attempt process approaches a Poisson process with rate $N\mu$ as $\lambda \to \infty$.

The UPCO estimate for the offered load a over the measurement period is

$$\hat{a} = \frac{u}{1 - o/p} = p\frac{u}{c} = \frac{p}{T}\left(\frac{Tu}{c}\right), \tag{8}$$

where $c \triangleq p - o$. Thus, $\hat{a}$ may be viewed as the product of separate estimators for the arrival rate (p/T) and for the average holding time (Tu/c). The approximation for $\mathrm{var}(\hat{a})$ is obtained by introducing an approximate treatment of the scanning error, and then by examining (8) for large T. However, while the *structure* of the approximation is motivated by asymptotic analysis, the *validity* of the approximation is based on its accuracy for realistic values of T.

3.1 Treatment of scanning error

The scanning error for usage affects only the value Tu in (8), which may be expressed as

$$Tu = \sum_{j=1}^{c} \hat{h}_j + r_0 - r_T, \tag{9}$$

where $\hat{h}_j$ is the sampled holding time estimate for the jth call to be accepted by the group, $\hat{h}_j \in \{0,s,2s,\cdots\}$, and r_0, r_T are end effects. In particular, if the jth call to be accepted by the group was hit by k_j scans, then $\hat{h}_j = k_j s$ may be viewed as the sampled holding time estimate for this call. The variable r_0 is the total measurement period usage attributable to calls already in progress at the beginning of the interval, while r_T is the total usage due to accepted calls that would be measured in the subsequent measurement period of length T.

Throughout this analysis, we make the following simplifying assumptions:

(*i*) $\hat{h}_j, j = 1,2,\cdots,c$ are independent random variables.

(*ii*) $\hat{h}_j = h_j + e_j$ where e_j is the scanning error that results when a call with exponential holding time h_j begins at a time which is uniformly distributed between two successive sampling instants.

These simplifying assumptions hold *exactly* for the case $B = 0$, $s = 0$ (no congestion and continuous scan) and any z, since all calls are carried and the holding times are i.i.d. exponential random variables. They also hold *exactly* for the case $B = 0$, $s > 0$, and $z = 1$, since for a Poisson process the arrivals in disjoint intervals are independent. Furthermore, given a *fixed* number of arrivals in an interval (in particular, an interval of length s), the arrival times are independent and uniformly distributed within the interval. Thus, the simplifying assumptions—while not always true—can be rigorously justified for some important cases. In general, they can be expected to be reasonable assumptions if the usage on each server in the group does not approach unity, i.e., if congestion is not too severe.

As a result of the simplifying assumptions, the scanning error need only be examined for an isolated call. The analysis for this situation is treated in the appendix, where it is shown that with $e = \hat{h} - h = ks - h$, i.e., the sampled holding time minus the true holding time,

$$E(e) = 0 \tag{10}$$

$$\text{cov}(h,e) = 0 \tag{11}$$

$$\text{var}(e) = \bar{h}^2 \left[v \frac{1 + e^{-v}}{1 - e^{-v}} - 2 \right] \triangleq \bar{h}^2 q, \tag{12}$$

where $v = s/\bar{h}$, $\bar{h} = \mu^{-1}$. For $s = 0$ (continuous scan), var$(e) = 0$ as expected, and hence these results cover both the continuous or the discrete scan case.

3.2 Asymptotic analysis of variance

Since p corresponds to the arrivals for a renewal process, $x \triangleq p/T$ is asymptotically normal with mean λ and variance of the form $O(1/T)$ (Ref. 9, p. 40). It is established in Ref. 10 that the variance can be approximately expressed in terms of the peakedness z

$$\text{var}(x) \cong (2z - 1)\lambda/T. \tag{13}$$

As noted in Ref. 4, this approximation has been found to be quite good for $a > z - 1$, and $T \geqq 10\bar{h}$. Although the carried calls c do not necessarily correspond to a renewal process (unless $c \equiv p$), c/T is also asymptotically normal with mean $\lambda(1 - B)$, (where $B \triangleq \lim_{T \to \infty}(o/p)$) and variance $O(1/T)$. This follows since if $B > 0$ the overflow process o is a renewal process, and the carried calls between overflows are independent for successive interoverflow periods. The only other asymptotic result needed is the following one, the proof of which is essentially the same as that for the function of sampling moments theorem given on p. 366 of Cramér:[11]

