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On the Independence Theory of Equalizer Convergence

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High-speed pulse amplitude modulated (PAM) data transmission over telephone channels is only possible when adaptive equalization is used to mitigate the linear distortion found on the (initially unknown) channel. At the beginning of the equalization procedure, the tap weights are adjusted to minimize the intersymbol interference between pulses. The "stochastic gradient" algorithm is an iterative procedure commonly used for setting the coefficients in these and other adaptive filters, but a proper understanding of the convergence has never been obtained. It has been common analytical practice to invoke an assumption stating that a certain sequence of random vectors which direct the "hunting" of the equalizer are statistically independent. Everyone acknowledges this assumption to be far from true, just as everyone agrees that the final predictions made using it are in excellent agreement with experiments and simulations. We take the resolution of this question as our main problem. When one begins to analyze the performance of the algorithm, one sees that the average mean-square error after the nth iteration requires knowing, as an intermediate step, the mathematical expectation of the product of a sequence of statistically dependent matrices. We transform the latter problem to a space of sufficiently high dimension where the required average may be obtained from a canonical equation $\mathcal{V}_{n+1} = \mathcal{A}(\alpha)\mathcal{V}_n + \mathcal{F}$. *Here* $\mathcal{A}(\alpha)$ *is a square matrix, depending on the "step-size"* α *of the original algorithm, and* $\mathcal{V}_n$ *and* $\mathcal{F}$ *are vectors. The mean-square error is calculable from the solution* $\mathcal{V}_n$.

Information about the solution of our equation is obtained by doing

matrix perturbation theory on $\mathcal{A}(\alpha)$ for small values of α. We show that the first two terms of the perturbation solution contain, among their terms, the terms of the independence theory. Since the parameter α needs to be small even for independence theory to converge, agreement with an exact theory and experiment is obtained if, in some sense, the additional terms which appear in the perturbation solution may be disregarded. This will usually be the case.

I. INTRODUCTION

Adaptive equalization of telephone channels in order to facilitate high-speed data transmission has been successful ever since its introduction by Lucky in the 1960s. This technique uses a linear filter (configured as a tapped delay line) to remove the harmful effects of the linear channel distortion. At the start of the equalization procedure, a set of parameters, the tap weights, are adjusted so that the final setting of these taps minimizes the intersymbol interference between pulses in the data train. Many theoretical studies have been made concerning steady-state equalization after the optimum tap weights have been achieved; little analysis has been done concerning the convergence of the equalizer tap weights to their final settings. Even in the best published study on this problem (Ungerboeck, Ref. 1), it is necessary to invoke an assumption stating that a sequence of random vectors which direct the operation of the equalizer are statistically independent.† This independence assumption will be explained more fully later; for the moment, we only indicate that it is not even approximately true. In fact, given the nth vector of the sequence, all but one component of the next vector will be exactly known. Yet if this assumption is made, surprising agreement with actual performance is obtained.[1] Clearly, because of its importance, this situation begs for clarification. Hopefully, what we learn in equalization can be used for other applications where similar adaptive algorithms are used. In particular, the areas of linear prediction and adaptive array processing, both electromagnetic and sonar, come to mind. We concentrate our presentation on equalization, however, for here the author is sure of the details.

We shall take as our performance criterion the expected value of the mean-square distortion, although the average error vector is also considered as a simpler problem. In particular, then, we are not concerned with the fluctuations which might occur in actual use.

† We are here concerned with convergence in random data, not with a known specially designed sequence. In usual startup operation, the data symbols are also assumed known, either by using a known sequence or by assuming that sufficiently accurate estimates are available.

Typically, the sample paths are close to the mean (see Ref. 1). In a nutshell, our contribution to this problem consists of two parts. We first establish a time-independent difference equation which governs the average in question. This step is accomplished in a space of much higher dimension than one would initially assume. Second, examining the solution of this equation in a perturbation sense (the small "step-size" of the algorithm being the essential perturbation parameter), we find the leading terms contain the independence theory solution.

Before delving into the abstract problem, we devote Section II to describing some more conventional aspects of data transmission and equalization and Section III to discussing the behavior of the mean-square error if the independence assumption is made.

II. DATA TRANSMISSION AND EQUALIZATION

For our own convenience, we confine the discussion to binary baseband transmission and neglect the effects of additive noise.

The equalizer, and in fact the entire detection procedure, operates on the samples of the baseband received signal $r(t)$, where

$$r(t) = \sum_m a_{m+K} h(t - mT).$$

If $1/T'$ is the sampling rate, $1/T$ the symbol rate, a_n the data symbols (iid, ± 1 with equal probability) and $h(t)$ the overall system impulse response, then these samples are†

$$r(nT') = \sum_{m=-\infty}^{\infty} a_{m+K} h(nT' - mT) \qquad n = 0, 1, 2 \cdots . \tag{1}$$

For a synchronous equalizer, $T' = T$ and for a fractionally spaced equalizer, typically $T' = T/2$. If the coefficients of the equalizer are denoted by c_i, $i = 1, \cdots, N$ (c_i being also the ith component of a vector $\mathbf{c}$) and the sequence of output samples of the equalizer are y_n, then

$$y_n = \sum_{s=1}^{N} c_s r[(s-1)T' + nT] \qquad n = 0, 1, 2, \cdots . \tag{2}$$

We call attention to the fact that, even when $T' \neq T$, the equalizer samples are only of interest at multiples of the signaling interval T, and the notation of (2) takes this into account. We define a sequence (in time) of vectors $\mathbf{X}^{(n)}$ such that the sth component of vector $\mathbf{X}^{(n)}$ is

$$X_s^{(n)} = r[(s-1)T' + nT] \qquad \begin{matrix} s = 1, 2, \cdots, N \\ n = 0, 1, 2, \cdots, \end{matrix} \tag{3}$$

† We call the bit which "goes with" the mth pulse a_{m+K} (instead of the usual a_m) for later convenience.

and thus

$$y_n = \mathbf{c} \cdot \mathbf{X}^{(n)}. \tag{4}$$

The implementation of (2) to (4) is shown in Fig. 1.

Later, when we consider an adaptive equalizer, the taps will vary with time and $\mathbf{c}^{(n)}$ will be used for the sequence of tap-weight vectors. Ideally we would like (at least when n is large enough) the sequence of equalizer outputs to be the sequence of data symbols, except, perhaps, for a shift. For a finite equalizer (i.e., N finite) this ideal is not achievable, and instead the available taps are adjusted to minimize the average square error Ee_n^2, where

$$e_n = y_n - a_{n+K} \tag{5}$$

and E denotes the mathematical expectation with respect to the data symbols $\{a_n\}$. If one introduces the $N \times N$ channel autocorrelation matrix† (which is positive definite),

$$A = E\mathbf{X}^{(n)}\mathbf{X}^{(n)T}, \tag{6}$$

and the vector,

$$\mathbf{v} = Ea_{n+K}\mathbf{X}^{(n)}, \tag{7}$$

both of which do not depend on the time index n, then, for fixed taps $\mathbf{c}$, the mean-squared error $\mathscr{E}$ is given by

$$\mathscr{E} \equiv E(y_n - a_{n+K})^2 = \mathbf{c}^T A\mathbf{c} - 2\mathbf{c}^T\mathbf{v} + 1. \tag{8}$$

Equation (8) shows $\mathscr{E}$ to be a convex quadratic function of $\mathbf{c}$. Any optimum choice of $\mathbf{c}$, say, $\mathbf{c}^*$, satisfies

$$A\mathbf{c}^* = \mathbf{v} \tag{9}$$

which has a unique solution if A^{-1} exists. We denote the minimum of $\mathscr{E}$ by $\mathscr{E}^*$.

It will make little difference physically, and it will be a great convenience mathematically, if we pretend that the impulse response $h(t)$ used in (1) has finite duration. Thus, assume

$$h(t) = 0 \quad \text{if} \quad |t| > HT.$$

Let N_1 and N_2 be the largest integers such that

$$N_1 T \leq HT \tag{10a}$$

$$(N-1)T' - N_2 T \geq -HT. \tag{10b}$$

Further, choose the integer K in (1) to be N_1 and set $M = N_1 + N_2 + 1$, and let $\mathbf{a}^{(n)}$ be an M-dimensional vector whose ith component is

† The superscript T always denotes transpose.

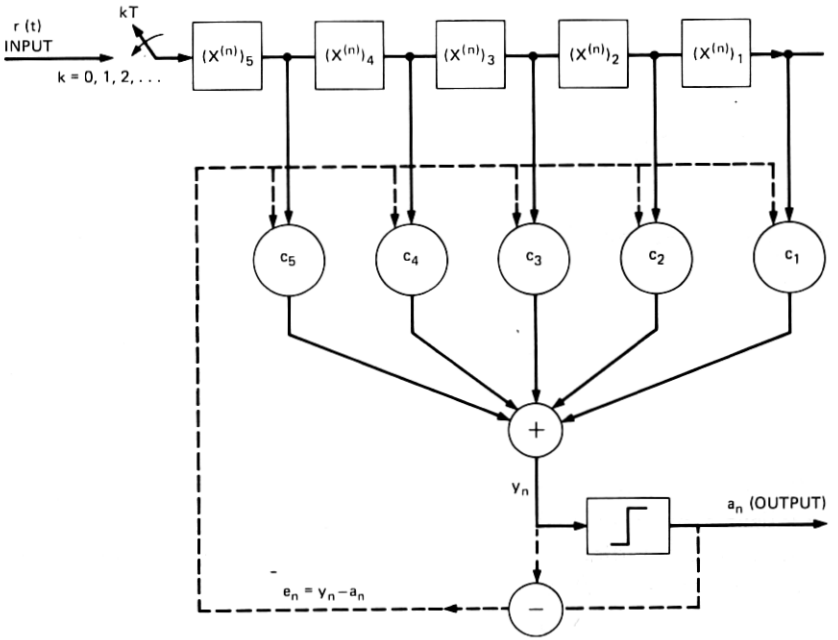

Fig. 1—Adaptive transversal equalizer, $N = 5$.

