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Offset Multireflector Antennas with Perfect Pattern Symmetry and Polarization Discrimination

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Conditions are derived that are useful for designing reflector antennas with excellent cross-polarization discrimination. These conditions ensure circular symmetry and absence of cross-polarization everywhere in the far field of an antenna, provided a suitable feed such as a corrugated horn is employed. The spherical wave radiated by the fundamental mode of such a feed has circular symmetry around the axis, and it is everywhere free of cross-polarization. An arbitrary sequence of N confocal reflectors (hyperboloids, ellipsoids, paraboloids) is combined with such a feed. It is shown that it is always possible to ensure circular symmetry (and absence of cross-polarization) in the antenna far field by properly choosing the feed axis orientation. If the final reflector is a paraboloid, a simple geometrical procedure can be used. It is also shown that the asymmetry caused by an arbitrary number of reflections can always be eliminated by properly introducing an additional reflection. An application to the problem of producing a horizontal beam using a vertical feed is discussed. Two arrangements are described that may be useful for radio relay systems.

Use of orthogonal polarizations is often required in radio systems to double transmission capacity. Antennas providing good discrimination between the two polarizations are then needed. The main purpose of this paper is to derive and discuss certain conditions that ensure excellent discrimination. When two or more reflectors and a suitable feed are arranged in accordance with these conditions, the antenna far field has, in all directions, the same polarization of the feed excitation. Furthermore, its pattern has circular symmetry. The above conditions also minimize astigmatism, and for this reason they are also useful* in the design of multibeam antennas (with several feeds).

* This is the subject of an article being prepared.

I. INTRODUCTION

A suitable feed for the antennas considered here is realized by properly corrugating the walls of a circular horn.[1–4] The spherical wave radiated by the horn then has circular symmetry and, by placing the feed at the focus of a paraboloid, an antenna with circular symmetry in the far field is obtained, provided the paraboloid is centered around the feed axis. Furthermore, the polarization of the plane wave reflected by the paraboloid then coincides with that of the feed excitation.

However, in the centered configuration the reflected wave is in part blocked by the horn.* To avoid this, the horn axis can be offset as in Fig. 1, but unfortunately this causes asymmetry in the pattern after reflection, resulting in an undesired cross-polarized component.[5,6] The same behavior occurs if, instead of a paraboloid, an arbitrary reflector system with a single axis of revolution is used. In Fig. 1, the asymmetry of the reflected wave increases with the angle of incidence α of the ray corresponding to the horn axis. This particular ray will be called *principal ray*.

Although a single offset reflection always causes some asymmetry, it

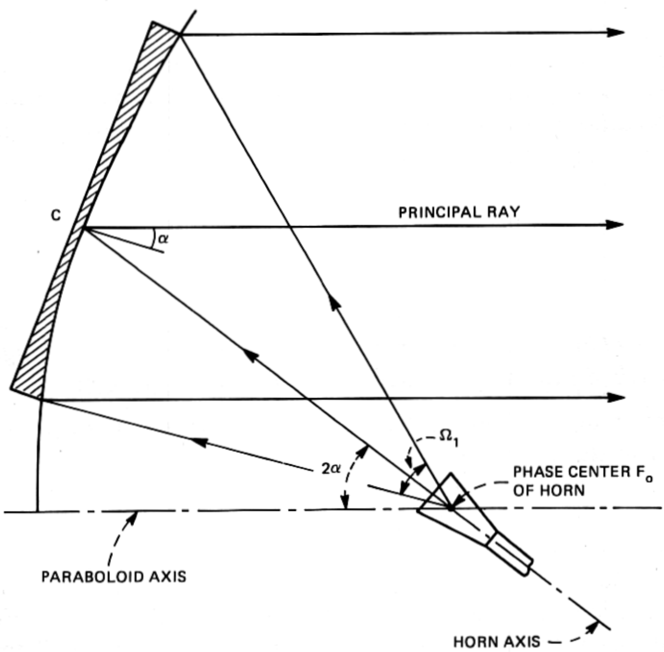

Fig. 1—The spherical wave radiated from F_0 by a corrugated feed is transformed by an offset paraboloid into a plane wave.

* This blockage impairs gain, side-lobes level, return loss, and cross-polarization discrimination.

is possible to combine two reflections with nonzero angles of incidence so as to ensure perfect symmetry after the two reflections.[7–10] In this paper we generalize and extend the results of Refs. 7 to 9 in several respects. First, the analysis here is not restricted to only two reflections, nor does it assume the final reflector is necessarily a paraboloid. Second, very simple conditions that guarantee symmetry after the final reflection are obtained. These conditions are shown to be direct consequences of a general principle of equivalence (see the appendix). Third, a general solution is given to the problem* of restoring the symmetry of a wave whose initial symmetry has been distorted by an arbitrary number of reflectors.

In Section III, two arrangements with excellent performance in cross-polarization are described. Both arrangements produce a horizontal beam using a vertical feed and may therefore be useful for microwave radio systems.

The following analysis is based on geometrical optics. Furthermore, the far field for the antennas of Figs. 12 and 13 is not derived in Section III, but it is important to note that the principle of equivalence of the following section allows the aperture field distribution for both antennas to be derived replacing the reflectors with a single paraboloid, centered around the feed axis. The aperture field distribution and far field of such a paraboloid are well known.[1–5] As pointed out at the beginning of this introduction, the entire aperture will be polarized in one direction if the feed is linearly polarized. The far field is thus free of cross-polarization, neglecting secondary effects such as edge diffraction.

II. THE EQUIVALENT REFLECTOR AND THE ORIENTATION OF ITS AXIS

Suppose a spherical wave from F_0, initially with symmetrical pattern, is successively reflected N times, using paraboloids, hyperboloids, and ellipsoids as shown in Fig. 2 for $N = 3$. The reflectors are properly arranged so that a spherical wave is produced after each reflection. Thus, if F_n is the focal point after the nth reflection, the nth reflector Σ_n transforms a spherical wave centered at F_{n-1} into a spherical wave centered at F_n. Note that some of the points $F_0, F_1, \cdots, F_N$ may be at ∞, in which case the corresponding spherical waves become plane waves. In Fig. 2, F_3 is at ∞, and therefore the last reflector is a paraboloid.

It is shown in the appendix that *such a sequence of confocal reflectors is always equivalent to a single reflector* which will be either an ellipsoid, a hyperboloid, or a paraboloid. This equivalent reflector produces, after a single reflection, the same reflected wave† as the given sequence of

* An interesting formulation of this problem is given in Ref. 10.

† Thus, if one considers the field distribution over a wavefront reflected by the equivalent reflector, it will coincide with the field distribution over the corresponding wavefront produced by the given sequence of reflectors.

