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A Characterization of the Invariance of Positivity for Functional Differential Equations

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For systems of functional differential equations that can take into account finite or infinite delays, a complete characterization is given of the invariance of positivity in the sense that all solution components are positive whenever the initial condition function is positive. A related result concerning a comparison of the solutions of pairs of initial value problems is also given. One application of the results described concerns a model for synchronizing geographically separated oscillators and another is in the area of economics.

I. INTRODUCTION

Consider a system of functional differential equations of the form

$$\dot{x} = f(t, x_t), \qquad t \geq t_0, \qquad x_{t_0} = \phi, \tag{1}$$

in which x is a real n-vector valued function of t, $\dot{x}$ denotes dx/dt, ϕ is an initial condition function, and x_t denotes the function defined by $x_t(s) = x(t + s)$ for $s \leq 0$.* (When $f(t, x_t)$ depends only on t and $x_t(0)$, (1) reduces to a system of ordinary differential equations.)

The main purpose of this paper is to give a solution to the problem of determining conditions under which (under certain typically very reasonable conditions on f), $x(t)$ of (1) has components that are all positive for $t \geq t_0$ whenever ϕ is positive in, for example, the sense that $\phi(s)$ has positive components for $s \leq 0$. The problem arises in connection with the mathematical modeling and analysis of economic processes, and it comes up in several other areas as well. (An example concerning the synchronization of geographically separated oscillators is described in Section 2.6.) In some instances, the invariance of positivity in the sense described above is crucial, in that the lack of

* For background material concerning equations of the type (1), see, for example, Refs. 1 and 2.

positivity of a component of $x(t)$ for some t and positive ϕ means that the associated model is inappropriate.

Our main result, Theorem 1 of Section II, is concerned with the case in which f is continuous and locally Lipschitz. It provides an explicit and useful condition under which positivity is invariant, and it also asserts that positivity is invariant if and only if (1) preserves nonnegativity in the sense that $x(t)$ has nonnegative components for $t \geq t_0$ whenever ϕ is nonnegative.

The nonnegativity-preservation problem has been considered in Refs. 3, 4, and 5, and the relationship between Theorem 1 and the earlier material is indicated in Section 2.3.

A corollary of Theorem 1 is as follows. Suppose that (1) is a system of *ordinary* differential equations, and that f is continuous and satisfies a global Lipschitz condition (in the usual sense). Let $g(t, x)$ denote $f(t, x_t)$. Then positivity is invariant for (1), by which we mean invariant for each starting point t_0, if and only if for each i, $g_i(t_0, v) \geq 0$ for each t_0 and each real n-vector v such that $v_i = 0$ and $v_j \geq 0$ for $j \neq i$. Notice that for the special case in which $f(t, x_t) = Ax$, where A is an $n \times n$ matrix of real constants, our condition is equivalent to the requirement that the off-diagonal elements of A are nonnegative. The corresponding proposition concerning nonnegativity preservation for this case is well known.[6] Of some interest is the fact that the corollary described above becomes false if the Lipschitz hypothesis is replaced with the assumption that (1) has exactly one solution for each initial condition (see Section 2.3).

A result related to Theorem 1 that provides a necessary and sufficient condition for the invariance of positivity, or of nonnegativity, of the difference of the solutions of a pair of equations of the type (1) is given in Section 2.4. Specific applications of that result, as well as of Theorem 1, are described in Section 2.6.

II. CHARACTERIZATION OF THE INVARIANCE OF POSITIVITY

2.1 Preliminaries

We use the following notation and definitions. With n an arbitrary positive integer, R^n denotes the set of real n-vectors $v = (v_1, v_2, \cdots, v_n)$, $R^n_+ = \{v \in R^n: v_i \geq 0 \text{ for each } i\}$, and $|v| = \max_i |v_i|$ for $v \in R^n$. For u and v in R^n, the inequality $u \geq v$ ($u > v$) means that $u_i \geq v_i$ ($u_i > v_i$) for each i. The zero n-vector of R^n is denoted by θ.