$\mathbf{a}_i^{(n)} = a_{n+i-1}$, $i = 1, \cdots, M$. Then using (3) and (1) we have

$$\mathbf{X}^{(n)} = B\mathbf{a}^{(n)}, \tag{11}$$

where in (11) B is an $N \times M$ matrix having elements

$$B_{ij} = h[(i-1)T' + (N_1 + 1 - j)T], \qquad \begin{matrix} 1 \le i \le N, \\ 1 \le j \le M. \end{matrix} \tag{12}$$

It follows from (10b) that $M \ge N$ if $T' = T$ and $M > (N+1)/2$ if $T' = T/2$.

The structure of the matrix B is illustrated below for the special case $T' = T$, $N = 3$, $M = 7$.

$$B = \begin{bmatrix} h_2 & h_1 & h_0 & h_{-1} & h_{-2} & 0 & 0 \\ 0 & h_2 & h_1 & h_0 & h_{-1} & h_{-2} & 0 \\ 0 & 0 & h_2 & h_1 & h_0 & h_{-1} & h_{-2} \end{bmatrix}.$$

This structure means that $\mathbf{X}^{(n)}$ has the same shifting property as $\mathbf{a}^{(n)}$. Thus, for example, in time sequence,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{bmatrix} b \\ c \\ d \end{bmatrix} \rightarrow \begin{bmatrix} c \\ d \\ e \end{bmatrix}, \quad \text{etc.}$$

Since

$$E\mathbf{a}^{(n)}\mathbf{a}^{(n)T} = I, \tag{13}$$

it follows from (6), (11), and (13) that

$$A = E\mathbf{X}^{(n)}\mathbf{X}^{(n)T} = BB^T. \tag{14}$$

For the special case $T' = T$, $h(nT) = \delta_{n0}$, then $M = N$, $A = I$, and

$$\mathbf{X}^{(n)} = \mathbf{a}^{(n)} = \begin{bmatrix} a_n \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n+N-1} \end{bmatrix}. \tag{15}$$

We now begin to describe the stochastic gradient algorithm used for equalizer convergence. But first we describe a different problem, the *deterministic* gradient algorithm, which is a method for finding the minimum on the surface $\mathscr{S}$, where

$$\mathscr{S} = \mathbf{c}^T A\mathbf{c} - 2\mathbf{c}^T\mathbf{v} + 1. \tag{16}$$

This provides some heuristics for writing down the stochastic algorithm, but should not be confused with it. We take pains to point out some differences as we proceed, since many people substitute discussion of this algorithm for the actual one.

Taking the gradient of (16) gives

$$\nabla\mathscr{S} = 2[A\mathbf{c} - \mathbf{v}]. \tag{17}$$

Hence, if we were searching for a minimum of the function (16) by taking steps in the gradient direction, we would write the following equation for our position $\mathbf{c}^{(n)}$ at the nth stage

$$\mathbf{c}^{(n+1)} = \mathbf{c}^{(n)} - \Delta(A\mathbf{c}^{(n)} - \mathbf{v}), \tag{18}$$

Δ being a step-size parameter. Equation (18) coupled with (6) and (7) motivates the actual stochastic gradient algorithm used, namely,

$$\mathbf{c}^{(n+1)} = \mathbf{c}^{(n)} - \alpha[\mathbf{X}^{(n)}(\mathbf{X}^{(n)T}\mathbf{c}^{(n)}) - a_{n+K}\mathbf{X}^{(n)}] \tag{19}$$

$$= \mathbf{c}^{(n)} - \alpha e_n\mathbf{X}^n, \tag{20}$$

e_n being the scalar error (5), and α the step-size†. Thus in N-dimensional tap space we move in directions $\mathbf{X}^{(n)}$, where $\mathbf{X}^{(n)}$ is [see (4)] the vector of values stored in the equalizer at time nT. Clearly, the allowed

† It is, of course, meaningless to speak of the "size" of α unless one fixes the size or scaling of the terms which multiply it in (20). We shall take the scaling of the latter so that, in the binary case, the matrix A [see (6)] has largest eigenvalue unity.

set of directions along which we "step" is, as (15) will testify, quite random and cannot be thought of as being gradient directions. Nevertheless, tradition dominates, and (19) and (20) are still referred to as a stochastic gradient algorithm.

For our purposes, (19) may be rewritten slightly by introducing the error vector

$$\boldsymbol{\epsilon}^{(n)} = \mathbf{c}^{(n)} - \mathbf{c}^*. \tag{21}$$

Subtracting $\mathbf{c}^*$ from both sides of (19) allows us to write

$$\boldsymbol{\epsilon}^{(n+1)} = (I - \alpha \mathbf{X}^{(n)} \mathbf{X}^{(n)T}) \boldsymbol{\epsilon}^{(n)} - \alpha(\mathbf{c}^{*T} \mathbf{X}^{(n)} - a_{n+K}) \mathbf{X}^{(n)}. \tag{22}$$

Note the quantity $\mathbf{c}^{*T}\mathbf{X}^{(n)} - a_{n+K}$ is the instantaneous error if the optimum taps were used. This is normally quite small and would be zero if perfect equalization were possible.

In terms of the $\boldsymbol{\epsilon}^{(n)}$, the mean-square error is

$$\mathscr{E}^{(n)} = \mathscr{E}^* + \boldsymbol{\epsilon}^{(n)T} A \boldsymbol{\alpha}^{(n)} \equiv \mathscr{E}^* + \mathscr{E}_{ex}^{(n)}. \tag{23}$$

In (23) the symbol $\mathscr{E}_{ex}^{(n)}$ has been introduced for the excess mean-square error over $\mathscr{E}^*$.

In (22) and (23), $\boldsymbol{\epsilon}^{(n)}$ is random, and in fact depends on the entire sequence of data symbols since the adaptation began. Our measure of the progress of the algorithm will be $E\mathscr{E}^{(n)}$, the average of the error at time n over all data sequences.

III. THE INDEPENDENCE THEORY

In this section we describe "independence theory," an approximation used to mathematically treat the stochastic gradient algorithm described by (22). Use of the approximation allows one (as we shall see) to determine bounds on the step-size α which will ensure stability and allows calculations to be made on convergence rates.

Independence theory treats the stochastic algorithm by assuming that the sequence $\mathbf{X}^{(n)}$ are statistically independent vectors. Since, from (22), $\boldsymbol{\epsilon}^{(n)}$ depends only on the sequence $\mathbf{X}^{(1)}, \cdots, \mathbf{X}^{(n-1)}$ (assuming we start with $\mathbf{X}^{(1)}$), we conclude $\boldsymbol{\epsilon}^{(n)}$ and $\mathbf{X}^{(n)}$ are independent. For an example as to how this is applied, we look at the average error vector $E\boldsymbol{\epsilon}^{(n)}$. We have, from (22), (6), (7), and (9),

$$E\boldsymbol{\epsilon}^{(n+1)} = (I - \alpha A) E\boldsymbol{\epsilon}^{(n)}. \tag{24}$$

If, for comparison, we introduce the error vector $\mathbf{c}_n - \mathbf{c}^*$ for the *deterministic* theory and call it $d^{(n)}$ so no confusion can arise, we would have, subtracting $\mathbf{c}^*$ from both sides of (18),

$$\mathbf{d}^{(n+1)} = (I - \Delta A) \mathbf{d}^{(n)}. \tag{25}$$

There is no question of an average in (25); $d^{(n)}$ is the error. In (24), $E\epsilon^{(n)}$ can be zero although the norm of $\epsilon^{(n)}$ can be quite large.

To emphasize the difference further, let us return to the simple model (15) which describes an undistorted channel, for which perfect equalization is possible. Only the initial setting of the taps is wrong. For this case, we have [note $A = I$ in (23)] using (22) and the independence assumption

$$E\epsilon^{(n+1)T}\epsilon^{(n+1)} = \epsilon^{(n)T}(I - \alpha\mathbf{X}^n\mathbf{X}^{(n)T})(I - \alpha\mathbf{X}^{(n)}\mathbf{X}^{(n)T})\epsilon^{(n)}$$
$$= (1 - 2\alpha + \alpha^2 N)\epsilon^{(n)T}\epsilon^{(n)}. \tag{26}$$

Thus the error decays to zero as

$$(1 - 2\alpha + \alpha^2 N)^n \mathscr{E}^{(0)}, \tag{27}$$

which is optimized if $\alpha = 1/N$ to give

$$\left(1 - \frac{1}{N}\right)^n \mathscr{E}^{(0)}. \tag{28}$$

Note how convergence is slowed as the number of taps N of the problem increases. By contrast, if $A = I$ in (25), choosing $\Delta = 1$ gives convergence in one step, independent of dimension.