If $g(.,.)$ is a twice continuously differentiable function in some neighborhood of the point $\lambda,\lambda(1 - B)$, then $g(p/T,c/T)$ is asymptotically normal with mean $g(\lambda,\lambda(1 - B))$ and variance $O(1/T)$. It follows that

$$E(g(p/T,c/T)) = g(\lambda,\lambda(1 - B)) + O(1/\sqrt{T}). \tag{14}$$

Now for large T, the end effects r_0,r_T in (9) can be ignored at the outset. In particular, we have $E(Tu) = O(T)$, var$(Tu) = O(T)$, whereas $E(r_0 - r_T) = o(1)$, var$(r_0 - r_T) = O(1)$. (In general, ignoring these end effects is valid when $T/\bar{h}$ is reasonably large, e.g., $T/\bar{h} \geqq 10$.) Thus, defining* $y = \Sigma_{j=1}^{c} \hat{h}_j/c$, where the $\hat{h}_j$ satisfy the simplifying assumptions made for handling the scanning error, it follows from (10) to (12) that

* While y can be defined to be 0 for $c = 0$, in order to simplify subsequent notation, we shall assume that $P(c = 0) = 0$. This is reasonable even for the typical values of T that are of interest in practical applications.

$$E(y) = E(\hat{h}) = \bar{h} \tag{15}$$

$$\text{var}(y) = \text{var}(\hat{h})E\left(\frac{1}{c}\right) = \frac{\bar{h}^2(1+q)}{\lambda T(1-B)} + o(1/T), \tag{16}$$

where we have used (14) to evaluate $E(1/c)$.

Turning our attention next to $\hat{a}$, we have

$$\hat{a} = xy \tag{17}$$

$$\text{var}(\hat{a}) = E(x^2y^2) - E^2(xy). \tag{18}$$

In order to simplify this expression, we first note that

$$E(y|c) = \bar{h}$$

and hence

$$E(xy) = E_{p,c}E(xy|p,c) = E_{p,c}(x\bar{h}) = \lambda\bar{h} = E(x)E(y); \tag{19}$$

i.e., x,y are uncorrelated, confirming that $\hat{a}$ is an unbiased estimate of a. By the same conditioning, we also obtain

$$E(x^2y^2) = \bar{h}^2(E(x^2) + (1+q)E(x^2/c)) \tag{20}$$

and since

$$E(x^2)E(y^2) = \bar{h}^2(E(x^2) + (1+q)E(1/c)E(x^2)), \tag{21}$$

$$E(x^2y^2) = E(x^2)E(y^2) + (1+q)\bar{h}^2w, \tag{22}$$

where $w = \text{cov}(x^2, 1/c)$. Substituting (19) and (22) into (18) and identifying terms, we have

$$\text{var}(\hat{a}) = E^2(x)\,\text{var}(y) + E^2(y)\,\text{var}(x) + \text{var}(x)\,\text{var}(y) + (1+q)\bar{h}^2w. \tag{23}$$

By direct substitution of the means and variances for x,y

$$\text{var}(\hat{a}) = \frac{a\bar{h}(1+q)}{T(1-B)} + (2z-1)\frac{a\bar{h}}{T} + o(1/T) + (1+q)\bar{h}^2w. \tag{24}$$

It remains to show that $w = o(1/T)$. But $Tw = \text{cov}(x^2, 1/(c/T))$ and hence by (14) it follows that $Tw = o(1)$, i.e., $w = o(1/T)$. This completes the analysis; the variance approximation given in eq. (5) corresponds to terms of $O(1/T)$ in (24).