We denote by C the Banach space of bounded continuous functions from $(-\infty, 0]$ to R^n, with norm given by

$$|w| = \sup_{t \in (-\infty, 0]} |w(t)|$$

for all $w \in C$.

The symbol T denotes any real interval of the form $[\alpha, \infty)$, (α, ∞), or $(-\infty, \infty)$, and t_0 is an element of T. For each $t \varepsilon T$ and each bounded continuous function w from $(-\infty, t]$ to R^n, w_t denotes the element of C defined by $w_t(s) = w(t + s)$ for $s \leq 0$.

Throughout Section II, f in (1) denotes a mapping of $T \times C$ into R^n. We say that f is *continuous in t* if $f(t, w_t)$ is a continuous function of t for $t \geq t_0$ whenever $t_0 \varepsilon T$ and w is a bounded continuous mapping of $(-\infty, \infty)$ into R^n, and we say that f is *locally Lipschitz* if for each $t_0 \varepsilon T$, each $\gamma \varepsilon [t_0, \infty)$, and each compact set B in R^n, there is a constant $\rho(t_0, \gamma, B)$ such that $|f(t, u) - f(t, v)| \leq \rho(t_0, \gamma, B)|u - v|$ for each $t \varepsilon [t_0, \gamma]$ and each u and v in C such that the range of u, and also of v, is contained in B.

A *solution* of (1) through a given $(t_0, \phi) \varepsilon T \times C$ means a continuous R^n-valued function x that is defined on $(-\infty, \infty)$, is differentiable on (t_0, ∞), and is such that (1) is satisfied (with the understanding that at $t = t_0$, $\dot{x}$ denotes the right-hand derivative).* As in the case of ordinary differential equations, if f is continuous in t and satisfies a uniform Lipschitz condition in the sense that f satisfies a local Lipschitz condition with $\rho(t_0, \gamma, B)$ independent of B, for each $(t_0, \phi) \varepsilon T \times C$ there is a unique solution of (1) through (t_0, ϕ) (see p. 409 of Ref. 2).

In the next section, we refer to the following hypothesis.

H.1: *There is a solution x of* (1) *through each* $(t_0, \phi) \varepsilon T \times C$, *and f is locally Lipschitz as well as continuous in t.* (*In particular, each solution of* (1) *is unique*.)

2.2 Our principal result

Under the assumption that H.1 holds, consider the following properties and condition.

Property 1 (*Invariance of Positivity, Version* 1): *For each* $(t_0, \phi) \varepsilon T \times C$ *such that* $\phi(0) > \theta$ *and* $\phi(s) \varepsilon R^n_+$ *for* $s \leq 0$, *we have* $x(t) > \theta$ *for* $t \geq t_0$.

Property 2 (*Invariance of Positivity, Version* 2): *For each* $(t_0, \phi) \varepsilon T \times C$ *such that* $\phi(s) > \theta$ *for* $s \leq 0$, *we have* $x(t) > \theta$ *for* $t \geq t_0$.

Property 3 (*Invariance of Nonnegativity*): *We have* $x(t) \geq \theta$ *for* $t \geq t_0$ *whenever* $(t_0, \phi) \varepsilon T \times C$ *with* $\phi(s) \varepsilon R^n_+$ *for* $s \leq 0$.

Condition 1: *For each i,* $f_i(t_0, \phi) \geq 0$ *whenever* $(t_0, \phi) \varepsilon T \times C$ *with* $\phi(s) \varepsilon R^n_+$ *for* $s \leq 0$ *and* $\phi_i(0) = 0$.

Theorem 1: *Let H*.1 *be satisfied. Then the following four statements are equivalent: Property* 1 *holds, Property* 2 *holds, Property* 3 *holds, and Condition* 1 *is met.*

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* It will become clear that our development can be extended at once to cover the case in which a solution need be defined on only an interval of the form $(-\infty, \beta)$ with $\beta > t_0$.