The convergence range of (24) for $A = I$ is $0 < \alpha < 2$, while for (27) it is $0 < \alpha < 2/N$. In practice, N ranges from about 7 to 64 and thus α is, by the requirement of convergence of the mean-square error, kept quite small.

In order to examine independence theory further, it will be convenient to discuss the (positive definite) error matrix

$$R^{(n)} = E\epsilon^{(n)}\epsilon^{(n)T}. \tag{29}$$

All the information we wish about $E\mathscr{E}_{ex}^{(n)}$, the average excess mean-square error, is contained in (29). Thus, from (23)

$$E\mathscr{E}_{ex}^{(n)} = E\epsilon^{(n)T}A\epsilon^{(n)} = \sum_{i,j} (a)_{ij}(E\epsilon^{(n)}\epsilon^{(n)T})_{ji}$$
$$= \operatorname{tr} AR^{(n)}. \tag{30}$$

Similarly, the average norm $E\,\|\,\epsilon^{(n)}\,\|^2 = \operatorname{tr} R^{(n)}$.

Our procedure for writing an equation for the time evolution of $R^{(n)}$ is simply to write the definition of $R^{(n+1)}$ using (29), substitute (22) for $\epsilon^{(n+1)}$, and do the average using the independence assumption. Various cross terms arise, and the computations naturally fall into three steps:

Step 1:

$$E[I - \alpha\mathbf{X}^{(n)}\mathbf{X}^{(n)T}]\epsilon^{(n)}\epsilon^{(n)T}[I - \alpha\mathbf{X}^{(n)}\mathbf{X}^{(n)T}]$$
$$= R^{(n)} - \alpha[AR^{(n)} + R^{(n)}A] + \alpha^2 E[\mathbf{X}^{(n)}\mathbf{X}^{(n)T}R^{(n)}\mathbf{X}^{(n)}\mathbf{X}^{(n)T}]. \tag{31}$$

Appendix A discusses the evaluation of the last term. For simplicity, we approximate the exact evaluation by $\alpha^2 A \text{ tr } AR^{(n)}$. When $A = I$, we have $\text{tr}[\alpha^2 A \text{ tr } AR^{(n)}] = \alpha^2 N \mathscr{E}_{ex}^{(n)}$, so in general this term plays the role of the $\alpha^2 N$ term in (26).

Step 2:

$$E\alpha[I - \alpha \mathbf{X}^{(n)} \mathbf{X}^{(n)T}] \boldsymbol{\epsilon}^{(n)} \mathbf{X}^{(n)T} (\mathbf{c}^{*T} \mathbf{X}^{(n)} - a_{n+K}). \tag{32}$$

This is considered further in Appendix A and, for reasons given there, is approximated by zero.

Step 3: As discussed in Appendix A,

$$E\alpha^2 (\mathbf{c}^{*T} \mathbf{X}^{(n)} - a_{n+K}) \mathbf{X}^{(n)} \mathbf{X}^{(n)T} (\mathbf{c}^{*T} \mathbf{X}^{(n)} - a_{n+K}) \approx \alpha^2 \mathscr{E}^* A. \tag{33}$$

Putting together these three steps, we have the following accurate approximation from independence theory:

$$R^{(n+1)} = R^{(n)} - \alpha[AR^{(n)} + R^{(n)}A] + \alpha^2 A \text{ tr } AR_n^{(n)} + \alpha^2 \mathscr{E}^* A. \tag{34}$$

Note that the last term prevents $R^{(n)} = 0$ from being a solution. Thus, $R^{(n)}$ is prevented from going to zero by the small forcing term. Thus, in particular, $\boldsymbol{\epsilon}^{(n)}$ only approaches zero but then executes small fluctuations about zero.

Since (34) is an approximation, we prove in Appendix B that the positive definite character of $R^{(n)}$ is preserved in (34).

We now introduce a more useful form of (34) when the mean-square error is of primary interest. Since A is hermitian, let U be the unitary transformation which diagonalizes A,

$$U^+ A U = D, \tag{35}$$

where we call the elements of the diagonal matrix D, by d_i. Further, let

$$U^+ R^{(n)} U = T^{(n)}. \tag{36}$$

In general, $T^{(n)}$ is not diagonal, but set $T_{ii}^{(n)} = t_i^{(n)}$. Further, note

$$\mathscr{E}_{ex}^{(n)} = \text{tr } AR^{(n)} = \text{tr } DT^{(n)} = \sum_{i=1}^{N} d_i t_i^{(n)}. \tag{37}$$

It follows from (34), (35), and (36) that

$$T^{(n+1)} = T^{(n)} - \alpha[DT^{(n)} + T^{(n)}D] + \alpha^2 D \text{ tr } DT^{(n)} + \alpha^2 \mathscr{E}^* D. \tag{38}$$

Noting from (37) that the mean-square error depends only on the $t_i^{(n)}$, we are motivated to look at the diagonal terms of (38). Happily, they decouple from the off-diagonal terms and we have

$$t_i^{(n+1)} = t_i^{(n)} - 2\alpha d_i t_i^{(n)} + \alpha^2 d_i \sum_{j=1}^{n} d_j t_j^{(n)} + \alpha^2 \mathscr{E}^* d_i. \tag{39}$$

If we introduce *vectors* $\mathbf{t}^{(n)}$ and $\mathbf{d}$ in the obvious way, (39) itself can be rewritten in matrix notation as

$$\mathbf{t}^{(n+1)} = M\mathbf{t}^{(n)} + \alpha^2 \mathscr{E}^* \mathbf{d}, \tag{40}$$

where the $N \times N$ matrix M has elements

$$M_{ij} = (1 - 2\alpha d_i)\delta_{ij} + \alpha^2 d_i d_j . \tag{41}$$

From (41) we note M is real and symmetric and thus has real eigenvalues.

The solutions to (40) will be stable if and only if the matrix M has all eigenvalues λ_i such that $-1 \leq \lambda_i \leq 1$. Let g be an eigenvector of M with eigenvalue λ. Then

$$M\mathbf{g} = \lambda \mathbf{g} \tag{42}$$

reads

$$g_i - 2\alpha d_i g_i + \alpha^2 (\sum_j d_j g_j) d_i = \lambda g_i$$

or

$$g_i = -\alpha^2 (\sum d_i g_j) \frac{d_i}{1 - \lambda - 2\alpha d_i}, \tag{43}$$

g_i denoting the components of $\mathbf{g}$.

In (41) we see that, whenever $d_i = 0$, there is a $\lambda = 1$ for all α. The eigenvector has $g_i = 1$ and $g_j = 0$, $j \neq i$. These eigenvalues do not change with α and are not of interest here. Set $\tilde{d}_i = d_i$ if $d_i \neq 0$. Then we are concerned with

$$\tilde{M}_{ij} = (1 - 2\alpha \tilde{d}_i)\delta_{ij} + \alpha^2 \tilde{d}_i \tilde{d}_j \tag{44}$$

in a space of appropriately reduced dimension $\tilde{N}$. For α small enough, the eigenvalues are approximately $1 - 2\alpha\tilde{d}_i < 1$ ($\alpha > 0$, of course). Now increase α until possibly one of the eigenvalues becomes ± 1. What is the critical value of α? Since all elements of (44) are strictly positive (except at most $\tilde{N}$ values of α), the magnitude of the largest eigenvalue may be taken to be associated with a positive eigenvalue.[2] Thus, in (43) [reinterpreted to match (44)], set $\lambda = 1$, multiply by d_i, and sum on i. We then obtain

$$\alpha_{crit} = \frac{2}{\sum \tilde{d}_i} = \frac{2}{\sum d_i}. \tag{45}$$

Thus, independence theory predicts a stable algorithm if

$$0 < \alpha < \frac{2}{\sum d_i} = \frac{2}{N\bar{d}}, \tag{46}$$

$\bar{d}$ being the average eigenvalue of the channel correlation matrix A.

The excess error $\mathscr{E}_{ex}^{(\infty)}$ after adaptation may be derived from (39) using (37). We get

$$\mathscr{E}_{ex}^{(\infty)} = \alpha \mathscr{E}^* \frac{\sum d_i}{2 - \alpha \sum d_i}. \tag{47}$$

The above discussion should provide the reader with an idea of what we hope to justify and why. The independence assumption, if it leads to valid results, provides a very workable theory for gaining insights about, and doing calculations on, the convergence procedure.

IV. AN EXACT DESCRIPTION

In this section, we put forth an exact description of how, in principle, the average mean-square error may be obtained. We begin, however, with the average error vector $E\boldsymbol{\epsilon}^{(n)}$, a simpler quantity, but one which requires essentially the same treatment. The exact dynamics of $\boldsymbol{\epsilon}^{(n)}$ is given in (22), and the independence theory for $E\boldsymbol{\epsilon}^{(n)}$ is given by (24).