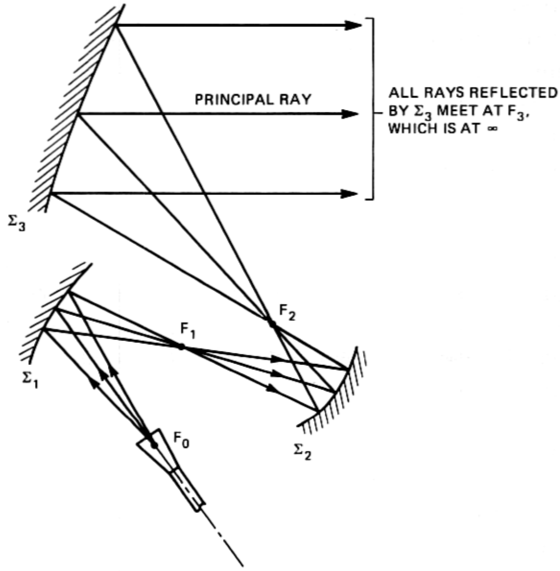

Fig. 2—The spherical wave from F_0 is transformed into a plane wave by three confocal reflectors. The nth reflector transforms the spherical wave from F_{n-1} into a spherical wave converging towards F_n.

reflectors. Thus, for the purpose of determining the properties of the reflected wave, one may replace the N reflectors with the equivalent reflector. This reflector has an axis of symmetry, which passes through F_0, and will be called the *equivalent axis.* It is clear that in order that the symmetry of the incident beam be preserved, *the principal ray must coincide with the equivalent axis.**

2.1 The central rays, their closed path, and the equivalent axis

Consider first $N = 1$. Suppose the reflector Σ_1 and one of its foci, F_0, are given, but the exact location of the axis of Σ_1 is not known and must be found. Then one may proceed as follows. Let a ray from F_0 be reflected twice by Σ_1, as shown in Fig. 3, and let $\vec{s}$ and $\vec{s}''$ be the initial and final directions of the ray. Then, from Fig. 3,

$$\vec{s} = \vec{s}'' \tag{1}$$

only when the ray coincides with the axis. Thus, the axis can be found by searching for a ray that satisfies this condition. Note from Fig. 3 there are two such rays, with opposite directions.

Next consider $N > 1$. Since a confocal sequence of reflectors $\Sigma_1, \cdots,$ Σ_N is equivalent to a single reflector Σ_e, the above procedure is appli-

* Since one can travel along the equivalent axis in two opposite directions, two opposite orientations can be chosen for the principal ray.

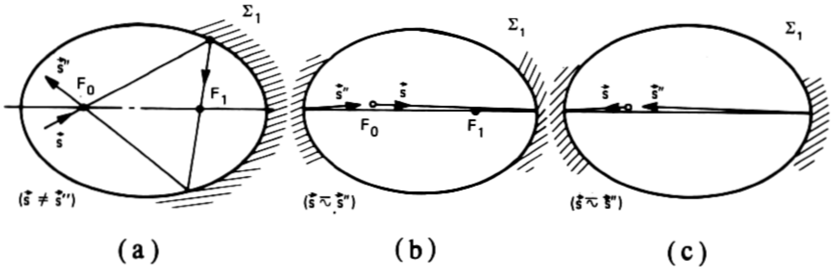

Fig. 3—The axis of Σ_1 is determined by varying $\vec{s}$ until $\vec{s} = \vec{s}''$.

cable also to this case. Thus, to determine the axis of Σ_e (equivalent axis), one must consider a ray from F_0, with initial directions $\vec{s}$. This ray must be reflected *twice* by Σ_e, and $\vec{s}$ must then be chosen so that $\vec{s}'' = \vec{s}$. Notice that the two reflections by Σ_e imply a total of $2N$ reflections in the original configuration. The first N reflections take place in the order Σ_1, $\cdots$, Σ_N, while the last N have the reverse order $\Sigma_N, \cdots, \Sigma_1$. The final ray passes again through F_0, with the same direction as the original ray. In Fig. 4a, $\vec{s} \neq \vec{s}''$. In Fig. 4b, on the other hand, condition (1) is satisfied, and therefore the ray through F_0 gives the correct orientation of the equivalent axis (and the principal ray for which symmetry is preserved).

Notice that if, after the above $2N$ reflections, the ray in Fig. 4a is reflected $2N$ more times it will not follow the same path of the first $2N$ reflections. In Fig. 4b, on the other hand, the path of the first $2N$ reflections is closed. This closed path, which determines the equivalent axis, will be called the *central path*. The two rays that proceed along the central path in opposite senses will be called the *central rays*.

We show next that condition (1) leads to a straightforward geometrical procedure for determining the equivalent axis when Σ_N is a paraboloid.

*2.2 The equivalent axis when the last reflector Σ_N is a concave paraboloid**

It is now shown that, when the last ellipsoid in Fig. 4a is replaced by a concave paraboloid, the final ray direction $\vec{s}''$ becomes independent of the initial direction $\vec{s}'$. This constant value of $\vec{s}''$ then gives the direction of the equivalent axis, which can thus be found straightforwardly.

Notice the path of Fig. 4a involves two successive reflections by the last ellipsoid Σ_N. Let ψ be the angle between the axis of Σ_N and the ray produced after the second reflection (see Fig. 5). The parameters of the ellipsoid Σ_N are now gradually modified, keeping the vertex V and the focus F_{N-1} fixed, increasing the distance between F_N and F_{N-1} until

* The following considerations apply also when Σ_N is a convex paraboloid, but this case is of little practical interest and will therefore be ignored.

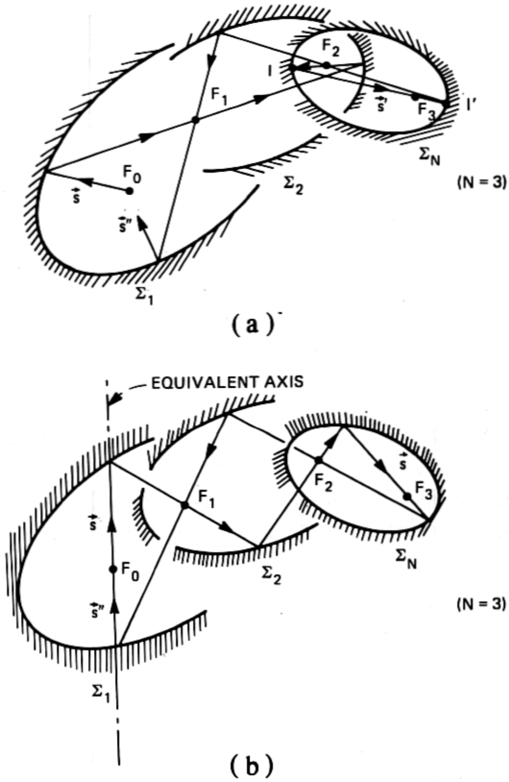

Fig. 4—(a) $2N$ successive reflections. (b) The central path. The equivalent axis through F_0 is obtained by varying in (a) the initial direction $\vec{s}$ until $\vec{s} = \vec{s}''$ as shown in (b).