Proof: We first show that Property 3 and Condition 1 are equivalent.

Suppose that Condition 1 is satisfied, that $(t_0, \phi) \in T \times C$ is given, with $\phi(s) \in R^n_+$ for $s \leq 0$, and that $x(t) \geq \theta$ for $t \geq t_0$ is violated. Then there is a $t' > t_0$ and an index ℓ such that $x_\ell(t') < 0$. Since f is continuous in t and locally Lipschitz, the Bellman-Grownwall Lemma[7] and Theorem 3 of Ref. 2 can be used to show that, given any $\tau > t'$, there are an $\epsilon > 0$ and an R^n-valued function w defined on $(-\infty, \tau]$, and differentiable on (t_0, τ), such that $\dot{w}_i = f_i(t, w_t) + \epsilon$, $t \in [t_0, \tau)$, $i = 1, 2, \cdots, n$, with $w_{t_0} = \phi$ *and* $w_\ell(t') < 0$.* Let $I = \{i: w_i(t) < 0 \text{ for some } t \in (t_0, \tau)\}$, and for each $i \in I$, let $t_i = \inf\{t \in (t_0, \tau): w_i(t) < 0\}$. Choose k so that $t_k = \min\{t_i: i \in I\}$. We have $w_k(t_k) = 0$, $\dot{w}_k(t_k) \leq 0$, and $w_{t_k}(s) \in R^n_+$ for $s \leq 0$. Thus, $f_k(t_k, w_{t_k}) + \epsilon \leq 0$, which contradicts Condition 1. Therefore, we have Property 3 when Condition 1 is met. On the other hand, if Condition 1 is not satisfied, there is an index ℓ and a $(t_0, \phi) \in T \times C$ with $\phi(s) \in R^n_+$ for $s \leq 0$ and $\phi_\ell(0) = 0$ such that $f_\ell(t_0, \phi) < 0$. Since there is a solution x through (t_0, ϕ), and it clearly satisfies $x_\ell(t) < 0$ for $(t - t_0)$ positive and sufficiently small, we see that Property 3 implies that Condition 1 is met. This proves the equivalence of Property 3 and Condition 1.

It is clear that Property 1 implies Property 2. To see that Condition 1 is satisfied when Property 2 holds, suppose once more that Condition 1 is not met. Then, as in the paragraph above, there is a $(t_0, \phi) \in T \times C$ with $\phi(s) \in R^n_+$ for $s \leq 0$ such that x satisfies $x_\ell(t') < 0$ for some ℓ and $t' > t_0$. By H.1 and Lemma 2 of Ref. 2 (which is a result concerning the continuous dependence of the solution on the initial data), there is a $(t_0, \tilde{\phi}) \in T \times C$ such that $\tilde{\phi}(s) > \theta$ for $s \leq 0$ and such that the corresponding solution $\tilde{x}$ meets $\tilde{x}_\ell(t') < 0$. Therefore, Condition 1 is satisfied when Property 2 holds. To complete the proof of the theorem, we now show that Property 3 implies Property 1.

Assume that Property 3 holds and that Property 1 does not hold. Then there is a $(t_0, \phi) \in T \times C$ and a corresponding solution x such that $\phi(0) > \theta$, $\phi(s) \in R^n_+$ for $s \leq 0$, $x(t) \in R^n_+$ *for* $t \geq t_0$, and $x_\ell(t') = 0$ for some ℓ and $t' > t_0$. We assume without loss of generality that $t' = \inf\{t > t_0: x_i(t) = 0 \text{ for some } i\}$. Thus, $x(t) > \theta$ for $t \in [t_0, t')$.