For simplicity, we rename the terms in (22)

$$(I - \alpha \mathbf{X}^{(n)} \mathbf{X}^{(n)T}) \boldsymbol{\epsilon}^{(n)} \equiv P_n \boldsymbol{\epsilon}^{(n)} \tag{48}$$

and

$$-\alpha(\mathbf{c}^{*T} \mathbf{X}^{(n)} - a_{n+K}) \mathbf{X}^{(n)} \equiv \mathbf{f}^{(n)}, \tag{49}$$

so (22) reads

$$\boldsymbol{\epsilon}^{(n+1)} = P_n \boldsymbol{\epsilon}^{(n)} + f^{(n)}, \tag{50}$$

which, by iteration starting with a fixed error vector $\boldsymbol{\epsilon}^{(0)}$, has the solution

$$\boldsymbol{\epsilon}^{(n+1)} = \prod_{i=0}^{n} P_i \boldsymbol{\epsilon}^{(0)} + \sum_{s=0}^{n-1} \left(\prod_{i=s+1}^{n} P_i \right) \mathbf{f}^{(s)} + \mathbf{f}^{(n)}. \tag{51}$$

Note in (51) the matrices P_i do not commute so that a product $\prod_1^n P_i$ means in the order $P_n \cdots P_2 P_1$.

We proceed to examine (51) in more detail. We remark first that, by their very definition, P_n and $\mathbf{f}^{(n)}$ depend on the data variables $\{a_n, a_{n+1}, \cdots, a_{n+M-1}\}$ [see (11), (15), (48), (45)], and thus $\boldsymbol{\epsilon}^{(n+1)}$ depends on the entire sequence $\{a_i\}_{i=0}^{n+M-1}$. If we formally average (51) making use of the stationarity of the basic Bernoulli sequence $\{a_i\}$, we have

$$E\boldsymbol{\epsilon}^{(n+1)} = \left(E \prod_{i=0}^{n} P_i \right) \boldsymbol{\epsilon}^{(0)} + \sum_{s=1}^{n} \left(E \prod_{i=1}^{s} P_i \mathbf{f}^{(0)} \right), \tag{52}$$

the expectation being taken over all binary variables which enter (52), namely, $a_0, a_1, \cdots, a_{n+M-1}$. The first term of (52) represents the decay of the initial error to zero (the transient); the second term is the forced response, causing a small but nonzero steady state error as $n \to \infty$.

We have not been able to work with (52) directly, and at this point our analysis takes a crucial turn. We average (51) again, only this time we do not average over all the binary variables which enter but only over the sequence $a_0, a_1, \cdots, a_n$. Call this conditioned average E_n. Then

$$E_n\epsilon^{(n+1)} = \left(E_n \prod_{i=0}^{n} P_i\right)\epsilon^{(0)} + \sum_{s=0}^{n-1}\left(E_n \prod_{i=s+1}^{n} P_i f^{(s)}\right) + E_n\mathbf{f}^{(n)}. \quad (53)$$

Now, however, (53) is not one vector equation but 2^{M-1} of them, since it is valid for any sequence of values of $\{a_{n+1}, \cdots, a_{n+M-1}\}$; these variables appear in (53) for arbitrary values. Thus the set of values just mentioned form a "super-index" which we may collectively call J, J taking 2^{M-1} values. For example, we might choose to call (for $M = 3$) the values $\{+1, +1\}$ to be $J = 1$, $\{+1, -1\}$ to be $J = 2$, $\{-1, +1\}$ to be $J = 3$, and $\{-1, -1\}$ to be $J = 4$. For the moment, however, the precise mapping from the $(M-1)$ binary variables to the integer J is unimportant.

We also want to consider the matrix

$$P_n = I - \alpha\mathbf{X}^{(n)}\mathbf{X}^{(n)T} \quad (54)$$

not as a $N \times N$ matrix, but as one consisting of $2^{M-1} \times 2^{M-1}$ blocks of $N \times N$ matrices so that it may act in (53) as a transition matrix between vector blocks.

Thus in (54) P_n is determined by $\mathbf{X}^{(n)}$, i.e., from (1), by

$$\mathbf{X}^{(n)} = B\begin{pmatrix} a_n \\ a_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ a_{n+M-1} \end{pmatrix}. \quad (55)$$

Hence the "super-index" J corresponding to the vector result of an operation by P_n would be the last $M-1$ components of $\mathbf{a}^{(n)}$, namely $(a_{n+1}, a_{n+2}, \cdots, a_{n+m-1})$. On the other hand, P_n acts on a quantity determined by

$$\mathbf{X}^{(n-1)} = B\begin{pmatrix} a_{n-1} \\ a_n \\ \cdot \\ \cdot \\ \cdot \\ a_{n+m-2} \end{pmatrix}; \quad (56)$$

that is, something with vector index $J' = (a_n, \cdots, a_{n+m-2})$. Thus if we call

$$I - \alpha\mathbf{X}^{(n)}\mathbf{X}^{(n)T} \equiv K(J, J'), \quad (57)$$

$K(J, J')$ can only act between index pairs (J, J'), which are "shift-compatible." Thus if

$$J \leftrightarrow (s_1, \cdots, s_{m-1})$$
$$J' \leftrightarrow (t_1, \cdots, t_{m-1}), \tag{58}$$

where the s_i and t_i are binary variables, then

$$K(J, J') = 0 \quad \text{unless} \quad s_i = t_{i+1}, \qquad i = 1, \cdots, M-2. \tag{59}$$

On the other hand, if (J, J') are shift-compatible, this is enough to determine the appropriate $\mathbf{X}^{(n)}$. Thus with (58), (59),

$$\mathbf{X}^{(n)} = B \begin{pmatrix} t_1 \\ s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_{m-1} \end{pmatrix}, \tag{60}$$

and we use (57) to define the appropriate $K(J, J')$. Having, in the manner thus described, achieved the block structure (57), we define the $N \times 2^{M-1}$ dimensional square matrix

$$A(\alpha) = \frac{1}{2} \begin{bmatrix} K(1,1) & K(1,2) & \cdots & K(1, 2^{M-1}) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ K(2^{M-1}, 1) & \cdots & & \end{bmatrix}. \tag{61}$$

There are, in fact, in any row of (61) only two nonvanishing blocks. Summing over the row thus corresponds, because of the factor of ½ in front, to averaging over the first component a_n of $\mathbf{A}^{(n)}$.

We write any N vector which is further labeled by our block index J [$\mathbf{v}(J)$, say] as an $N \times 2^{M-1}$ vector $\mathbf{V}$

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}(1) \\ \mathbf{v}(2) \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{v}(2^{M-1}) \end{pmatrix}. \tag{62}$$

To tie this all together, it is now easy to convince oneself that, if we let V_{n+1} correspond to $E_n \epsilon^{(n+1)}$ as in (62) and, similarly, let F correspond to $E_n \mathbf{f}^{(n)}$, then, by making use of the stationarity of the averages which appear, (53) represents the solution of the equation

$$V_{n+1} = A(\alpha)\mathbf{V}_n + \mathbf{F} \tag{63}$$

with initial condition

$$V_0 = \begin{pmatrix} \epsilon^{(0)} \\ \epsilon^{(0)} \\ \cdot \\ \cdot \\ \cdot \\ \epsilon^{(0)} \end{pmatrix} \equiv [\epsilon^{(0)}]. \tag{64}$$

In (64), the notation [v] has been introduced to represent an n-vector "stacked" 2^{M-1} times.

The solution to (63) and (64) contains all the information we want. In fact, once $\mathbf{V}_n$ is known we, by definition, know $E_{n-1}\epsilon^{(n)}(J)$, where we have modified the notation slightly to make explicit the dependence on $J \leftrightarrow (a_{n+1}, \cdots, a_{n+M-1})$. To regain $\epsilon^{(n)}$, we simply average:

$$\epsilon^{(n)} = E[E_{n-1}\epsilon^{(n)}(J)] = \frac{1}{2^{M-1}} \sum_{J=1}^{2^{M-1}} E_{n-1}\epsilon^{(n)}(J). \tag{65}$$

The average in (65) can be put in another form if we introduce the matrix†

$$P_1 = \frac{1}{2^{M-1}} \begin{bmatrix} I & I & I & \cdots \\ I & I & I & \cdots \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ I & I & I & \cdots \end{bmatrix}, \tag{66}$$

having each $N \times N$ block equal to the identity matrix. Then

$$[\epsilon^{(n)}] = P_1 \mathbf{V}_n. \tag{67}$$

We may already note that P_1 is an orthogonal projection operator ($P_1^2 = P_1$, $P_1^T = P_1$) and (67) thus states that $[\epsilon^{(n)}]$ is a projection of $\mathbf{V}_n$ into an appropriate subspace. Further, note that

$$[Ef^{(n)}] = [\mathbf{0}] = P_1\mathbf{F} \tag{68}$$

and thus $\mathbf{F}$ belongs to the orthogonal subspace.

The formal solution of (63) (including the final projection) is

$$P_1\mathbf{V}_n = P_1A^n(\alpha)[\epsilon^{(0)}] + P_1 \sum_{s=0}^{n-1} A^s(\alpha)\mathbf{F} \tag{69}$$

having the limit

$$P_1\mathbf{V}_\infty = P_1[I - A(\alpha)]^{-1}\mathbf{F}. \tag{70}$$

† We hope a warning that the symbol P_1 is being used for different things in (66) and (48) will eliminate confusion.

Both (69) and (70) can be computed using the spectral decomposition for functions of a matrix A. If A has all its eigenvalues λ_i of index one, that is, if its eigenvectors U_i span the space (all Jordan blocks one-dimensional) and if W_i are corresponding eigenvectors of A^T, chosen so that

$$W_i^T U_j = \delta_{ij}, \tag{71}$$

then for (almost) any function $h(\cdot)$,

$$h(A) = \sum_i h(\lambda_i) \mathbf{U}_i \mathbf{W}_i^T. \tag{72}$$

Roughly, $h(\cdot)$ is restricted so that $h(\lambda_i)$ is defined. A similar but more complicated theorem holds if the U_i do not span the space. If $\alpha \neq 0$, it may be reasonable to assume that the U_i do indeed span the space, but for $\alpha = 0$ they do not.

