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Critical Section Methods for Loop Plant Allocation

By A. E. GIBSON and D. B. LUBER

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Allocation is the planning of current and future commitments of spare feeder capacity in the loop plant network. The purpose of allocation is to make the most efficient use of existing and future feeder facilities to reduce network operating costs and to defer capacity expansion. In this paper, new heuristic allocation algorithms are described that can be applied to multigauge feeder routes with complex topologies. Algorithms are described for both routes with high growth in customer demand and for routes with substantial customer movement but with little net growth. A combined algorithm that can be applied to routes with general growth characteristics is also described. A detailed application example is also included.

1. INTRODUCTION

The purpose of allocation in the loop plant network is to use existing and future facilities efficiently to reduce the cost of providing customer service and to defer expanding the capacity of the network (with new feeder facilities).

Allocation procedures for the loop plant network were originally described by Marsh.[1] These procedures were developed specifically for feeder routes experiencing significant annual customer growth. The objective was to improve the utilization of existing facilities to defer placing new cable facilities. This is accomplished by apportioning space feeder pairs along a route in proportion to forecasted growth rates.

Not all routes, however, have high growth, Many, particularly in large urban centers, are characterized by considerable customer movement with little or no net growth. Customers move from one location to another with no effect on total demand. On routes of this type, the costs associated with inward and outward movement predominate. Models of network operating costs associated with customer movement have been described by Koontz[2] and Freedman.[3] When there is little net growth, the objective is to allocate the available facilities to minimize total operating costs. A polynomial-bound allocation algorithm for low-growth routes has been described by Elken et al.[4]

Both approaches to loop feeder allocation assume a network model like the one in Fig. 1. In this model, the loop network is divided into straight line feeder routes emanating from the central office (CO) that feed M units of distribution plant (and associated geography) called allocation areas. Each allocation area, a_i, is assumed to be connected to the feeder route at a single point via lateral cable.

In this model, every feeder route is also subdivided into N individual segments called feeder sections. Each section, fs_j, is assumed to have a uniform cross-sectional capacity, s_j (number of pairs). Each is studied as a separate entity, independent of the rest of the route, when considering capacity expansion alternatives.[5]

The point of connection between a lateral cable and a feeder section is assumed to occur at the far field (as opposed to central office) end of the section. The current and projected demand for cable facilities in allocation area a_i, $w_i(t)$, forms the "load" on its associated feeder section (an allocation area will never be connected to more than one feeder section). The total demand for facilities within a feeder section will be the sum of the load on that section plus the load on all sections on the route beyond it.

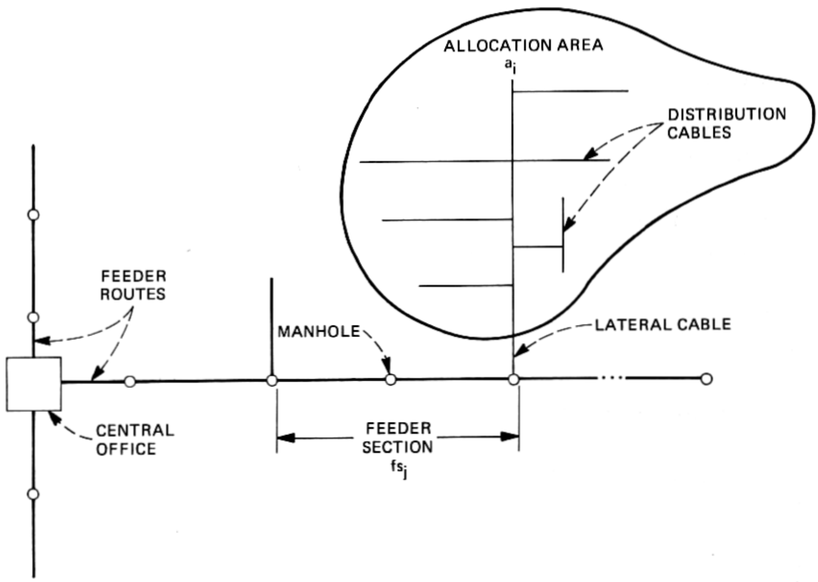

Fig. 1—Loop network model.

For many routes, this model is overly simplistic. Most routes do not consist of a single linear path, but rather of a number of different paths and branches. A typical example of a real route is in Fig. 2. We will use this route for the example presented in Section IV of this paper.