Let

$$B = \{v \in R^n_+: v_i \leq \sup_{t \in (-\infty, t']} |x(t)|, i = 1, 2, \cdots, n\}.$$

From $x_\ell(t') = 0$ and the observation that

$$\int_{t''}^{t} \frac{\dot{x}_\ell(s)}{x_\ell(s)}\, ds = \ln[x_\ell(t)] - \ln[x_\ell(t'')],\ t \in [t'', t')$$

* At $t = t_0$, the unique solution with which Theorem 3 of Ref. 2 is concerned has a right-hand derivative equal to the value of the functional that corresponds here to f. (See the proof of Theorem 2 of Ref. 2.)

for any $t'' \varepsilon (t_0, t')$, we see that $\dot{x}_\ell(t)\, [x_\ell(t)]^{-1}$ is not bounded from below on (t_0, t'). Let $\sigma \varepsilon (t_0, t')$ be chosen so that $\dot{x}_\ell(\sigma) + \rho(t_0, t', B)\, x_\ell(\sigma) < 0$, in which ρ is the Lipschitz constant in Section 2.1.

Define $w(\sigma) \varepsilon C$ by $[w(\sigma)]_\ell(s) = \max[0, (x_\sigma)_\ell(s) - (x_\sigma)_\ell(0)]$ for $s \le 0$, and $[w(\sigma)]_i(s) = (x_\sigma)_i(s)$ for $s \le 0$ and $i \ne \ell$. It can be verified that $w(\sigma)(s) \varepsilon B$ for $s \le 0$, and that $|x_\sigma - w(\sigma)| \le x_\ell(\sigma)$.

Therefore,

$$\begin{aligned} f_\ell[\sigma, w(\sigma)] &= \dot{x}_\ell(\sigma) - f_\ell(\sigma, x_\sigma) + f_\ell[\sigma, w(\sigma)] \\ &\le \dot{x}_\ell(\sigma) + \rho(t_0, t', B)\, |x_\sigma - w(\sigma)| \\ &\le \dot{x}_\ell(\sigma) + \rho(t_0, t', B)\, x_\ell(\sigma). \end{aligned}$$

We have $f_\ell[\sigma, w(\sigma)] < 0$, $(\sigma, w(\sigma)) \varepsilon T \times C$, $w(\sigma)(s) \varepsilon R^n_+$ for $s \le 0$, and $[w(\sigma)]_\ell(0) = 0$, which, in view of the equivalence of Condition 1 and Property 3, contradicts our assumption that Property 3 holds. Thus Property 3 implies Property 1, which completes the proof of Theorem 1.

2.3 Notes

The Condition 1-implies-Property 3 assertion of the theorem becomes false if the hypothesis that f is locally Lipschitz is dropped and Property 3 is modified in the natural way so that it concerns *all* solutions that correspond to the indicated type of initial condition.[4]

The following example shows that the theorem becomes false if the Lipschitz hypothesis is replaced with the assumption that (1) has at most one solution for each $(t_0, \phi) \varepsilon T \times C$. Let $n = 1$, and let f be defined for all t by $f(t, x_t) = -(x(t))^{1/2}$ for $x(t) \ge 0$, and $f(t, x_t) = 0$ for $x(t) < 0$. Observe that f is continuous, and that a solution (in the usual sense) of $\dot{x} = f(t, x_t)$ for $t \ge 0$, $x(0) = x^0$, is given by $x(t) = x^0$ for $t \ge 0$ if $x^0 \le 0$, and $x(t) = ((x^0)^{1/2} - \tfrac{1}{2}t)^2$ for $t \varepsilon [0, 2(x^0)^{1/2}]$ with $x(t) = 0$ for $t > 2(x^0)^{1/2}$ if $x^0 > 0$. It can be verified that there are no other solutions, even though f is not locally Lipschitz. While here $x(t)$ is nonnegative for $t \ge 0$ whenever $x^0 \ge 0$, it is obviously not true that $x(t)$ is positive for all $t \ge 0$ whenever x^0 is positive.