We may already note that asymptotic stability of the full-fledged algorithm is guaranteed if all eigenvalues of $A(\alpha)$ are less than unity in magnitude. In fact, only those eigenvalues which are associated with a U_i such that $PU_i \neq 0$ need have magnitude less than unity.

In general, because of the very large dimension ($N2^{M-1}$) encountered in practical use, the above theory would be more useful if workable approximations could be found. We present one such approach in Section VI which is based on a perturbation approach for small step-size α. Before doing that, we retreat a bit to demonstrate how the mean-square error may be brought into essentially the same form just developed for the average error vector.

We again find it more convenient to discuss the error matrix $R^{(n)}$ defined in (29). We substitute (50) directly into (29) and perform our trick of taking the average E_n (which involves averaging only over a_0, $a_1, \cdots, a_n$ leaving $a_{n+1}, \cdots, a_{n+M-1}$ fixed) to obtain

$$E_n R^{(n+1)} = \frac{1}{2} \sum_{a_n} P_n (E_{n-1} R^{(n)}) P_n + \frac{1}{2} \sum_{a_n} \mathbf{f}^{(n)} \mathbf{f}^{(n)T}$$

$$+ \frac{1}{2} \sum_{a_n} (P_n (E_{n-1} \boldsymbol{\epsilon}^{(n)}) \mathbf{f}^{(n)} + \mathbf{f}^{(n)} (E_{n-1} \boldsymbol{\epsilon}^{(n)})^T P_n). \tag{73}$$

In (73), $\sum_{a_n}$ refers to summing over $a_n = \pm 1$. Note that in (73) the sequence of quantities $E_{n-1}\boldsymbol{\epsilon}^{(n)}$ may be regarded as known (or calculable) since they are the N dimensional subvectors which make up the $N \times 2^{M-1}$ dimensional solution V_n to (63) and (64).

We will rewrite (73), but first we need some notation. If R is any $N \times N$ matrix, we may make an N^2 dimensional vector out of it by

writing the quantity

$$\xi(R) = \begin{bmatrix} R_{11} \\ R_{12} \\ \cdot \\ \cdot \\ \cdot \\ R_{1N} \\ R_{21} \\ R_{22} \\ \cdot \\ \cdot \\ \cdot \\ R_{2N} \\ \cdot \\ \cdot \\ \cdot \\ R_{N1} \\ \cdot \\ \cdot \\ \cdot \\ R_{NN} \end{bmatrix}. \tag{74}$$

We call $\xi(R)$ the vector made out of R.

In this trivial sense, we use $\xi(\cdot)$ as an operator. We use this to turn some of the terms in (73) into vectors. Introducing the "J-index" for emphasis (it is, of course, implicit when we use E_n) we define

$$\mathbf{w}^{(n+1)}(J) = \xi[E_n R^{(n+1)}(J)] \tag{75a}$$

$$\mathbf{g}(J) = \xi[E_n \mathbf{f}^{(n)} \mathbf{f}^{(n)T}] \tag{75b}$$

$$\mathbf{g}(V_n, J) = \xi[E_n \mathbf{f}^{(n)} \mathbf{f}^{(n)T} + E_n P_n \boldsymbol{\epsilon}^{(n)} \mathbf{f}^{(n)T} + E_n \mathbf{f}^{(n)} \boldsymbol{\epsilon}^{(n)T} P_n]. \tag{75c}$$

Next we note that if A, R, and B are $N \times N$ matrices, then

$$\xi(ARB) = C\xi(R), \tag{76}$$

where C is an $N^2 \times N^2$ matrix. In fact, C is the direct product $A \otimes B^T$ where $A \otimes B$ (not $A \otimes B^T$) is given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1N}B \\ a_{21}B & a_{22}B & \cdots & a_{2N}B \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{N1}B & a_{N2}B & \cdots & a_{NN}B \end{bmatrix}. \tag{77}$$

In the notation (75) and (77), (73) may be rewritten as

$$\mathbf{w}^{(n+1)}(J) = \frac{1}{2}\sum_{a_n} P_n \otimes P_n \mathbf{w}^{(n)}(J') + \mathbf{g}(V_n, J). \tag{78}$$

In (78) J' is the compatible pair of indices that are allowed with J. As in (63), we form $N^2 \times 2^{M-1}$ dimensional vectors $\mathbf{W}_n$, $\mathbf{G}$, and $\mathbf{G}(V_n)$ from $\mathbf{w}^{(n)}(J)$, $\mathbf{g}(J)$, and $\mathbf{g}(V_n, J)$, respectively. And finally, using the definition of $K(J, J')$ in (57) to (60) we write

$$B(\alpha) = \frac{1}{2}\begin{bmatrix} K(1,1)\otimes K(1,1) & K(1,2)\otimes K(1,2) & \cdots \\ \vdots & & \\ K(2^{M-1},1)\otimes K(2^{M-1},1) & \cdots & \end{bmatrix}. \tag{79}$$

The collection of equations (78) reads

$$\mathbf{W}_{n+1} = B(\alpha)\, W_n + G(V_n). \tag{80}$$

Equation (80) with (63), (64), and the initial condition

$$\mathbf{W}_0 = \begin{bmatrix} \xi(\epsilon^{(0)}\epsilon^{(0)T}) \\ \xi(\epsilon^{0}\epsilon^{(0)T}) \\ \vdots \\ \xi(\epsilon^{0}\epsilon^{(0)T}) \end{bmatrix} \tag{81a}$$

provide an exact description of the error matrix.

To simplify matters, we replace (80) by the approximate version

$$\mathbf{W}_{n+1} = B(\alpha)\mathbf{W}_n + \mathbf{G}, \tag{81b}$$

where $\mathbf{G}$, as already defined, is formed from (75b) as $\mathbf{G}(V_n)$ was formed from (75c). When more is understood about the solutions of our equations, we see that the replacement of (80) by (81b) is not a serious matter.†

Again, we are not interested so much in W_n as the projected version

$$[\xi(R^{(n)})] = P_1 \mathbf{W}_n. \tag{82}$$

In (82) the bracket notation is the same as (64) except that $\xi(R^{(n)})$ is a vector of dimension N^2 instead of N. Also, in (32) P_i has the same meaning as in (66) except that the identity matrices are all in N^2 dimensions instead of N.

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† In most situations, $\mathbf{G}(V_n)$ is small compared to the initial error and the associated transient. The main effect of the forcing term is to give a nonzero error as $n \to \infty$. But $V_n \to 0$, and $\mathbf{G}(V_n)$ reduces to $\mathbf{G}$.

In summary, we thus see that both eq. (63) for $\mathbf{V}_n$, which represents $E_{n-1}\epsilon^{(n)}$, and (81b) for $\mathbf{W}_n$, which represents $E_{n-1}R^{(n)}$, have the form

$$\mathscr{V}_{n+1} = \mathscr{A}(\alpha)\mathscr{V}_n + \mathscr{F}, \tag{83}$$

with

$$\mathscr{V}_0 = [\phi], \tag{84}$$

and the quantity of interest being

$$\mathscr{P}_1\mathscr{V}_n \tag{85}$$

for the appropriate dimension and projection $\mathscr{P}_1$.

V. THE CASE $\alpha = 0$

The equalization problem is uninteresting when the step-size is taken to be zero, i.e., nothing happens. However, since we soon intend to do a perturbation analysis about $\alpha = 0$ we must be familiar with our formalism when $\alpha = 0$. This is not trivial, and we devote this section to it.

To display matrices explicitly, we need a labeling procedure. We let the "super-index" $\mathbf{J}$ run from 0 to $(2^{M-1} - 1)$.† The $\mathbf{J}$ value which labels $(a_1, \cdots, a_{M-1})(a_i = \pm 1)$ is gotten as follows: Change +1 to 0, and −1 to 1, obtaining then binary representation of J. Thus, for $M = 3$, $J = 0, 1, 2, 3$ correspond respectively to (+, +), (+−), (−+), and (−, −). With this labeling we have

$$\mathscr{A}(0) = \frac{1}{2}\begin{bmatrix} 1 & & 1 & \\ 1 & & 1 & \\ 01 & 0 & 01 & 0 \\ 01 & & 01 & \\ 001 & & 001 & \\ 001 & & 001 & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ 000 \cdots 1 & & 00 \cdots 1 & \\ 000 \cdots 1 & & 00 \cdots 1 & \end{bmatrix} \otimes I \equiv \Gamma \otimes I. \tag{86}$$

Let S be vector space of dimension N or N^2 accordingly as $\mathscr{V}_n$ in (83) refers to $\mathbf{V}_n$ or $\mathbf{W}_n$. Then in (86) I refers to the identity in S.

The matrix $\mathscr{A}(\alpha)$ has the same structure as (86), with each identity being replaced by the appropriate $I - \alpha XX^T$ or $(I - \alpha XX^T) \otimes (I - \alpha XX^T)$.

† This labeling is for descriptive convenience here. We hope the reader is forgiving if we later let $\mathbf{J} = 1, 2, \cdots, 2^{M-1}$. We will be explicit about the convention when it matters.