$F_N \to \infty$. The ellipsoid then becomes a paraboloid with focus F_{N-1} and from the figure $\psi = 0$, which shows that

> If a ray from the focus F_{N-1} of a paraboloid is reflected twice by the paraboloid, so that the second reflection occurs at ∞, the final ray coincides with the paraboloid axis and it has the direction going from F_{N-1} towards the vertex V of the paraboloid. (2)

This implies that, when in Fig. 4 the last ellipsoid Σ_N is replaced by a paraboloid, the direction of $\vec{s}''$ becomes independent of $\vec{s}$, and it can be determined by tracing the ray $F_{N-1}V$ as shown in Fig. 6. The direction $\vec{s}''$ so obtained gives the equivalent axis, as one may verify considering a ray with *initial* direction given by the above value of $\vec{s}''$. One can see from Fig. 6 the path of this ray closes, after $2N$ reflections. Thus,

To obtain the equivalent axis of a sequence of $N - 1$ re-

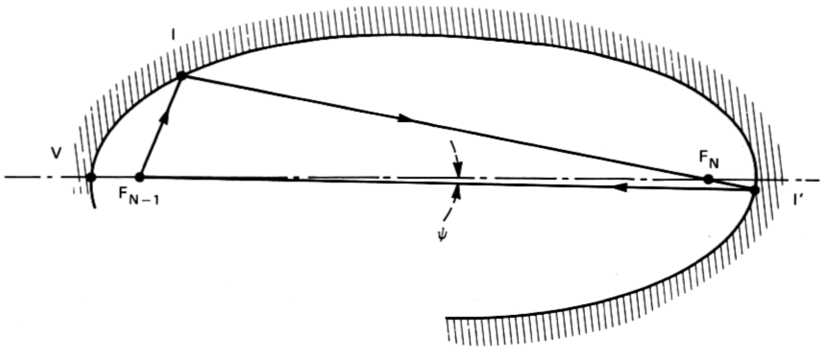

Fig. 5—As the distance of F_N from the other focus F_{N-1} is increased, keeping V and F_{N-1} fixed, the ellipsoid approaches a paraboloid with vertex V and focus F_{N-1}; for the ray reflected at I' one has $\psi \to 0$.

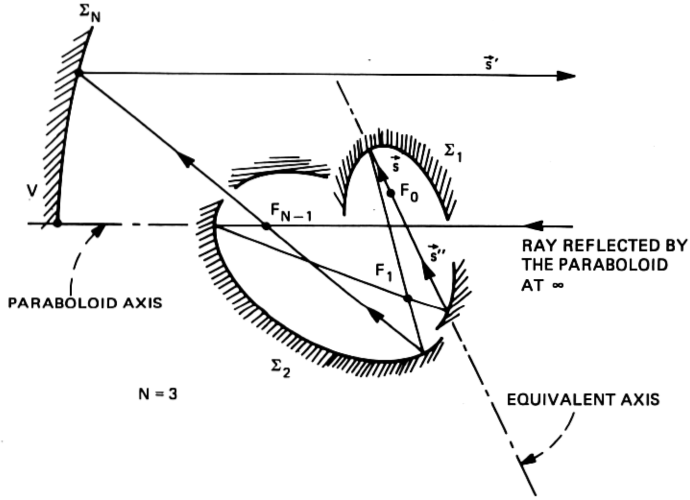

Fig. 6—By tracing from ∞ the path of the ray defined by the paraboloid axis one obtains after $N-1$ reflections the equivalent axis through F_0. If a symmetrical feed is placed at F_0, centered around the equivalent axis, a symmetrical pattern will be reflected by the paraboloid.

flectors $\Sigma_1, \Sigma_2, \cdots, \Sigma_{N-1}$ followed by a paraboloid Σ_N with focus F_{N-1} and vertex V, simply reflect $N-1$ times the ray $F_{N-1}V$ by $\Sigma_{N-1}, \Sigma_{N-2}, \cdots, \Sigma_1$. The final ray through F_0 is the equivalent axis and, therefore, the principal ray along which symmetry is preserved. (3)

As an example, consider $N = 2$, and assume the first reflector is not a paraboloid.* Then four different arrangements are obtained depending on whether the first reflector is an ellipsoid or an hyperboloid, and is convex or concave. In each case (see Figs. 7 and 8), the equivalent axis† is determined by the intersection I' of the paraboloid axis with the first reflector. The equivalent axis is the line F_0I'. Note the axis of the paraboloid intercepts the first reflector Σ_1 in two points, but only one, I', is acceptable.‡ The acceptable point is the point of reflection of the ray F_1V. Since only one side of the surface Σ_1 is reflecting, only one of the above two points can be considered a point of reflection for the above ray.

From Figs. 7 and 8, since in all cases the equivalent axis and the paraboloid axis meet on Σ_1, the angles 2α and 2β giving their inclinations from the axis of Σ_1 are related,

$$\tan \alpha = m \tan \beta, \tag{4}$$

where m is the axial magnification of Σ_1 given by the distances of the reflector vertex V_0 from the two focal points F_0 and F_1,

$$m = \frac{|F_0V_0|}{|F_1V_0|}. \tag{5}$$

Note that if e is the eccentricity of the reflector, in Figs. 7 and 8,

$$m = \frac{e+1}{e-1}, \frac{e-1}{e+1}, \frac{e+1}{1-e}, \frac{1-e}{1+e}, \tag{6}$$

respectively. In Fig. 7 one has $e > 1$, whereas in Fig. 8, $0 < e < 1$.

In the two cases of Figs. 7a and 8a, eq. (4) is equivalent to eq. (1) of Ref. 9. In the other two cases, on the other hand, eq. (1) of Ref. 9 is not applicable [to obtain a correct relation, one has to replace α with β in eq. (1)].

Another useful relation, derived in the following section, is

$$\tan i = \frac{M}{1-M} \tan p. \tag{7}$$

It relates the angles of incidence i and p of the central ray on the two

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* The case where Σ_1 is a paraboloid is treated in Section 2.6.

† That is, the beam orientation for which symmetry is preserved.

‡ Notice for the purpose of deriving the equivalent axis that the entire surfaces of the various ellipsoids, hyperboloids, and paraboloids must be considered to be reflecting. Thus, both branches of an hyperboloid must be considered. Of course, an actual antenna will use only certain sections of the various surfaces.