Essentially, the fact that Condition 1 and Property 3 are equivalent for ordinary differential equations is proved in Ref. 4, and in Ref. 5 (and in the setting provided by the results in Ref. 2) that result is extended to cover the more general case. At the time Ref. 4 was written, this writer was unaware of Ref. 3, which contains a theorem (proved in a very different way) from which the result in Ref. 4 can be obtained. Our proof of the equivalence of Condition 1 and Property 3 is basically the same as the proof in Ref. 4 for the ordinary differential equations case. We did not omit the proof mainly because a modification of it is referred to in the next section. Also, for the case in which

(1) takes into account only finite delays (i.e., for equations of "retarded" type), a direct variation of the proof, using the continuous dependence result in, for example, Ref. 1, p. 41, shows that Condition 1 and Property 3 are equivalent without the Lipschitz hypothesis, provided that (1) has exactly one solution for each $(t_0, \phi) \varepsilon T \times C$.

Our proof of Theorem 1 shows also that, when H.1 is met, Condition 1 is necessary and sufficient that positivity is invariant in the sense that Property 3 holds *and* we have $x_l(t) > 0$ for $t \geq t_0$ whenever $(t_0, \phi) \varepsilon T \times C$ is such that $\phi(s) \varepsilon R^n_+$ for $s \leq 0$ and the index l is such that $\phi_l(0) > 0$.

2.4 The comparison theorem

The proof of Theorem 1 can be modified to establish a corresponding theorem concerning a comparison of the solutions of two initial value problems. To describe that result, let g be a function from $T \times C$ into R^n, and consider together with (1) the equation

$$\dot{y} = g(t, y_t), \qquad t \geq t_0, \qquad y_{t_0} = \psi, \tag{2}$$

as well as the following hypothesis, properties, and condition.

H.2: For each $(t_0, \phi) \varepsilon T \times C$, (1) has a solution x, and similarly, for each $(t_0, \psi) \varepsilon T \times C$, (2) has a solution y. The mappings f and g are continuous in t, and at least one of the mappings f or g is locally Lipschitz.

Property 4: For each $(t_0, \phi, \psi) \varepsilon T \times C \times C$ such that $\phi(0) > \psi(0)$ and $\phi(s) \geq \psi(s)$ for $s \leq 0$, we have $x(t) > y(t)$ for $t \geq t_0$ (i.e., we have $x(t) > y(t)$ for $t \geq t_0$ for any solution x of (1) through (t_0, ϕ) and any solution y of (2) through (t_0, ψ)).

Property 5: For each $(t_0, \phi, \psi) \varepsilon T \times C \times C$ such that $\phi(s) > \psi(s)$ for $s \leq 0$, we have $x(t) > y(t)$ for $t \geq t_0$.

Property 6: We have $x(t) \geq y(t)$ for $t \geq t_0$ whenever $(t_0, \phi, \psi) \varepsilon T \times C \times C$ such that $\phi(s) \geq \psi(s)$ for $s \leq 0$.

Condition 2: For each i, $f_i(t_0, \phi) \geq g_i(t_0, \psi)$ whenever $(t_0, \phi, \psi) \varepsilon T \times C \times C$ with $\phi_i(0) = \psi_i(0)$ and $\phi(s) \geq \psi(s)$ for $s \leq 0$.

Our result is the following.

Theorem 2: If H.2 is met, then Property 4, Property 5, Property 6, and Condition 2 are equivalent to one another.

Proof: The proof is similar to the one given of Theorem 1. In fact, straightforward modifications show that Property 6 and Condition 2 are equivalent, and that Property 5 implies Condition 2. Since it is clear that Property 4 implies Property 5, it therefore suffices to use the equivalence of Condition 2 and Property 6 to prove that Property 6 implies Property 4. We do that as follows.