More important than topology is the fact that the model of Fig. 1 ignores the transmission complexities of most feeder routes. In a typical feeder cross section, one can find as many as four different gauges of telephone cable. These contain, in order of increasing resistivity, wire of 19, 22, 24, or 26 gauge. An allocation area that can be satisfactorily fed by a given minimum coarse gauge cable cannot be fed by finer gauge cables. This implies that on most routes it is not enough just to allocate bulk feeder facilities; on most routes, facilities must be allocated by gauge.

In Section II of this paper we review the algorithms described in Refs. 1 and 4. These are then extended for multigauge, complex routes in Section III. We then show how the two algorithms can be combined into a single heuristic which can be applied to routes with both high and low growth components. A comprehensive example is presented in Section IV.

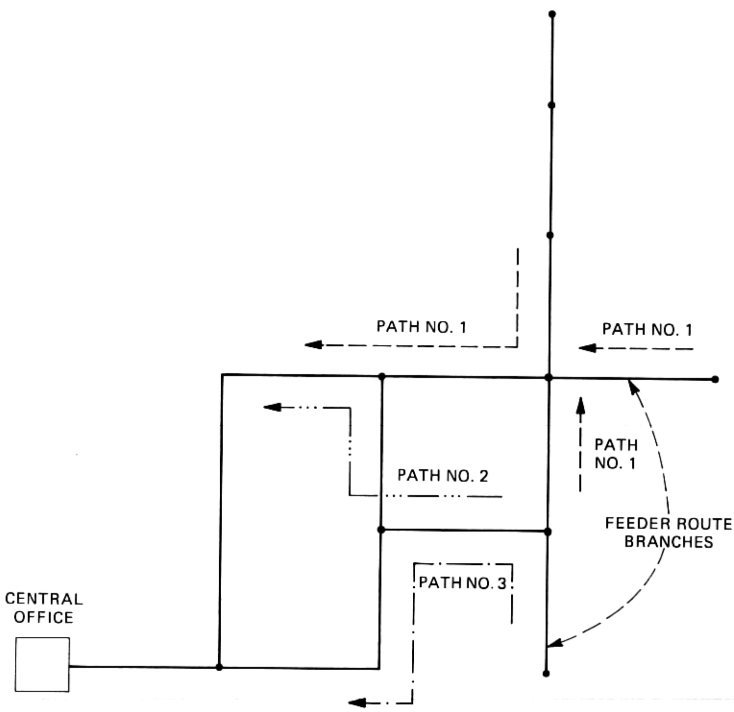

Fig. 2—Example of a route with complex topology.

II. ALLOCATION CONCEPTS

In this section, we review the basic concepts of feeder allocation. First, we describe allocation for routes with growing customer demand, and then allocation for slowly growing or nongrowing routes.

2.1 Allocation on growing routes

On growing routes, the goal is to defer the need to place new facilities. Facilities will be required first in certain "critical sections"—those sections with the shortest projected lifetimes—on the route. Section fs_3 in Fig. 3 is an example of a critical section. The section has 30 spare pairs which must satisfy customer demand that is growing at a rate of 30 pairs per year (20 pairs per year in a_4 + 10 pairs per year in a_3). The maximum possible lifetime for this section is one year. Since one year is the shortest lifetime on the route, fs_3 is designated the first critical section on the route.

Now in order for fs_3 to actually last one year, the 30 spare pairs must be distributed such that 10 are connected to the lateral feeding a_3 and 20 are connected to pairs in fs_4 which are connected to the lateral feeding a_4. That is, the 30 pairs must be allocated to the two allocation areas such that each allocation area will last one year—the lifetime of *their* most critical section.

Although fs_4 has the next shortest lifetime on this route, it is not considered a critical section. Rather, we define fs_1 as the nextmost, or second, critical section on the route. The assumption is that the relief of the first critical section, fs_3, will impact on the distribution of facilities to all sections beyond it. Therefore it is not really necessary to consider the allocation of facilities in fs_4 until the time fs_3 is relieved. On the other hand, relief of fs_3 will not impact on the distribution of facilities between it and the co. Thus we can profitably consider allocation possibilities that would defer relief in fs_1 at this time.