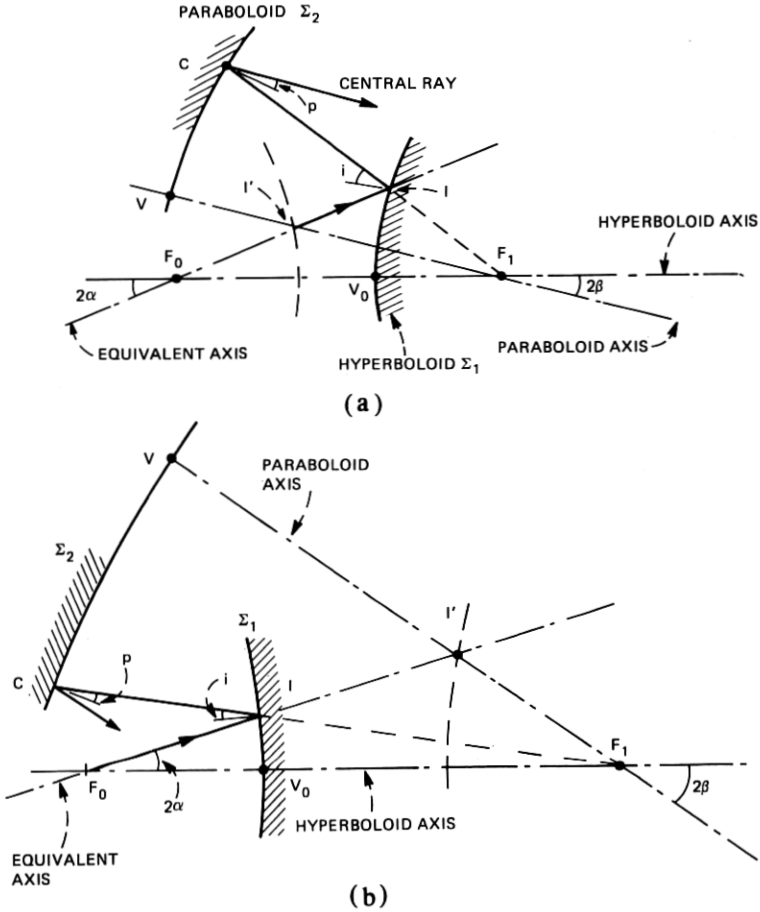

Fig. 7—How to determine the central path and the equivalent axis of a paraboloid combined in (a) with a convex hyperboloid and in (b) with a concave hyperboloid.

reflectors (see Figs. 7a and 8) to the magnification M, defined as

$$M = \pm\frac{|F_0 I|}{|IF_1|}, \tag{8}$$

I being the point of incidence of the central ray on the first reflector. In eq. (8) one has to take the positive sign when F_0 and F_1 are on opposite sides of the tangent plane at I, as in Fig. 8; otherwise, as in Fig. 8, $M < 0$. The angles of incidence must be taken with opposite sign in Figs. 7a and 8, where the two reflections have opposite senses; in Fig. 7b, on the other hand, i and p have the same sign.

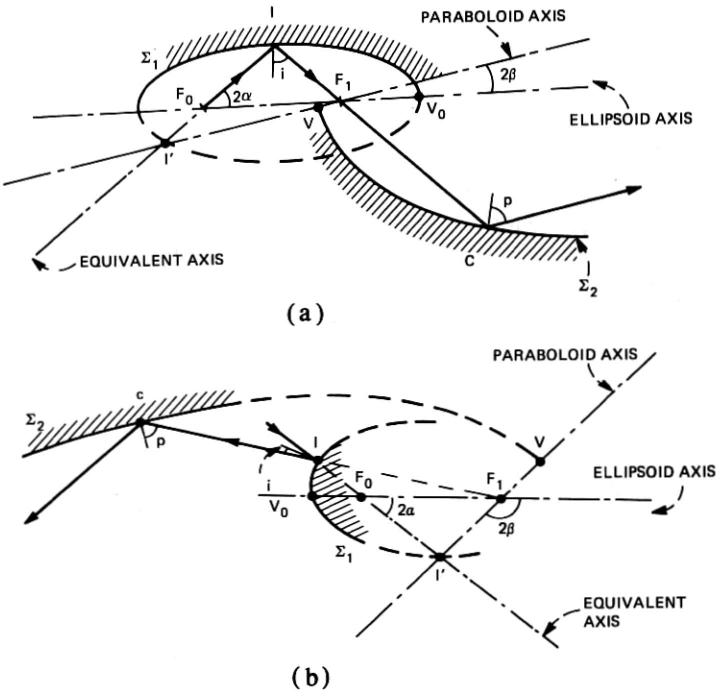

Fig. 8—How to determine the central path and the equivalent axis of a paraboloid combined in (a) with a concave ellipsoid and in (b), with a convex ellipsoid.

The magnification* M determines the ratio between the angular width Ω_0 of the beam incident as I and the width Ω_1 of the reflected beam. More precisely,† for small Ω_0,

$$M = \frac{\Omega_1}{\Omega_0}. \tag{9}$$

If M is specified, eq. (7) gives the angles of incidence i and p that result in a symmetrical beam after two reflections.

A very general relation, which reduces to eq. (7) in the particular case where Σ_N is a paraboloid, is derived in Section 2.4.

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* Another important significance of M is that the paraxial focal length f_e, for any of the arrangements of Figs. 7 and 8, in the vicinity of the central ray, is $f_e = Mf_p$, where f_p is the paraboloid focal length $f_p = CF_1$; f_e has the significance that a small lateral displacement δs of a feed initially placed at F_0 will cause an angular displacement $\delta\theta = \delta s/f_e$ of the beam reflected by the paraboloid.

† Thus, if a beam of small angular width Ω_0 is transformed by a sequence of N reflectors with magnifications $M_1, \cdots, M_N$, the final beam has angular width

$$\Omega_1 = M_t \Omega_0,$$

where $M_t = M_1 M_2 \cdots M_N$.

2.3 Relations governing the reflections of a central ray by the first or the last reflector

The restriction that Σ_N must be a paraboloid is now removed. The closed path of the central ray in Fig. 4 involves two successive reflections by Σ_1. Consider these two reflections and assume for the moment Σ_1 is a concave ellipsoid as shown in Fig. 9a. The central ray in Fig. 9a first passes through F_1 with direction $\vec{a}$, it is successively reflected at I' and I, and it then passes again through F_1 with direction $\vec{c}$.

Let $2i$ and $2i'$ be the angles of the two reflections and M and M' the corresponding magnifications,

$$M = -\frac{\ell_1}{\ell_2}, M' = -\frac{\ell_1'}{\ell_2'}. \tag{10}$$

ℓ_1, ℓ_2, etc. being defined in Fig. 9a. Then, if $2\gamma = 2i + 2i'$ is the total angle of reflection (given by the angle between the final and initial rays $\vec{c}$ and $\vec{a}$) it is shown in Section A.3 of the appendix that

$$\tan i = \frac{M}{M-1} \tan \gamma \tag{11}$$

and

$$\tan i' = \frac{1}{1-M'} \tan \gamma. \tag{12}$$

Thus, if the parameters (M,i, or M',i') of either reflection are given, the total angle of reflection for a central ray can be calculated. Note that eqs. (11) and (12) apply also to the two consecutive reflections of the central ray by the last reflector Σ_N.

In Fig. 9a, the reflector Σ_1 is a concave ellipsoid, but eqs. (11) and (12) are valid also if Σ_1 is an hyperboloid or is concave, as shown in Figs. 9b, c, and d. Note in cases 9c and 9d the central ray is first reflected at I', then passes through the point at ∞ and is then reflected again at I. Figs. 7a,b and 8a,b correspond to Figs. 9b, 9c, 9a, and 9d, respectively.