Assume that Property 6 holds, but that Property 4 fails to hold. Thus there are $(t_0, \phi, \psi) \in T \times C \times C$, a number $t' > t_0$, and an index ℓ such that $\phi(0) > \psi(0)$, $\phi(s) \geq \psi(s)$ for $s \leq 0$, $x_\ell(t') = y_\ell(t')$, and $x(t) > y(t)$ for $t \in [t_0, t')$. Let

$$B = \{v \in R^n : |v| \leq \sup[\max_{t \in (-\infty, t']} (|x(t)|, |y(t)|)]\}.$$

Assume for the moment that f is locally Lipschitz. Using the fact that $[\dot{x}_\ell(t) - \dot{y}_\ell(t)][x_\ell(t) - y_\ell(t)]^{-1}$ is not bounded from below on (t_0, t') (see the proof of Theorem 1), choose $\sigma \in (t_0, t')$ so that $\dot{x}_\ell(\sigma) - \dot{y}_\ell(\sigma) + \rho(t_0, t', B)\,[x_\ell(\sigma) - y_\ell(\sigma)] < 0$.

Define $u(\sigma) \in C$ by the conditions $[u(\sigma)]_\ell(s) = \max[(y_\sigma)_\ell(s), (x_\sigma)_\ell(s) - (x_\sigma)_\ell(0) + (y_\sigma)_\ell(0)]$, $s \leq 0$, and $[u(\sigma)]_i(s) = (x_\sigma)_i(s)$, $s \leq 0$, $i \neq \ell$. We have $[u(\sigma)]_\ell(0) = (y_\sigma)_\ell(0)$, and $y_\sigma(s) \leq u(\sigma)(s) \leq x_\sigma(s)$ for $s \leq 0$. In particular, $u(\sigma)(s) \in B$ for $s \leq 0$. Also, if $(x_\sigma)_\ell(s) - (x_\sigma)_\ell(0) + (y_\sigma)_\ell(0) \geq (y_\sigma)_\ell(s)$, then $(x_\sigma)_\ell(s) - [u(\sigma)]_\ell(s) = (x_\sigma)_\ell(0) - (y_\sigma)_\ell(0)$. On the other hand, if $(x_\sigma)_\ell(s) - (x_\sigma)_\ell(0) + (y_\sigma)_\ell(0) < (y_\sigma)_\ell(s)$, then $(x_\sigma)_\ell(s) - (y_\sigma)_\ell(s) < (x_\sigma)_\ell(0) - (y_\sigma)_\ell(0)$. Consequently, $|x_\sigma - u(\sigma)| \leq (x_\sigma)_\ell(0) - (y_\sigma)_\ell(0)$. Therefore,

$$f_\ell[\sigma, u(\sigma)] - g_\ell(\sigma, y_\sigma) = \dot{x}_\ell(\sigma) - \dot{y}_\ell(\sigma) + f_\ell[\sigma, u(\sigma)] - f_\ell(\sigma, x_\sigma)$$

$$\leq \dot{x}_\ell(\sigma) - \dot{y}_\ell(\sigma) + \rho(t_0, t', B)[x_\ell(\sigma) - y_\ell(\sigma)],$$

which shows that $f_\ell[\sigma, u(\sigma)] - g_\ell(\sigma, y_\sigma) < 0$. This contradicts Condition 2 and hence Property 6. A similar contradiction can be obtained when g rather than f is locally Lipschitz. (In the analogous argument, the function $v(\sigma) \in C$ that plays the role of $u(\sigma)$ is defined by $[v(\sigma)]_\ell(s) = \min[(x_\sigma)_\ell(s), (y_\sigma)_\ell(s) - (y_\sigma)_\ell(0) + (x_\sigma)_\ell(0)]$ for $s \leq 0$, and $[v(\sigma)]_i(s) = (y_\sigma)_i(s)$ for $s \leq 0$ and $i \neq \ell$.) This shows that Property 6 implies Property 4, and it completes the proof.

2.5 Comments

Since Property 6 does not hold when $f = g$ and (1) has more than one solution through some $(t_0, \phi) \in T \times C$, we see that the Condition 2-implies-Property 6 part* of Theorem 2 becomes false if the hypothesis that at least one of the functions f or g is locally Lipschitz is omitted. The example given in Section 2.3 shows that the theorem becomes false even if the hypothesis is replaced with the assumption that (1) and (2) have at most one solution for each $(t_0, \phi) \in T \times C$ and each $(t_0, \psi) \in T \times C$, respectively. On the other hand, the equivalence of Condition 2 and Property 6 holds for equations of retarded type when the Lipschitz hypothesis is replaced with the assumption of uniqueness of solutions for at least one of the equations (1) and (2).†

* For ordinary differential equations with $f = g$, this part of Theorem 2 is along the lines of a well-known result (Ref. 8).