The facilities in fs_1 will be allocated to *all* the allocation areas on the route such that each will have enough to last as long as the section, i.e., two years. However, some pairs allocated to allocation areas a_3

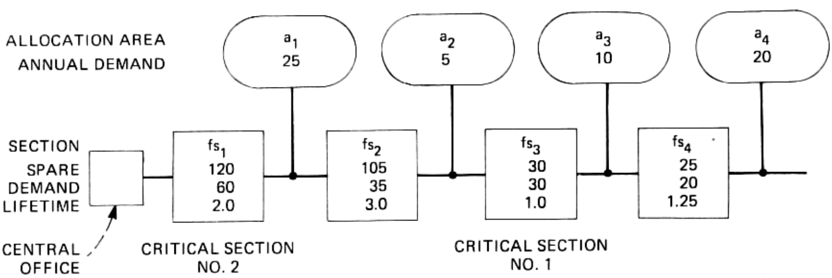

Fig. 3—Allocation example.

and a_4 cannot actually be connected to the laterals feeding the allocation area, since there are only enough pairs in fs_3 to last one year. The additional pairs—the pairs needed to satisfy the second year's growth in these allocation areas—will be held in reserve in fs_2. We will refer to these pairs as being allocated to fs_2. They will be used to energize the new facilities placed in fs_3 (to serve a_3 and a_4) at the time that section is relieved.

2.2 Allocation on slowly growing routes

On routes where there is little net growth, the first critical section may have a very long lifetime. If there is no net growth, there might not even be a critical section in the sense we have just described. In such cases, the present worth of relief costs will be very small (or even nonexistent). For routes of this type, the question of deferring capacity expansion is meaningless.

This does not imply that allocation cannot profitably be applied to slow-growing routes. Although there is little growth, there can still be significant customer movement. This movement generates costs. A significant component of these costs are the costs of the network rearrangements necessary to get a spare pair from where it is presently available to where it is required in the distribution network (not all pairs are accessible to every location).

Obviously, the more spare pairs in a particular allocation area, the less likely a rearrangement will be required to provide service. This probability of needing a rearrangement at time t, $(P_r(t))$, is generally modeled by a function such as (see Krone[6]):

$$P_r(t) = (w_i(t)/x_i)^{\lambda_i}, \tag{1}$$

where

- $w_i(t)$ is the number of pairs in a_i assigned to a customer at time t,
- x_i is the number of feeder pairs available for assignment in a_i at that time (including in-service and vacant pairs), and
- λ_i is a constant parameter representing the general accessibility of pairs in a_i (typically, values are around 10).

The expected cost in an allocation area would be equal to the probability that a customer who requests service requires a rearrangement multiplied by the number of customers requesting service multiplied by the cost resulting from a blocked service request.

A good allocation of the available feeder facilities would be one that would minimize the present worth of this cost over a given interval of time, say, 0 to T. For allocation area a_i, this cost function can be expressed as

$$g_i(x_i^1) = (x_i^0\alpha_i)^{\lambda_i+1}(x_i^1)^{-\lambda_i} \tag{2}$$

where x_i^1 denotes the number of feeder cable pairs to be allocated to a_i and x_i^0 equals the number of pairs currently allocated to a_i. The coefficient α_i is defined as

$$\alpha_i = \left(\gamma_i \int_0^T e^{-rt} \left[\frac{w_i(t)}{w_i(0)} \right]^{\lambda_i} dt \right)^{1/(\lambda_i+1)}, \tag{3}$$

where

$\gamma_i =$ the current annual operating costs per pair allocated to a_i
$r =$ the discount rate.

Thus, the objective of an allocation algorithm for a low-growth route would be to find the solution vector $(x_1^{1*}, x_2^{1*}, \ldots, x_M^{1*})$ which minimizes

$$\sum_{i=1}^{M} g_i(x_i^1), \tag{4}$$

subject to the constraints

$$\sum_{i \in I(j)} x_i^1 \leq s_j, j = 1, 2, \cdots N,$$

where s_j is the number of pairs available for service in fs_j and $I(j)$ is the set of indices of allocation areas fed by or through fs_j. Since no (section) relief is assumed to be required, there will be no section allocations; only allocations to allocation areas will be developed.

The optimal solution to this problem is derived in Ref. 4. As in the previous case, this solution also involves the identification of critical sections, only now a critical section is one that has the fewest facilities s_j with respect to the requirements $w_i(t)$ over the interval $(0, T)$.

III. ALGORITHMS FOR MULTIGAUGE, COMPLEX ROUTES

For this discussion we modify the route model introduced in the first section. First we define the capacity within each section in terms of the pairs available within the section, by gauge. And, second, we identify the demand for facilities in each section, by gauge, for each allocation area.