2.4 How to arrange two reflectors

Consider Fig. 10a showing a principal ray from F_0 reflected by two reflectors Σ_1 and Σ_2. We wish to show that, in order that this ray be a central ray, i.e., that symmetry be preserved after these two reflections, their parameters M, M', i, and i' must satisfy the condition

$$\tan i = M \frac{1-M'}{1-M} \tan i'. \tag{13}$$

Consider the ray reflected by Σ_1. Let this ray be reflected *twice* by Σ_2, and then again *twice* by Σ_1, as in Fig. 10b. If 2γ denotes the total angle of the first two reflections by Σ_2 and $2\gamma'$ the angle of the other two re-

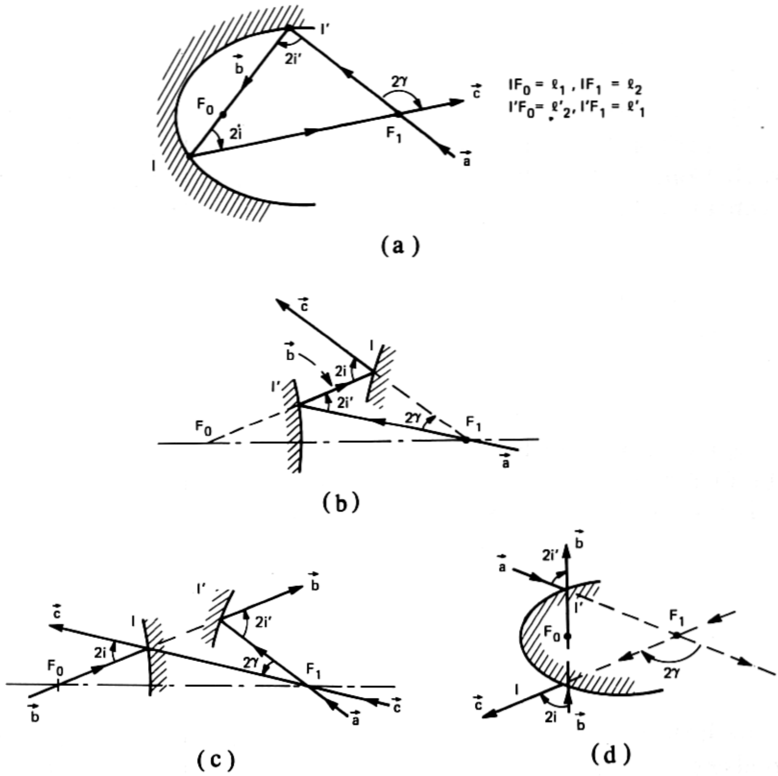

Fig. 9—Two successive reflections. (a) By concave ellipsoid. (b) By convex hyperboloid. (c) By concave hyperboloid. (d) By convex ellipsoid.

flections, one must have

$$2\gamma + 2\gamma' = 2\pi, \tag{14}$$

if the path of the ray is to close (which is necessary for it to be a central ray) after the four consecutive reflections. Now $\tan \gamma$ is given by eq. (11), and $\tan \gamma'$ by eq. (12) with γ replaced by γ'. Thus, by requiring condition (14), one obtains condition (13). In the particular case where the second reflector is a paraboloid,

$$M' = 0$$

and eq. (13) give Eq. 7 (with $i' = p$).

2.5 Restoration of beam symmetry after an arbitrary number of reflections

Suppose an arbitrary sequence of $N - 1$ reflections $\Sigma_1, \cdots \Sigma_{N-1}$ have distorted the initial symmetry of a spherical wave originating from F_0. We wish to restore symmetry by introducing an additional reflector Σ_N. Let the principal ray through F_0 be reflected $N - 1$ times by the given reflectors as shown in Fig. 11a for $N = 3$. The reflector Σ_N must be chosen

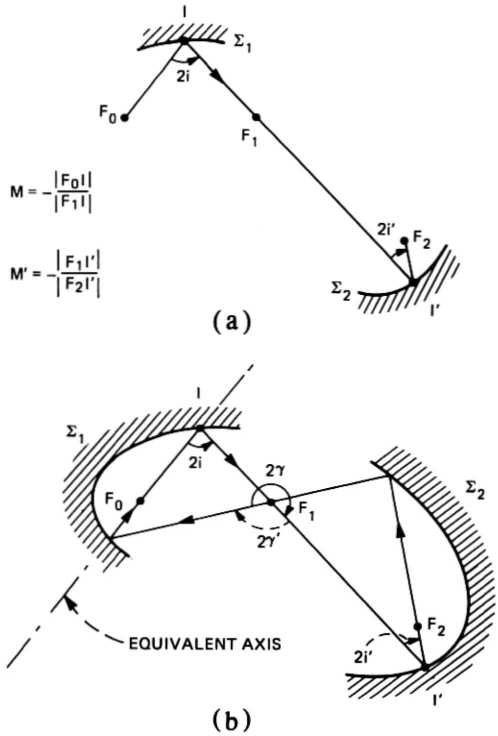

Fig. 10—Central path and equivalent axis of a combination of two ellipsoids.

so that this ray is one of the two *central rays* of the sequence $\Sigma_1, \cdots, \Sigma_N$. This means the path of the ray must *close* after $2N$ successive reflections. Now a part of this path, the section determined by the reflections of Σ_1, $\Sigma_2, \cdots, \Sigma_{N-1}$, is fixed in advance. Therefore let this part of the central ray be determined first. It starts at F_{N-1} and, after $2(N-1)$ reflections, it ends again at F_{N-1} with direction $\vec{a}$ as shown in Fig. 11a. Since its final direction $\vec{a}$ is given, its initial direction $\vec{c}$ can be found by tracing the ray backwards. Once $\vec{c}$ is known, the condition that Σ_N must satisfy is simply eq. (12), with γ given by the angle between $\vec{c}$ and $\vec{a}$, shown in Fig. 11.

2.6 How to determine the first reflector if the remaining ones are given

The above argument applies also to the problem where the first reflector, rather than the last, is to be found and the remaining reflectors are given. The only difference in this case is that one must use eq. (11), instead of eq. (12), as shown by the following example. To consider a situation of practical interest, suppose the last reflector Σ_N is a paraboloid as shown in Fig. 11b. Assume that all the reflectors except the first one are given and that Σ_1 must be chosen so that the central ray passes

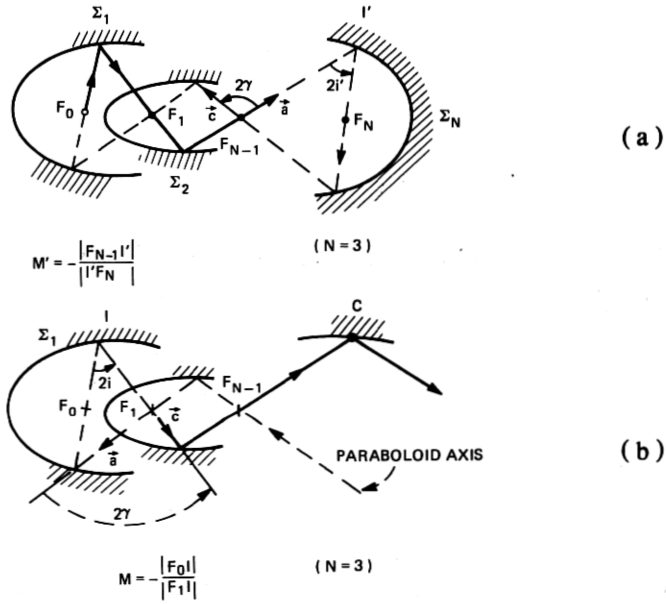

Fig. 11—(a) How to determine the last reflection if the first $N-1$ are given. (b) How to determine the first reflection if the last $N-1$ are given.