Table I—Zero eigenvalue structure of Γ

Index	# Blocks of This Index
$M-1$	1
$M-2$	1
$M-3$	2
$M-4$	4
$M-5$	8
$\vdots$	
$M-l$	$2^{l-2}(l \geq 3)$
$\vdots$	
$M-(M-2)=2$	2^{M-4}
1	2^{M-3}

The matrix Γ in (86) is basic to our study and we now concentrate on it; it has dimension 2^{M-1}. Clearly, the all-ones vector is an eigenvector of Γ having eigenvalue one. The reader may convince himself that Γ^{M-1} is proportional to the matrix consisting of all ones, which has $(2^{M-1}-1)$ eigenvectors perpendicular to the all-ones vector. These eigenvectors are associated with eigenvalue zero. Using the fact that the eigenvalues of a power of a matrix are the powers of the eigenvalues, we conclude that Γ has one unity eigenvalue and $(2^{M-1}-1)$ zero ones. The zero eigenvalues are not of index one however (index, recall, is the dimension of the Jordan block). Table I summarizes the structure of the zero eigenvalues of Γ.

While it is not crucial for the sequel, we also give the eigenvectors and generalized eigenvectors of Γ. These are the columns, albeit permuted, of Hadamard matrices H_n constructed according to $H_2 = 1$,

$$H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix} = H_{2n}^T. \tag{87}$$

Rows and columns of H_n are labeled from 0 to $n-1$. Our claim is that the columns of $H(2^M-1)$ are the (unnormalized) generalized eigenvectors of Γ. Recall that a sequence of vectors x_l, $l = 1, \cdots, k$ forms a chain of generalized eigenvectors corresponding to a k-dimensional Jordan block when

$$\Gamma X_l = X_{l+1} \qquad l = 1, \cdots, k-1$$

$$\Gamma X_k = \lambda X_k.$$

Clearly, the last 2^{M-2} columns of H satisfy $\Gamma X_k = 0$ and these are the only ones. If c_k is the kth column, $2^{M-2}+1 \leq k \leq 2^{M-1}$, then the chain that ends with it is, in reverse order,†

† For (88) to hold, it is essential that the first column be labeled c_0. Also, of course, the c_k of this section is different from c_k in Section II where it signified equalizer taps. No confusion should arise.

$$(c_k, c_{k/2}, c_{k/4}, \cdots). \tag{88}$$

These notions may be verified for

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}. \tag{89}$$

The chains are (4, 2, 1), (5), (6, 3), (7). If we rearrange the columns of H_8 to give

$$\tilde{H}_8 = (c_0, c_5, c_7, c_6, c_3, c_4, c_2, c_1), \tag{90}$$

then

$$\frac{1}{8}\tilde{H}_8^T \Gamma \tilde{H}_8 = \begin{bmatrix} 1 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & & & & & \\ & & & 0 & 1 & & & \\ & & & 0 & 0 & & & \\ & & & & & 0 & 1 & 0 \\ & & & & & 0 & 0 & 1 \\ & & & & & 0 & 0 & 0 \end{bmatrix}. \tag{91}$$

From the direct product structure in (86) we conclude that if Φ_i are a complete o.n. set for S, then the generalized eigenvectors of $\mathscr{A}(0)$ are

$$\frac{1}{\sqrt{2^{M-1}}} c_k \otimes \Phi_i, \tag{92}$$

c_k being the columns of the Hadamard matrix just described. In particular, $\mathscr{A}(0)$ has N (or N^2) unity eigenvalues of index one, having eigenvectors

$$U_i = \frac{1}{\sqrt{2^{M-1}}} \begin{bmatrix} \Phi_i \\ \Phi_i \\ \cdot \\ \cdot \\ \cdot \\ \Phi_i \end{bmatrix}; \tag{93}$$

the remaining eigenvalues are zero. The projection operator onto the

space spanned by the eigenvectors having $\lambda = 1$ is, using (93),

$$\sum_i U_i U_i^T = \mathscr{P}_1, \tag{94}$$

where $\mathscr{P}_1$ has already been introduced in (66), I being the identity of S. Since $\mathscr{P}_1 = \mathscr{P}_1^T$, the projection is orthogonal. We call the projection onto the "zero eigenvalue subspace" of $\mathscr{A}(0)$ by $\mathscr{P}_0$ and $\mathscr{P}_0 = I - \mathscr{P}_1$.

Thus, when we solve (83), we really desire, according to (85), not $\mathscr{V}_n$ but $\mathscr{P}_1 \mathscr{V}_n$, its projection onto the unity eigenvalue subspace of $\mathscr{A}(0)$.

A standard spectral representation of $\mathscr{A}(0)$ is

$$\mathscr{A}(0) = \mathscr{P}_1 + \mathscr{D}_0, \tag{95}$$

where $\mathscr{D}_0^{M-1} = 0$. This defines (for us) $\mathscr{D}_0$. It may be shown that $\mathscr{P}_1 \mathscr{D}_0 = 0$.

We remark here that our basic equalization problem is unchanged if any infinite sample sequence of data values $\{a_n\}$ is replaced by their negatives. This follows from the quadratic nature (in the a_n) of the algorithm (19). As a consequence, we have

$$E_n \boldsymbol{\epsilon}^{(n)}(I \leftrightarrow s_1, \cdots, s_{m-1}) = E \boldsymbol{\epsilon}^{(n)}(J \leftrightarrow -s_1, \cdots, -s_{m-1}) \tag{96}$$

and similarly for $\mathbf{w}^{(n)}(J)$. We have not exploited this symmetry, but if we had, the dimension of $\mathscr{A}(\alpha)$ could be reduced by a factor of 2. $\mathscr{A}(0)$ would then, in particular, have a different form, but would have many of the same properties discussed here.

Finally, we take this opportunity to get some notational problems out of the way. We introduce a convenient way of labeling matrices C with block structure as in (86). Label rows by μ, $\mu = 1, 2, \cdots, 2^{M-1}$ and likewise columns by ν. If we write

$$\begin{aligned} \mu &= (i-1)n + k \qquad 1 \le i, j \le 2^{M-1} \\ \nu &= (j-1)n + l \qquad 1 \le k, l \le N \text{ (or } N^2\text{)} \\ & \qquad\qquad n = N \text{ (or } N^2\text{)}, \end{aligned} \tag{97}$$

then the pair (i, j) specifies which block we are concerned with, while the pair (k, l) are the usual matrix indices for the $N \times N$ (or $N^2 \times N^2$) matrix in that block. Thus, for example, in (77),

$$(A \otimes B)_{\mu\nu} = a_{ij} b_{kl}. \tag{98}$$

Likewise, in (92) the vector $c \otimes \Phi$ has components

$$(\mathbf{c} \otimes \boldsymbol{\Phi})_\mu = c_i \, \boldsymbol{\Phi}_k. \tag{99}$$

The orthonormal basis for S where the kth basis vector has a one in the kth position and zeros elsewhere is denoted by $\{e_k\}$.

VI. THE PERTURBATION THEORY

We begin the next stage of analysis by writing our matrices in a new basis. Consider the orthogonal transformation matrix U which brings $\mathscr{A}(0)$ to Jordan form, namely, the matrix U whose columns are of the form

$$\frac{1}{\sqrt{2^{M-1}}} \mathbf{c}_i \otimes e_k, \tag{100}$$

where $\mathbf{c}_i$ are columns of the Hadamard matrix of appropriate dimension, and $\mathbf{e}_k$ are the basis vectors of S. In (100), i and k range over all possible values. The columns of U are assumed to be arranged so that the result on $\mathscr{A}(0)$ comes out "nice." We will not bother to be too explicit, except to say that the first N(or N^2) columns of U are

$$\frac{1}{\sqrt{2^{M-1}}} c_0 \otimes \mathbf{e}_k \qquad k = 1, \cdots, N(N^2). \tag{101}$$

Then†

$$U^T \mathscr{A}(\alpha) U = \begin{bmatrix} \beta & \nu \\ \gamma & \delta \end{bmatrix} = \tilde{\mathscr{A}}(\alpha). \tag{102}$$

In (102), β is an $N \times N$ matrix, ν is $N \times (2^{M-1} - N)$ matrix, etc. If $\alpha = 0$, (102) takes the form

$$\begin{bmatrix} I & 0 \\ 0 & \mathscr{J} \end{bmatrix}, \tag{103}$$

$\mathscr{J}$ being a Jordan block exemplified by (91), i.e., "nice." Note that $\mathscr{J}^l = 0$ if $l \geq M-1$.

In general, when $\alpha \neq 0$, all blocks in (103) have added terms which are linear in α, or linear and quadratic, depending on whether (61) or (79) applies.‡

We shall be especially concerned with the matrix β, for it is here that the germ of independence theory appears. To calculate it, we want

$$\beta_{kl} = \left[\frac{1}{\sqrt{2^{M-1}}} \mathbf{c}_0 \otimes \mathbf{e}_k\right] \mathscr{A}(\alpha) \left[\frac{1}{\sqrt{2^{M-1}}} \mathbf{c}_0 \otimes \mathbf{e}_l\right]. \tag{104}$$

Calling the (m, n) element of the (i, j) block of $\mathscr{A}(\alpha)$ by $\frac{1}{2}\theta^{ij}_{mn}$, (104) becomes

† Henceforth, we denote transformed quantities by a tilde.