IV. CONCLUSIONS

In this paper, we have developed a simple approximation for the variance of the UPCO offered load estimate commonly used in offered load estimation. This approximation shows clearly the role of source load variation, switch count error, peakedness, congestion, and length of the measurement period. Relative to previous work, the main contribution is the explicit inclusion of congestion. Thus the results are of particular interest for high congestion situations such as occur in measuring loads on high usage groups.

While the basic approximation is developed here for a single measurement interval, it can be easily applied in analyzing load estimates based on the average load over a number of single measurement intervals.

V. ACKNOWLEDGMENTS

Discussions with S. R. Neal and D. W. Hill and D. L. Jagerman are gratefully acknowledged.

APPENDIX

Analysis of Switch Count Error

In this appendix we analyze, using methods similar to Hayward,[3] the following switch count error model: (*i*) a call with holding time h begins at a time uniformly distributed between two successive sampling instants, (*ii*) the sampling interval is of length s, (*iii*) the holding time is exponentially distributed with rate parameter μ.

For an arbitrary call, the error e between the true holding time h for the call, and the "sampled holding time," is given by $e = ks - h$, where k represents the scan count for the call, $k \epsilon \{0,1,2,\cdots\}$. The scan count for the call is simply the total number of scans that occur during the time the call is in progress.

Since $e\ \epsilon[-s,s]$, it is convenient to define a normalized error $e' = k - h'$, where $h' = h/s$ is exponentially distributed with rate parameter $\mu' = \mu s = s/\overline{h}$. The density of h' is therefore given by

$$f(t) = \begin{cases} 0 & t < 0 \\ \mu' e^{-\mu' t} & t \geq 0 \end{cases}. \tag{25}$$

Define $x' = x/s$, where x is uniformly distributed in $[0,s]$ and represents the time from a sampling instant to the beginning of a call. Given $x' \epsilon [0,1]$, it is straightforward to show that the conditional probability density of e' at $e' = y$ is

$$g(y|x') = \begin{cases} 0, & y \not\epsilon [-(1-x'),x'] \\ \sum_{k=0}^{\infty} f(k-y), & y \epsilon [-(1-x'),x']. \end{cases} \tag{26}$$

The only case for which a negative argument can occur in any term in the preceding sum is for $k = 0$, $y > 0$. Thus,

$$\sum_{k=0}^{\infty} f(k-y) = \frac{e^{-\mu'}}{1-e^{-\mu'}} \mu' e^{\mu' y} \text{ for } y > 0$$

$$\sum_{k=0}^{\infty} f(k-y) = \frac{1}{1-e^{-\mu'}} \mu' e^{\mu' y} \text{ for } y < 0.$$

Defining $r = e^{-\mu'}$, (26) becomes

$$g(y|x') = \begin{cases} 0 & x' < y \le 1 \\ \dfrac{r}{1-r}\mu' e^{\mu' y} & 0 \le y \le x' \\ \dfrac{1}{1-r}\mu' e^{\mu' y} & -(1-x') \le y < 0 \\ 0 & -1 \le y < -(1-x'). \end{cases} \tag{27}$$

To simplify obtaining of moments for e', we define $G(\alpha) = E(e^{\alpha e'}) = E_{x'}E(e^{\alpha e'}|x')$. Using (27),

$$G(\alpha) = \frac{1}{1-r} E_{x'}\left[r\int_0^{x'} \mu' e^{(\mu'+\alpha)y} dy + \int_{-(1-x')}^{0} \mu' e^{(\mu'+\alpha)y} dy \right]. \tag{28}$$

After integration, one obtains

$$G(\alpha) = \left(\frac{\mu'}{\mu'+\alpha}\right) - \left(\frac{1+r}{1-r}\right)\frac{\mu'}{(\mu'+\alpha)^2} + \frac{(e^{\alpha} + re^{-\alpha})}{1-r}\frac{\mu'}{(\mu'+\alpha)^2}. \tag{29}$$

We have $G(0) = 1$, $G'(0) = 0$, and

$$G''(0) = \frac{1+r}{1-r}\frac{1}{\mu'} - 2\frac{1}{(\mu')^2}; \tag{30}$$

hence,

$$E(e) = 0 \tag{31}$$

$$\mathrm{var}(e) = \bar{h}^2 \left(\frac{1+e^{-s/\bar{h}}}{1-e^{-s/\bar{h}}}\cdot\frac{s}{\bar{h}} - 2\right), \tag{32}$$

which establishes (10) and (12) of the main section.