through the center C of the paraboloid aperture. Then, as in the previous problem, one notices that a part of the desired central path is fixed in advance. This part starts as F_1 with direction $\vec{c}$ and, after $2(N-1)$ reflections, it ends at F_1 with direction $\vec{a}$ as shown in Fig. 11b. Once $\vec{a}$ is found (by ray tracing), the condition that Σ_1 must satisfy is given by eq. (11), with γ given by the angle shown in the figure between $\vec{c}$ and $\vec{a}$.

2.7 The first and the last reflector are paraboloids

Consider first $N = 2$, in which case eq. (13) with $M = M' = \infty$ demands that the angles of incidence on the two paraboloids be identical, except for a difference in sign. For this to happen, the axes of the two paraboloids must coincide, in which case one can show that the two angles of incidence coincide *for any* choice of the principal ray. These remarks apply also to $N > 2$, since the last $N-1$ reflectors can always be replaced by an equivalent paraboloid. Thus,

> In order that symmetry be preserved, when both Σ_1 and Σ_N are paraboloids, the axis of Σ_1 must coincide with the equivalent axis of $\Sigma_2, \cdots, \Sigma_N$, in which case symmetry is preserved by any choice* of the principal ray. (15)

* A little thought shows that there is another case where the central ray is undetermined: namely, when the equivalent reflector is a flat plate.

III. AN APPLICATION

The most important example of an offset arrangement is perhaps the horn reflector,[11] an antenna consisting of a horn combined with a paraboloid. The excellent properties of this antenna (negligible return loss, very low level of the far sidelobes, etc.) are well known. However, the angle of incidence on the paraboloid is 45 degrees, and this causes in the far field a cross-polarized component of about −20 dB in certain directions.[11] The 45-degree angle of incidence is required to produce a beam orthogonal to the feed axis, which is an important requirement* for radio relay systems. In this section it is shown, with two examples given in Figs. 12 and 13, how this requirement can be fulfilled using two or more reflectors satisfying condition (7). In both Figs. 12 and 13, the feed is of the type described in Refs. 1 to 4, and therefore the antenna beam is essentially free of cross-polarization *everywhere* (see the last remark in the introduction).

Figure 12 shows two large reflectors, a paraboloid and an hyperboloid, arranged to satisfy simultaneously condition (7) and the requirement $i + p = 90°$, without aperture blockage. This arrangement is of the type shown in Fig. 8b of Ref. 7. In Fig. 13, three reflectors, a large paraboloid Σ_3, and two small hyperboloids Σ_2 and Σ_1 are used. This arrangement is more compact, and it requires less total reflecting area, than the one of Fig. 12. It is thus particularly attractive when the antenna aperture is large, i.e., the far-field beamwidth is small. The angle of incidence i and the magnification M of the first reflector Σ_1 satisfy condition (7), with p given by the angle shown in Fig. 12. To understand the significance of p, replace the last two reflectors Σ_2 and Σ_3 by their equivalent paraboloid. According to (3), the axis of this paraboloid is obtained from the axis of Σ_3 by reflecting it once, onto Σ_2, as shown in Fig. 13. Then $2p$ is the angle the central ray makes with this equivalent axis. Note that p is equal to the angle of incidence on this equivalent paraboloid (not shown in Fig. 13). This angle of incidence must satisfy eq. (7). One can verify from the figure that

$$\tan p = \frac{\tan \alpha + m_2 \tan \beta}{1 - m_2 \tan \alpha \tan \beta}, \tag{16}$$

α and β being the angles (see Fig. 7a) of the central ray and the axis of Σ_3 with respect to the axis of Σ_2, and

$$m_2 = \frac{|V_2F_1|}{|V_2F_2|} = \frac{e_2 + 1}{e_2 - 1}, \tag{17}$$

e_2 being the eccentricity of the hyperboloid Σ_2. Also,

$$2i = 90° + 2\beta - 2\alpha, \tag{18}$$

* Of course, this is not the only requirement that must be satisfied. Other requirements will be discussed in an article being prepared.

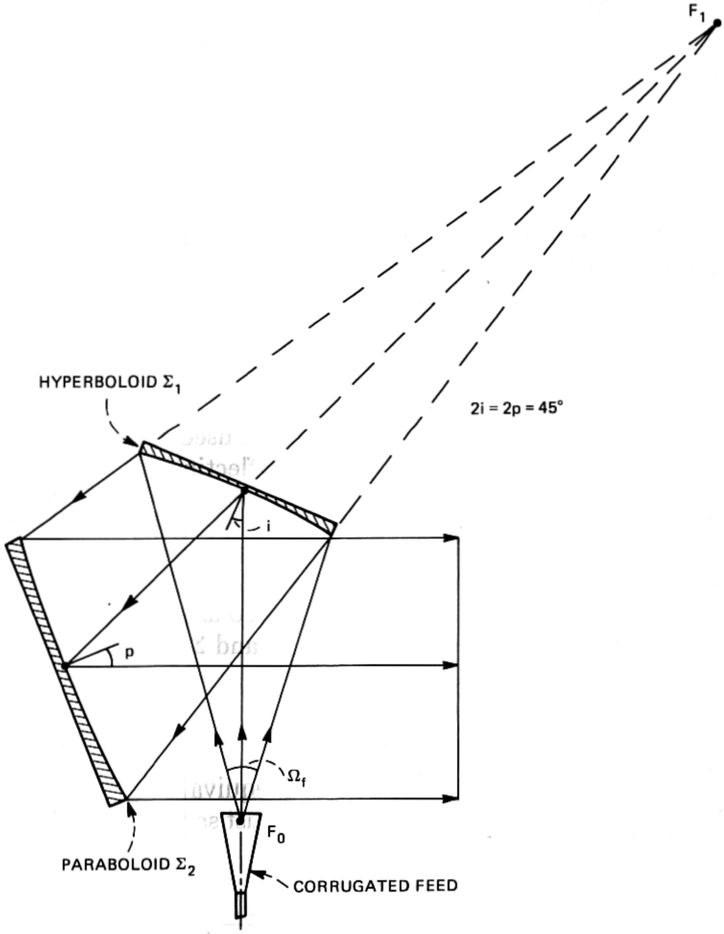

Fig. 12—A vertical feed and two reflectors with $i + p = 45$ degrees producing a horizontal beam without symmetry distortion.

and from eq. (7), solving for M,

$$M = \frac{\tan i}{\tan i + \tan p}. \tag{19}$$

Using eqs. (16) to (19), one can express M directly in terms of α, β, m_2.