‡ The reader should note that the simple equations (27) and (28) suggest that the linear and quadratic α-terms are of equal importance for ranges of α of interest.

$$\beta_{kl} = \frac{1}{2^M} \sum_{\substack{ij \\ mn}} (\mathbf{c}_0)_i (\mathbf{e}_k)_m \, \theta_{mn}^{ij} \, (\mathbf{c}_0)_j (\mathbf{e}_l)_n. \tag{105}$$

Now $\theta_{mn}^{ij} = 0$ whenever Γ_{ij} in (86) is. Thus, for fixed mn there are only 2^m possible θ_{mn}^{ij} which are nonzero. Denote the sum over these as $\sum_{\text{nonzero}}$. Then (105) becomes, using $(\mathbf{e}_k)_m = \delta_{km}$, $(\mathbf{c}_0)_i = 1$

$$\beta_{kl} = \frac{1}{2^M} \sum_{\text{nonzero}} \theta_{kl}^{ij}. \tag{106}$$

Equation (106) gives β_{kl} as the average of the (k, l) elements of all 2^M blocks in $\mathscr{A}(\alpha)$ which are not *a priori* zero. This, however, is nothing but

$$E(I - \alpha \mathbf{X}\mathbf{X}^T) = I - \alpha A, \tag{107}$$

precisely the matrix which enters in the independence theory! Likewise, if $\mathscr{A}(\alpha) = \mathrm{B}(\alpha)$

$$E(I - \alpha \, XX^T) \otimes (I - \alpha XX^T) \tag{108}$$

is the matrix by which we would solve independence theory had we rewritten (31) giving $R^{(n)}$ its vector form rather than its matrix form.

What do vectors look like with our new o.n. basis? If $\mathscr{V}$ is a column vector of numbers in the original basis, then in the new basis the numbers are $U_i^T \mathscr{V}$. Let $\mathscr{V}$ be considered as blocks of $N(N^2)$ vectors Φ^i; the kth component of each is Φ_k^i. Then the inner product of a particular row of U with $\mathscr{V}$, namely,

$$\sum_{\mu} (c_i \times \mathbf{e}_k)_\mu \mathscr{V}_\mu$$

is a generic term of $U^T \nu$ which evaluates to

$$\frac{1}{\sqrt{2^{M-1}}} \sum_{l=1}^{2^{M-1}} (\mathbf{C}_i)_l \Phi_k^l. \tag{109}$$

Thus the first $N(N^2)$ components (the first blocks) is simply $\sqrt{2^{M-1}}$ times the average of the blocks of $\mathscr{V}$. In other words,

$$\begin{bmatrix} \Phi \\ \Phi \\ \cdot \\ \cdot \\ \cdot \\ \Phi \end{bmatrix} \rightarrow \sqrt{2^{M-1}} \begin{bmatrix} \Phi \\ \\ 0 \end{bmatrix}. \tag{110}$$

The right member of (110) is, of course, written in a notation compatible with (102). Likewise, a vector with zero average transforms to a

vector which may be written

$$\sqrt{2^{M-1}} \begin{bmatrix} 0 \\ \\ \Phi \end{bmatrix}. \tag{111}$$

Thus the initial condition for (63) or (80) is of type (110) unlike the driving term for (63), which is of type (111). The driving term for (80) has both types.

Finally, we note that the projection operator onto the unity eigenspace of $\mathscr{A}(0)$ is

$$\mathscr{P}_1 = \begin{bmatrix} I & 0 \\ \\ 0 & 0 \end{bmatrix} \tag{112}$$

while

$$\mathscr{P}_0 = \begin{bmatrix} 0 & 0 \\ \\ 0 & I \end{bmatrix}. \tag{113}$$

It will also be convenient to write

$$U^T \mathscr{V}_n = \tilde{\mathscr{V}}_n = \frac{1}{\sqrt{2^{M-1}}} \begin{bmatrix} x_n \\ \\ y_n \end{bmatrix}. \tag{110b}$$

Putting together the pieces just described in this section, the contrast between the mathematics of the exact theory and independence theory is as follows. The former problem is the following: solve for x_n where

$$\begin{bmatrix} x_{n+1} \\ \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \beta & \nu \\ \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x_n \\ \\ y_n \end{bmatrix} + \begin{bmatrix} \Phi \\ \\ \Psi \end{bmatrix}, \tag{114}$$

where x_0 is given, $y_0 = 0$. The latter problem is: Solve for x_n where

$$x_{n+1} = \beta x_n + \Phi, \tag{115}$$

x_0 is given. Note if ν and γ in (114) were zero, the solution to the two problems would be identical. Since ν and γ vanish when $\alpha = 0$, we may hope a perturbation approach will be useful for small α. More specifically, we treat

$$\begin{bmatrix} \beta - I & \nu \\ \\ \gamma & \delta - \mathscr{J} \end{bmatrix}$$

as a perturbation of (103), the matrix $\tilde{A}(\alpha)$ when $\alpha = 0$.

We begin by considering the eigenvalue problem for $\mathscr{A}(\alpha)$. When $\alpha = 0$, the eigenvalues of β are unity while these of δ are zero, and these eigenvalues vary continuously as α is increased. Consider solving for the large eigenvalues. In general, we have to solve

$$\begin{aligned} \beta x + \nu y &= \lambda x \\ \gamma x + \delta y &= \lambda y, \end{aligned} \tag{116}$$

where λ is one of these eigenvalues, presumed close to one. Since the eigenvalues of δ will be presumed smaller than λ, $(\lambda - \delta)^{-1}$ exists and we conclude from the second equation of (116) that

$$y = (\lambda - \delta)^{-1}\gamma x.$$

Substituting this into the first equation yields

$$\left[\beta + \nu \frac{1}{\lambda - \delta}\gamma\right] x = \lambda x. \tag{117}$$

Consistent with the perturbation spirit, we replace the λ (on the left) by 1 and δ by its value when $\alpha = 0$, namely, $\mathscr{J}$ [see (103)].

Thus the large λ's are (approximately) solutions to

$$\left[\beta + \nu \frac{1}{I - \mathscr{J}}\gamma\right] x = \lambda x \tag{118}$$

and the corresponding eigenvector to $\mathscr{A}(\alpha)$ is, approximately,†

$$\begin{bmatrix} x \\ \dfrac{1}{I - \mathscr{J}}\gamma x \end{bmatrix}. \tag{119}$$

Using these approximations and applying the spectral decomposition discussed in (72) to evaluate $\mathscr{A}^n(\alpha)$, it is now straightforward to show that the desired solution to (114) is, at least if we neglect the small eigenvalues.

$$x_n = \left[\beta + \nu \frac{1}{I - \mathscr{J}}\gamma\right]^n x_0 + \sum_{s=0}^{n-1} \left[\beta + \nu \frac{1}{I - \mathscr{J}}\gamma\right]^s \left[\Phi + \nu \frac{1}{I - \mathscr{J}}\Psi\right]. \tag{120}$$

† With the present representation, the perturbation theory has been painless. More formal and more thorough approaches to perturbation theory of matrices may be found in Refs. 2 and 3.

From (120) we conclude

$$\lim_{n\to\infty} \mathbf{x}_n = [\mathrm{I} - \beta - \nu \frac{1}{\mathrm{I} - \mathscr{J}} \gamma]^{-1}[\Phi + \nu \frac{1}{I - \mathscr{J}} \Psi]. \tag{121}$$

In fact, the steady-state error can also be computed exactly from (114) as

$$[I - \beta - \nu \frac{1}{I - \delta} \gamma]^{-1}[\Phi + \nu \frac{1}{1 - \delta} \Psi]. \tag{122}$$

Within the spirit of our approximations, (122) is consistent with (121). The neglect of the small eigenvalues is justified by the fact that their contribution will damp out quickly, and also that they operate in a subspace approximately orthogonal to the one we are interested in. Thus in (119) the "second half" of the large eigenvector is small because of the γ factor. The corresponding form for the "small" eigenvectors would have the first portion small.

We take (120) and (121) as our approximate solution. The terms

$$\nu \frac{1}{I - \mathscr{J}} \gamma \quad \text{and} \quad \nu \frac{1}{I - \mathscr{J}} \Psi \tag{123}$$

are higher order terms in the perturbation, and neglecting them we obtain†

$$x_n = \beta^n x_0 + \sum_{s=0}^{n-1} \beta^s \Phi \tag{124}$$

$$x_\infty = \frac{1}{I - \beta} \Phi, \tag{125}$$

exactly what independence theory would predict.