† See the corresponding comment in Section 2.3.

The proof of Theorem 2 described in the preceding section can be used to verify that C in Theorem 2 can be replaced with the set of continuous bounded functions from $(-\infty, 0]$ to D, where D is any open rectangular interval of R^n, provided that by a solution of (1) or (2) is meant a solution whose values are contained in D for $t \geq t_0$. (In this connection, note that $u(\sigma)$ of the proof of Theorem 2 satisfies $u(\sigma)(s) \in D$ for $s \leq 0$ whenever $D \subset R^n$ is a rectangular interval and both $x_\sigma(s)$ and $y_\sigma(s)$ are contained in D for $s \leq 0$.) Of course, the case in which $D = \{v \in R^n: v > \theta\}$ is of particular interest.

2.6 Applications

There are many applications of Theorems 1 and 2. As a simple example for the purpose of illustration, consider the delay-differential equations

$$\dot{x}_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{n} h_{ij}(t)[x_j(t - \tau_{ij}) - x_i(t)], \; t \geq 0$$

$$i = 1, 2, \cdots, n, \quad (3)$$

which arise[9] in the study of models of systems for synchronizing geographically separated oscillators. In (3), each τ_{ij} is a nonnegative constant, each h_{ij} is nonnegative, continuous, and bounded on $[0, \infty)$, $x_i(t)$ denotes the frequency of the ith oscillator, and the h_{ij} can depend on x as well as on certain fixed nonlinear functions. Under certain reasonable hypotheses concerning the h_{ij} (see Ref. 9), given a continuous $x(t)$ for $t \in \tau$, where $\tau = [-\max_{j \neq i} \tau_{ij}, 0]$, there is a constant ρ such that for each i, $x_i(t) \to \rho$ as $t \to \infty$. Assume here that there is such a ρ for each initial-condition function.

Theorem 1 shows that each $x_i(t)$ in (3) is positive for $t \geq 0$ whenever $x(t) > \theta$ for $t \in \tau$.* (The nature of the dependence of the h_{ij} on x is not of consequence at this point. If it were not true that $x(t) > \theta$ for $t \geq 0$ whenever $x(t) > \theta$ for $t \in \tau$, we would have a contradiction to the theorem.) Assuming now that the h_{ij} are independent of x, it follows from Theorem 2 that, for example, ρ is either increased or unchanged when $x(t)$ for $t \in \tau$ is replaced with any continuous $\bar{x}(t)$ for which $\bar{x}(t) \geq x(t)$ for $t \in \tau$.

Two related observations concern the equations

$$\dot{x}_i(t) = -b_{i0}[x_i(t)] + \sum_{\substack{j=1 \\ j \neq i}}^{n} b_{ij}[x_j(t - \tau_{ij})] + u_i(t), \; t \geq 0$$

$$i = 1, 2, \cdots, n \quad (4)$$

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* This proposition is a special case of Lemma 1 of Ref. 9 whose proof is very different.

of a model of a compartmental system with delays,[10] in which the b_{i0} and the b_{ij} for $i \neq j$ are locally Lipschitz monotone-nondecreasing functions such that $b_{i0}(0) = b_{ij}(0) = 0$, each u_i is continuous and satisfies $u_i(t) \geq 0$ for $t \geq 0$, and, as in the preceding example, the τ_{ij} are nonnegative constants. In (4), $x_i(t)$ denotes the amount of material in the ith compartment.