An important property of Figs. 12 and 13 is that there is no aperture blockage even for relatively large values (as large as 30 degrees) of the angular width Ω_f of the beam radiated by the feed. Another important property, to be discussed in a future article, is that, if the feed is slightly displaced so as to cause a small angular displacement of the antenna beam, the resulting aberrations are very small. This is a consequence of

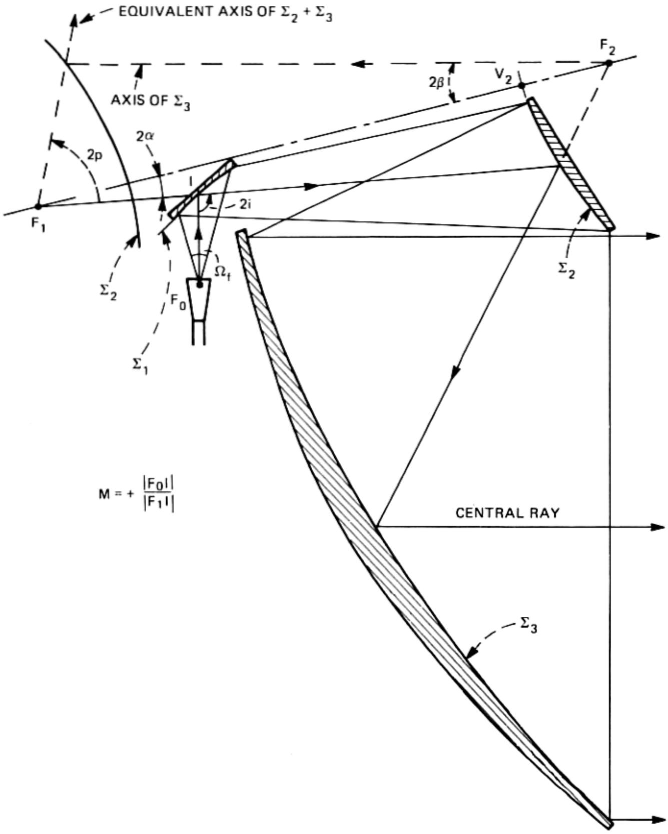

Fig. 13—A vertical feed and three reflectors producing a horizontal beam without symmetry distortion.

condition (7), and it implies that several beams can be produced efficiently by placing several feeds in the focal plane.

IV. CONCLUSIONS

The transformation of a symmetrical beam by an arbitrary arrangement of N confocal reflectors has been studied. It has been shown that it is always possible to choose the principal ray (i.e., the axis of the input beam) so that symmetry is preserved by the transformation. This is a consequence of the principle of equivalence shown in the appendix, according to which an arrangement of several reflectors can always be replaced by a single reflector producing the same transformation. Thus, in order that symmetry be preserved, the principal ray must coincide with the axis of symmetry of this equivalent reflector, i.e., the equivalent axis. A property of the equivalent axis is that the path of a ray having initially its direction becomes closed after $2N$ successive reflections. Because of this property, the equivalent axis can be found by a straightforward geometrical procedure if the last reflector is a paraboloid. A simple relation [eq. (11) or (12)] has been given for determining the angle of incidence and the magnification of the first or last reflector so as to guarantee symmetry. In Section III, the problem of modifying the horn reflector to eliminate the asymmetry and cross-polarization due to the paraboloid has been discussed. Two solutions have been described.

APPENDIX

General Properties of a Sequence of N Confocal Reflectors

The results of this paper are consequences of the principle of equivalence stated at the beginning of Section II. This principle is now derived.

As pointed out in the introduction, the reflectors we consider are ellipsoids, hyperboloids, or paraboloids; let $F_0, F_1, \cdots, F_N$ be $N + 1$ arbitrary points, let a point source be placed at F_0, and let a sequence of N reflectors $\Sigma_1, \cdots, \Sigma_N$ be used to successively transform the spherical wave from F_0 into spherical waves through $F_0, F_1, \cdots, F_N$. The nth reflector, Σ_n, with its focal points of F_{n-1} and F_n then transforms the spherical wave incident from F_{n-1} into a spherical wave through F_n.

Draw two spheres S and S' centered at F_0 and F_N. For each point P of S, there is, on S', a corresponding point determined by the ray through P. This mapping has the following properties.

A circle on S′ corresponds to each circle on S. In fact, it is well known[12,13] that a circular cone of rays from F_{n-1} is transformed by the nth reflector into a circular cone of rays through F_n.

The mapping is conformal,* and therefore two orthogonal curves of S are transformed into two orthogonal curves of S'.

Another property is that, if the point source at F_0 is linearly polarized and the lines of the electric field $\vec{E}$ on S are given, then the corresponding lines defined on S' by the above mapping give correctly the lines of $\vec{E}$ on S'. This result is true in general[14] for arbitrary reflectors, not necessarily paraboloids, hyperboloids, or ellipsoids. It allows the polarization of S' to be determined straightforwardly once the relationship between corresponding rays through F_N and F_0 is known.

A.1 The central rays

Draw a line through F_0, to cut the sphere S at two antipodal points. We show that it is always possible to choose the line orientation so that the corresponding points of S' are also antipodal points.

* This property is valid in general for an arbitrary wavefront S which is transformed by an arbitrary number of reflections (by arbitrary reflectors, not necessarily of the type considered here) into a wavefront S'. The mapping determined between S and S' by the rays orthogonal to S (and S') is a conformal mapping.

Let L_1, L_2 and M_1, M_2 be antipodal points of S (see Fig. 14; the sphere S is not shown). Let L_1', L_2' and M_1', M_2' be their corresponding points on S'. Through L_1', L_2' M_1', M_2' draw two great circles. The two circles will intersect in two antipodal points $0_1'$ and $0_2'$, as shown in Fig. 14. We show that the corresponding points are also antipodal points. In fact, 0_1 and 0_2 are the points of intersection of the two circles of S that correspond to the two circles of S'. Since the circles of S contain the antipodal points L_1, L_2 and M_1, M_2, they are great circles and therefore their intersections 0_1 and 0_2 are antipodal points. Q.E.D.

An important significance of the points 0_1, 0_2, $0_1'$, and $0_2'$ is the following. Let a ray from F_0 be reflected by the sequence of N reflectors *twice*, first in the order $\Sigma_1, \Sigma_2, \cdots, \Sigma_N$ and then in the reverse order Σ_N, $\Sigma_{N-1}, \cdots, \Sigma_1$. After these $2N$ reflections, the ray will pass again through F_0, but its direction will in general differ from the direction given initially, and therefore the ray will not in general follow the same path if reflected $2N$ more times. However, a little thought shows that, since the three points $0_1, F_0, 0_2$ are collinear and so also are $0_1'F_N0_2'$, the path of a ray from 0_1 (or from 0_2) will become closed after $2N$ reflections. The same observation applies to the ray from 0_2, which will follow, in the opposite direction, the same path of the ray from 0_1.