To examine further the key expression

$$\beta + \nu \frac{1}{I - \mathscr{J}} \gamma, \tag{126}$$

some more concrete expression for the $\nu\gamma$ type terms is needed. For example, consider an initial error matrix R_0. Then

$$R_1 = E[I - \alpha X_1 X_1^T] R_0 [I - \alpha X_1 X_1^T]. \tag{127}$$

This must correspond to β and so, as we already know,

$$\beta = E[I - \alpha \mathbf{X}_1 \mathbf{X}_1^T] \otimes [I - \alpha \mathbf{X}_1 \mathbf{X}_1^T]. \tag{128}$$

In general, then (neglecting the forcing terms), independence theory

† Noting that $(\lambda + \epsilon)^n \approx \lambda^n$ for $n = 0(1/\epsilon)$ but not for $n \to \infty$, we expect the approximation to break down after a while. This may very well happen only after the taps have, for practical purposes, converged to the desired solution.

can be written (letting $P_n = I - \alpha \mathbf{X}_n \mathbf{X}_n^T$)

$$R_{n+1} = EP_1 R_n P_1. \tag{129}$$

If we consider two iterations

$$R_2 = E[I - \alpha X_2 X_2^T][I - \alpha X_1 X_1^T] R_0 [I - \alpha X_1 X_1^T][I - \alpha X_2 X_2^T], \tag{130}$$

this corresponds, on squaring the matrix in (102), to $\beta^2 + \nu\gamma$. Thus $\nu\gamma$ corresponds to

$$E[P_2 P_1 R_0 P_1 P_2 - P_\infty P_1 R_0 P_1 P_\infty], \tag{131}$$

where P_∞ is simply a notation denoting that it (P_∞) is to be treated independently of P_1. The matrix R_0 is not statistical. The proper way to write (131) is†

$$\nu\gamma = E[(P_2 \otimes P_2)(P_1 \otimes P_1) - \beta^2]. \tag{132}$$

In general, it can be shown

$$\nu \frac{1}{I - \mathscr{J}} \gamma = \sum_{s=0}^{M-2} \nu \mathscr{J}^s \gamma = \sum_{s=2}^{M} [E(P_s \otimes P_s)(P_1 \otimes P_1) - \beta^2]. \tag{133}$$

Using (133) in (126) provides us with the next correction to the eigenvalues by way of (118).

Furthermore, (133) suggests a simplified "dynamics" for R_n, namely,

$$R_{n+1} = EP_1 R_n P_1 + E \sum_{s=1}^{M-1} [P_{1+s} P_1 R_{n-s} P_1 P_{1+s} - P_\infty P_1 R_{n-s} P_1 P_\infty]. \tag{134}$$

A general discussion of these correction terms seems out of the question. In fact, the expectations are not trivial to do. Instead, we resort again to the simple model of (15), where $A = I$, and $\mathscr{E}_{ex}^{(n)} = \operatorname{tr} R_n$, and set $N = 3$. For this case, we have been able to do the expectations and compute the eigenvalues of β and $\beta + \nu[1/(I - \mathscr{J})]\gamma$. The eigenvalues results are given in Table II. Certainly, in this case the perturbation philosophy seems well justified.

VII. CONCLUSIONS

We conclude (as explained above) that a perturbation analysis suggests that the difference between independence theory and one which takes into account the correlations between the "gradient"

† Using $(A \otimes B)(C \otimes D) = AC \otimes BD$, other forms are, of course, possible.

Table II—A comparison of the eigenvalues of β and its perturbation for a special situation

β	$\beta + \nu \frac{1}{I - \mathcal{J}} \gamma$
0.667	0.674
0.555	0.543
0.555	0.555
0.555	0.555
0.333	0.337
0.333	0.333
0.333	0.333
0.333	0.333
0.333	0.333

directions is slight. Our early worry was that the shifting property

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \rightarrow \begin{bmatrix} x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n+1} \end{bmatrix} \tag{135}$$

in going from one gradient direction to the next could cause trouble with independence theory. Any notion that this particular dependence must result in mathematics completely foreign to that of independence theory has been shown to be false. Independence theory is an inherent part of the exact description.

The situation in (135) does, however, have the rigorous property that the "new" component (x_{n+1}) is independent of the others. For real problems, this situation may well be violated in certain cases of severe intersymbol interference. Examining the $N = 1$ case leads us to propose the following criterion to measure this dependence. Namely, if, in the synchronous case, the received pulse $h(t)$ [see (1)] is normalized so that $\sum_{-\infty}^{\infty} h_n^2 = 1$, then we might expect

$$\sum_{s=1}^{\infty} \left(\sum_{l=-\infty}^{\infty} h_l h_{l+s} \right)^2 \ll 1$$

to be a good measure of independence for the new component.

Our effort has been a long and tedious one, and our attempts to pull insights from complicated equations have sometimes been nonrigorous and no doubt occasionally colored by the previous experimental results and simulation results of others.[1] Thus, while the ultimate justification of independence theory must remain empirical, we hope that our

efforts at least make mathematically plausible the successes of independence theory.

Finally, it is a pleasure to say that the present work has benefited from discussions with J. Salz, L. A. Shepp, N. J. A. Sloane, and H. S. Witsenhausen.

APPENDIX A

Evaluation of Some Averages

For the purposes of this appendix, we drop the superscript in (11), labeling things as if $n = 1$. For application to (31) we consider the average

$$E\mathbf{X}\mathbf{X}^T R\mathbf{X}\mathbf{X}^T \tag{136}$$

for an arbitrary $N \times N$ matrix R. Here (11) holds, and we are averaging over the binary variables in $\mathbf{a}$. Expanding (136) using (11) we have to do the key average

$$E\mathbf{a}\mathbf{a}^T Q\mathbf{a}\mathbf{a}^T \equiv EC, \tag{137}$$

where $Q = B^T RB$. Thus from (137),

$$\begin{aligned}(EC)_{ij} &= E \sum_{k,l} (\mathbf{a}\mathbf{a}^T)_{ik} Q_{kl} (\mathbf{a}\mathbf{a}^T)_{lj} \\ &= E \sum_{k,l} a_i a_k a_l a_j Q_{kl}. \end{aligned} \tag{138}$$

Using the fact that for independent binary variables

$$Ea_i a_k a_l a_j = \delta_{ik}\delta_{lj} + \delta_{il}\delta_{kj} + \delta_{ij}\delta_{kl} - 2\delta_{ik}\delta_{il}\delta_{ij} \tag{139}$$

we obtain, upon using (139) in (138)

$$(EC)_{ij} = Q_{ij} + Q_{ji} + (\text{tr } Q)\delta_{ij} - 2Q_{ii}\delta_{ij}. \tag{140}$$

In matrix notation, (140) becomes

$$EC = Q + Q^T + (\text{tr } Q)I - 2 \text{ diag } Q, \tag{141}$$

with the definition

$$(\text{diag } Q)_{ij} = (Q_{ii})\delta_{ij}. \tag{142}$$

Note that if the a_i were unit-variance Gaussian, the last term in (141) (diag Q) would not arise. It will be dropped because it is small in usual cases. Finally, multiplying (141) on the left by B and on the right by B^T we recover (136), obtaining (since Q is symmetric now)

$$E\mathbf{X}\mathbf{X}^T RXX^T = 2\,ARA + (\text{tr } RA)A - 2B(\text{diag } B^T RB)B^T. \tag{143}$$

Now we recall that all terms in (143) are multiplied by α^2. We would

neglect them all unless one can be large. In fact, (tr RA)A can be N times larger and hence this is the only term we need keep.

We move on to consider (32), rewritten as

$$-E\alpha(I - \alpha xx^T)\epsilon[\mathbf{c}^{*T}xx^T - a_{n+J}X^T]. \tag{144}$$

The term linear in α in (144) vanishes as a correspondence of (6), (7), and (9). One of the α^2 terms is

$$\alpha^2 EXX^T\epsilon\mathbf{c}^{*T}XX^T. \tag{145}$$

Evaluating (145) using (143), we check to see if the dominant term can be large. It is given by

$$A \text{ tr } \epsilon\mathbf{c}^{*T}A = A(\epsilon^T A\mathbf{c}^*). \tag{146}$$

If we introduce the $(M - 1)$ vector $\mathbf{u}$, having all zeros except a one in the $(1 + J)$ place, then

$$a_{n+J} = \mathbf{u}\cdot\mathbf{a} = u^T\mathbf{a}, \tag{147}$$

and the other α^2 term is proportional to

$$\alpha^2 EXX^T\epsilon\mathbf{u}^T\mathbf{a}X^T = \alpha^2 EB[\mathbf{a}\mathbf{a}^T(B^T\epsilon\mathbf{u}^T)\mathbf{a}\mathbf{a}^T]B^T. \tag{148}$$

Evaluating (148) using (138) and (141), we get

$$A(\epsilon^T Bu). \tag{149}$$

However, using (7) we readily verify $B\mathbf{u} = \mathbf{v}$, and a final use of (9) shows that the two dominant α^2 terms (146) and (149) cancel. The other terms are truly α^2 terms (as opposed to $\alpha^2 N$) and are neglected, leading us to replace (32) by zero.

We have introduced enough tricks now so that the reader may easily reproduce (33).

APPENDIX B

Definiteness of Solution to (34)

We give here an explicit demonstration that the solution to (34) retains its positive definite character. By induction on n, it is sufficient to show that $R^{(n+1)}$ is positive definite (≥ 0) if $R^{(n)}$ is.

We make repeated use that $R \geq 0$ if R is hermitian and $\phi^T R\phi \geq 0$ for any vector ϕ.

We recall $A \geq 0$ (and therefore hermitian) and hence $R^{(n+1)}$ is hermitian.

Rewrite the right member of (34) as

$$(I - \alpha A)R(I - \alpha A) + \alpha^2[A \text{ tr } AR - ARA] + \alpha^2\mathscr{E}^*A. \tag{150}$$

Each term in (150) is positive definite; the only nonobvious one is the

second. However, it may be rewritten as

$$\sqrt{A}\,[\text{tr}\,\sqrt{A}\,R\,\sqrt{A} - \sqrt{A}\,R\,\sqrt{A}]\,\sqrt{A}, \tag{151}$$

since tr AB = tr BA. The matrix $\sqrt{A}\,R\,\sqrt{A}$ is, of course, positive definite. Now observe that if $B \geq 0$ then tr $B - B \geq 0$. This concludes the proof.

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