To determine section requirements, the paths between the CO and the individual allocation areas are identified. If a route is thought of as a graph with each section an arc and each point of connection as a node, then a path will be any acyclic graph in the route that terminates at the CO (node). We will now assume that each allocation area is fed via a single path, p. If a single allocation area is fed by more than one path, we assume that it can be subdivided into the appropriate number of smaller suballocation areas. After all paths have been identified and the distances along them computed, it is an easy matter to determine the gauge requirements for each allocation area in each section using standard resistance design concepts.[7]

We store the relationship between allocation areas and section requirements by the set of allocation area indices $I(j, l)$ where $i \in I(j, l)$ implies that a_i requires pairs of gauge $l(l = 26, 24, 22, 19)$ or *coarser* in fs_j. Note that the demand for facilities in a section can be satisfied by pairs of the required gauge or any gauge coarser. We also define a capacity matrix $S_{n\times4}$ where the elements $s_{j,l}$ equal the number of available pairs of gauge *l or coarser* in fs_j.

3.2 Allocation on routes with increasing customer demand

We now show how the concepts described in Section 2.1 can be extended into an allocation algorithm for multigauge complex routes.

First the projected requirements for cable facilities in each section are accumulated by allocation area, *by gauge*. Critical sections are then identified by section and gauge. The first critical section on the route will be the section in which the first (in time) facility shortage will occur. The gauge in which the shortage will occur is identified as the "problem" gauge. The second critical section on the route will be the section that will have the next shortage and either is not fed by the first critical section or has a problem gauge that is finer than the problem gauge of the first critical section. Subsequent critical sections are defined in a similar manner.

After all critical sections have been identified, each allocation area is associated with its "most" critical section: the critical section with the earliest projected shortage time in which the allocation area has requirements (in the problem or coarser gauge). This association of allocation areas and critical sections will be used to compute the pair allocations for the allocation areas. These allocations will equal the number of pairs required to satisfy the growth requirements in the allocation areas up to the shortage time of their most critical section.

In addition to the most critical section for each allocation area, all other critical sections (with later shortage times) for which an allocation area has facility requirements are also identified. These associations are used to compute the number of pairs that should be held in reserve to energize future relief, i.e., to compute the section allocations.

3.2 Identification of critical sections

The first step in the algorithm is to accumulate facility requirements for each section and to identify future section shortages. If $w_i(t)$ equals the projected demand for facilities in a_i at time t and f_j is the fill-at-relief factor for fs_j (typically, f_j will be somewhere between 0.8 and 0.9), then a shortage is said to occur in that section, in gauge l, at time t if the inequality

$$\sum_{i \in I(j,l)} w_i(t) < f_j s_{j,l} \tag{5}$$

is first upset at time t. Note that this implies that, at time t, fs_j will not have sufficient facilities to satisfy additional growth in gauge l *or coarser*. All excess coarse gauge pairs are assumed to have been used to fill the need for facilities in the problem gauge. Thus, if a section is determined to be a critical section in gauge l, then it will be a critical section for all allocation areas with requirements in the section of gauge l or coarser. If the above inequality is first upset for two different values of l at the same time, the finer of the two gauges should be considered the problem gauge for that section.

After the shortage time(s) and gauge(s) have been determined for each section on a route, the next step is to identify and rank the critical sections from $k = 1$ to K. The first critical section on the route will be the one with the first shortage time. If two or more sections have the same shortage time and gauge and are in the same path, then the one nearer the co should be selected. The second critical section will be the section with the next (or same) time to exhaust on the route which either is not fed by the first critical section or has a shortage in the same or a finer gauge. If the section with the next time to exhaust is fed by the first critical section and the problem gauge is the same, then there must be at least one gauge change (to the problem gauge) between this section and the first, as shown in Fig. 4, for it to be the second critical section. If there is not, then it is not a critical section.