The path of the rays from 0_1 and 0_2 will be called the *central path* and the two rays *central rays.* This definition is consistent with the one given in Section II. As we shall see, there is in general only one central path, except when both Σ_1 and Σ_N are paraboloids (see Section 2.6) or when the equivalent reflector is a flat plate [$m_e = 1$ in eq. (21)].

The axial ray F_00_1 is now chosen as reference axis. Let a particular plane through this ray be chosen as reference plane. Consider a particular ray from F_0, and let θ be its angle with respect to the axis and ϕ the angle its plane makes with the reference plane. After N reflections, both the ray in question and the axial ray pass through F_N. Let θ' be the angle between the two rays at F_N, let ϕ' be the angle their plane makes with an arbitrary reference plane (chosen through the axial ray). We wish to

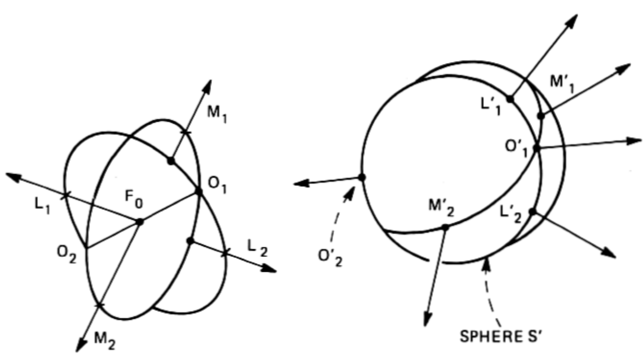

Fig. 14—How to determine the central rays.

show that

$$\phi' = \pm\phi + \phi_0 \tag{20}$$

and

$$\tan\frac{\theta'}{2} = m_e \tan\frac{\theta}{2}, \tag{21}$$

where m_e is a constant determined by the N reflectors and ϕ_0 is determined by the orientation of the two reference planes which will be chosen so that

$$\phi_0 = 0. \tag{22}$$

A.2 Derivation of eqs. (20) and (21)

First consider on S a great circle, through the two axial points 0_1 and 0_2, given by

$$\phi = a, \tag{23}$$

where a is a constant. Since the corresponding circle on S' must pass through $0_1'$ and $0_2'$, it is a great circle, given by

$$\phi' = a', \tag{24}$$

where a' is a constant. This shows that ϕ' depends only on ϕ, not on θ. We now recall that the mapping of S' must be conformal and therefore the angle between two circles through $0_1'$ must equal the angle between the corresponding circles of S. This implies eq. (20).

Next we derive eq. (21). Since the sign in front of ϕ in eq. (20) depends on the definition of ϕ', and can therefore be chosen arbitrarily, we choose for the following derivation

$$\phi' = \phi.$$

Since a circle θ = constant is orthogonal to a circle ϕ = constant, the corresponding circles on S' must be orthogonal. This implies θ' is a function of θ only. To determine this function, consider on S three points of coordinates:

$$(\theta,\phi), \quad (\theta + d\theta,\phi), \quad (\theta,\phi + d\phi).$$

Let

$$(\theta',\phi), \quad (\theta' + d\theta',\phi)(\theta',\phi + d\phi)$$

be the corresponding coordinates on S. Let $d\ell_1$ and $d\ell_2$ denote on S the distances of the first point from the other two. Then

$$d\ell_1 = r d\theta, \quad d\ell_2 = r \sin\theta d\phi, \tag{25}$$

r being the radius of the sphere S. Similarly, for the corresponding distances on S',

$$d\ell_1' = \mathrm{r}' d\theta', \quad d\ell_2' = \mathrm{r}' \sin\theta' d\phi. \tag{26}$$

Since the mapping is conformal, one must have

$$\frac{d\ell'_1}{d\ell'_2} = \frac{d\ell_1}{d\ell_2},$$

which gives the condition

$$\frac{d\theta}{\sin\theta} = \frac{d\theta'}{\sin\theta'}. \tag{27}$$

Integrating this gives eq. (21), where m_e is a constant of integration.

When $N = 1$, eqs. (20) and (21) are nothing new. In fact, then the reflector system reduces to a single reflector whose eccentricity determines the parameter m_e. When $N > 1$, eqs. (20) and (21) show the N reflectors are equivalent to a single reflector with eccentricity specified* by m_e.

A.3 Derivation of eqs. (11) and (12)

Consider the ellipsoid shown in Fig. 15. Then

$$\tan\alpha\tan\alpha' = 1 \tag{28}$$

and

$$\tan\alpha'\tan\psi' = \tan\alpha\tan\psi = m,$$

where

$$m = \frac{|F_0V_0|}{|F_1V_0|}. \tag{29}$$

Therefore, taking into account that $\gamma = 90° - \psi - \psi'$,

$$\tan\gamma \doteq \frac{1 - \tan^2\psi\tan^2\alpha}{\tan\psi(1 + \tan^2\alpha)} \tag{30}$$

Also, $i = 90° - \alpha - \psi$, and therefore

$$\tan i = \frac{1 - \tan\alpha\tan\psi}{\tan\alpha + \tan\psi}. \tag{31}$$

Now the magnification M of I is defined as

$$M = -\frac{|IF_0|}{|IF_1|}, \tag{32}$$

and from Fig. 15 is related to the angles ψ and α,

$$M = -\frac{\sin 2\psi}{\sin 2\alpha}$$

$$= -\frac{\tan\psi}{\tan\alpha}\frac{1 + \tan^2\alpha}{1 + \tan^2\psi}, \tag{33}$$

* The value of m_e can be calculated using the formula

$$m_e = \pm M_1 \cdot M_2 \cdots M_N,$$

where $M_1, \cdots, M_N$ are the magnifications calculated for the N reflections of the central ray chosen as reference axis. The sign of m_e depends on the sign convention for ϕ in eq. (20).

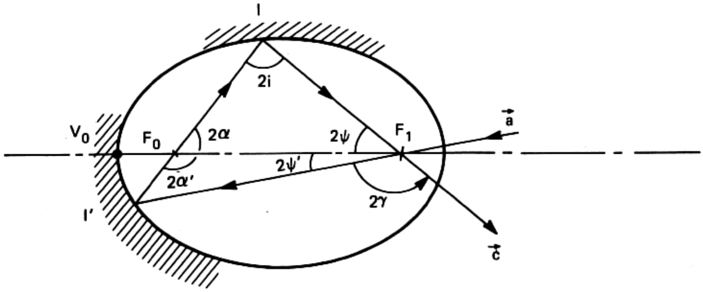

Fig. 15—Two successive reflections by a concave ellipsoid.

which gives

$$\frac{M}{M-1} = \frac{\tan\psi(1+\tan^2\alpha)}{(\tan\psi+\tan\alpha)(1+\tan\psi\tan\alpha)}. \tag{34}$$

From eqs. (30), (31), and (34), one obtains eq. (11). The derivation of eq. (12) is entirely analogous. The case where the reflector is convex, or is a hyperboloid, can be treated in the same way.

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