To establish the covariance between h, e, we note that because of (31), $\mathrm{cov}(h,e) = E(h\,e) = s^2E(h'e')$. But

$$\begin{aligned} E(h'e') &= E_{x'}\left[\int_{-(1-x')}^{x'} \sum_{k=0}^{\infty} y(k-y)f(k-y)dy\right] \\ &= E_{x'}\left[\int_{-(1-x')}^{x'} \sum_{k=0}^{\infty} (-y^2)f(k-y)dy\right] \\ &\quad + E_{x'}\left[\int_{-(1-x')}^{x'} \sum_{k=0}^{\infty} kyf(k-y)dy\right]. \end{aligned} \tag{33}$$

The first term is $-\mathrm{var}(e')$. To evaluate the second term, we note that

$$\sum_{k=0}^{\infty} kyf(k-y) = \sum_{k=0}^{\infty} ky\mu' e^{-\mu'(k-y)} = ye^{\mu' y}\mu' \sum_{k=0}^{\infty} kr^k$$

$$= ye^{\mu' y}\mu' r \sum_{k=1}^{\infty} k r^{k-1} = ye^{\mu' y}\mu' r \frac{d}{dr}\left(\frac{1}{1-r}\right), r = e^{-\mu'}.$$

Therefore

$$E_{x'}\left[\int_{-(1-x')}^{x'} \sum_{k=0}^{\infty} kyf(k-y)dy\right]$$

$$= \frac{r}{(1-r)^2} E_{x'}\left[\int_{-(1-x')}^{x'} y\mu' e^{\mu' y} dy\right]. \quad (34)$$

Thus, we are led to define the function

$$H(\alpha) = E_{x'}\left[\int_{-(1-x')}^{x'} \mu' e^{(\mu+\alpha)y} dy\right].$$

Carrying out the integration yields

$$H(\alpha) = -\frac{2\mu'}{(\mu'+\alpha)^2} + \frac{\mu'}{(\mu'+\alpha)^2}\frac{e^{\alpha} + r^2 e^{-\alpha}}{r}. \quad (35)$$

The expectation in (34) is now evaluated as

$$H'(0) = \frac{1}{\mu'}\frac{(1-r)(1+r)}{r} + \frac{1}{(\mu')^2}\left(4 - 2\frac{1+r^2}{r}\right),$$

giving

$$E_{x'}\left[\int_{-(1-x')}^{x'} \sum_{k=0}^{\infty} kyf(k-y)dy\right] = \frac{1}{\mu'}\left(\frac{1+r}{1-r}\right) - \frac{2}{(\mu')^2} = \text{var}(e').$$

Therefore, $E(h'e') = -\text{var}(e') + \text{var}(e') = 0$, i.e., h' and e' are uncorrelated random variables and

$$\text{var}(\hat{h}) = \text{var}(ks) = \text{var}(h) + \text{var}(e). \quad (36)$$

Remark: Hayward[3] treats switch count error and source load variation separately, assumes independence, and adds the separate variances to obtain an approximate result. He noted that the errors were probably correlated, though weakly, and that (at that time) no method to take this into account was evident (Ref. 3, p. 363). Since $\text{cov}(h,e) = 0$, it follows from this analysis that (for the same model studied by Hayward) the errors are in fact uncorrelated. It was also pointed out by the referee that an alternate proof that $\text{cov}(h,e) = 0$ can be obtained by noting that the scan count k is geometrically distributed for $k \geq 1$. Thus, by directly evaluating $\text{var}(ks)$, one finds that $\text{var}(ks) = \text{var}(h) + \text{var}(e)$, which implies $\text{cov}(h,e) = 0$.

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