From Theorem 2, we see that if x^a is a solution of (4) corresponding to $(u_1, u_2, \cdots, u_n) = u^a$ and $x(t) = x^a(t)$ for $t \in \tau$, where again $\tau = [-\max_{i \neq j} \tau_{ij}, 0]$, and similarly with regard to x^b and u^b, then we have $x^a(t) \geq (>)\ x^b(t)$ for $t \geq 0$ when $u^a(t) \geq u^b(t)$ for $t \geq 0$ and $x^a(t) \geq (>)\ x^b(t)$ for $t \in \tau$. The "$\geq$ part" of this proposition was given in Ref. 10. From Theorem 1, it is clear that we have $x(t) > \theta$ for $t \geq 0$ whenever $x(t) > \theta$ for $t \in \tau$, which does not seem to have been proved earlier, even for the case in which $\tau_{ij} = 0$ for all $i \neq j$.*

Consider now the case in which f in (1) is given by $f_i(t, x_t) = h_i(x_t)$ for each i, where each h_i is a functional on C with the property that $h_i(u) \geq h_i(v)$ for all u and v in C such that $u_i(s) = v_i(s)$ and $u(s) \geq v(s)$ for $s \leq 0$. Functions f of this form are generalizations of time-invariant quasimonotone (or Wazewski-type) functions[11] which are of interest in several areas, including economics. In economics applications, the $x_i(t)$ often denote prices (see, for example, Refs. 12 and 13). Observe that (4) is of the form considered here when the u_i are independent of t.

Assuming that H.1 is met, Theorem 1 provides the following simple necessary and sufficient condition for the invariance of positivity in the sense of Property 1 or Property 2.

For each i, $h_i(w) \geq 0$ for each w in C such that $w_i(0) = 0$, $w_i(s) \geq 0$ for $s \leq 0$, and $w_j(s) = 0$ for $s \leq 0$ and $j \neq i$.

REFERENCES

1. J. Hale, *Theory of Functional Differential Equations*, New York: Springer Verlag, 1977.
2. R. D. Driver, "Existence and Stability of Solutions of a Delay Differential System," Arch. Rational Mech. Anal., *10* (1962), pp. 401–426.
3. M. Nagumo, "Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen," Proc. Phys.-Math. Soc. Japan, Ser. 3, *24* (1942), pp. 550–559.
4. I. W. Sandberg, "A Nonnegativity-Preservation Property Associated with Certain Systems of Nonlinear Differential Equations," Proc. Int. Conf. Syst. Man. Cyber., Dallas, Texas, 1974, pp. 230–233.
5. G. Seifert, "Positively Invariant Closed Sets for Systems of Delay Differential Equations," J. Differential Equations, *22* (1976), pp. 292–304.
6. R. Bellman, *Introduction to Matrix Analysis*, New York: McGraw-Hill, 1970, p. 176.
7. V. V. Nemytskii and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton; Princeton U. Press, 1960.
8. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. I, New York: Academic Press, 1969, pp. 28–29.

* The monotonicity of the b_{i0} and b_{ij} is not needed for this result. It suffices that $b_{ij}(\alpha) \geq 0$ for $\alpha > 0$ and $i \neq j$.

9. I. W. Sandberg, "Some Properties of a Nonlinear Model of a System for Synchronizing Digital Transmission Networks," B.S.T.J., *48*, No. 9 (November 1969), pp. 2975–2997.
10. R. M. Lewis and B. D. O. Anderson, "Insensitivity of a Class of Non-linear Compartmental Systems to the Introduction of Arbitrary Time Delays," Preprint, February 1979.
11. A. N. Michel and R. K. Miller, *Qualitative Analysis of Large Scale Dynamical Systems*, New York: Academic Press, 1977, p. 59.
12. K. J. Arrow and L. Hurwicz, "Competitive Stability Under Weak Gross Substitutability: The 'Euclidean Distance' Approach," Internat. Econ. Rev., *1* (1960), pp. 38–49.
13. J. G. Kemeny and J. L. Snell, *Mathematical Models in the Social Sciences*, Cambridge, Mass.; MIT Press, 1972, pp. 37–38.