After the first two critical sections are identified, additional critical sections should be identified using the above criteria (substitute "any

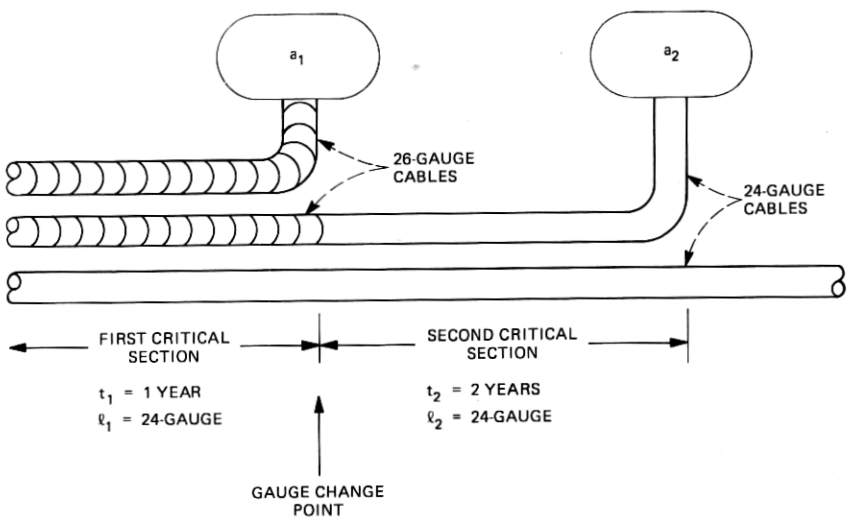

Fig. 4—Example of a second critical section fed by a first critical section.

other critical section" for "first critical section") until the section adjacent to the CO is identified as being a critical section in the finest available gauge.

3.3 Association of allocation areas with critical sections

The next step is to associate each allocation area with its most critical section. We start with the first critical section on the route; assume this section to be section fs_j^1 with problem gauge l^1. This will be the firstmost critical section for all allocation areas a_i such that $i \in I(j^1, l^1)$. These allocation areas are included in the set $A(1, 1)$.

After all the elements of $A(1, 1)$ have been identified, the allocation areas that have facility requirements in the problem gauge or coarser in the second critical section are identified. As they are identified, these allocation areas are assigned to one of two sets $A(2, 1)$ or $A(2, 2)$.

The first set, $A(2, 1)$, will contain all allocation areas, a_i, such that $i \in I(j^2, l^2)$ and $i \notin I(j^1, l^1)$, where fs_j^2 is the second critical section with problem gauge l^2. For these allocation areas, fs_j^2 will be their most critical section.

The second set, $A(2, 2)$, will contain all allocation areas a_i such that $i \in I(j^2, l^2)$ that are not elements of $A(2, 1)$. For these allocation areas, fs_j^2 will be their second most critical section.

The preceding process is repeated for the entire set of critical sections on the route. For each critical section fs_j^k, the elements of the sets $A(k, 1)$ and $A(k, 2)$ will be identified in turn.

3.3.1 Computation of individual allocations

The next step is to compute the allocation area allocations for each critical section. In order for a critical section, fs_j^k, to actually last until its shortage time t_k, every allocation area in the set $A(k, 1)$ must have sufficient facilities to satisfy demand until t_k. Thus we compute the allocation x_i^1 for each allocation area $a_i \in A(k, 1)$ where

$$x_i^1 = w_i(t_k)/f_k \tag{6}$$

and f_k is the fill at relief factor for fs_j^k,

In addition to the above, a second set of allocations are computed for all but the first critical section. These allocations are denoted as x_i^n and are given by

$$x_i^n = w_i(t_k)/f_k, \tag{7}$$

where

$$a_i \in A(k, 2).$$

The ordering, n, represents the number of allocations determined for a_i. Thus x_i^n is the allocation associated with the nth most critical section for a_i. We can interpret x_i^n as the number of pairs that would be allocated to a_i if relief were immediately provided to its $n-1$ most critical sections.

Together, the two sets of allocations form the complete set of allocations associated with the kth critical section. This set is denoted as the set $X(k)$. The difference between x_i^n and x_i^{n-1} represents the number of pairs that should be held in reserve in the section feeding the $(n-1)$th most critical section for a_i. When this section, assume it to be the kth critical section (i.e., $x_i^{n-1} \in X(k)$), is relieved, these pairs will be used to energize part of the relief cable. They will provide new spare facilities for a_i, and will be a_i's portion of the total number y_k that will be held allocated to this section.

The final step in the algorithm will be to compute the section allocations

$$y_k = \sum_{x_i^{n-1} \in X(k)} x_i^n - x_i^{n-1}. \tag{8}$$

These are computed after all the $X(k)$ have been first determined.

3.4 Allocation on routes with nonincreasing customer demand

On multigauge low growth routes, we want to minimize

$$\sum_{i=1}^{M} g_i(x_i^1) \tag{9}$$

subject to the constraints

$$\sum_{i \in I(j,l)} x_i^1 \le s_{j,l}; \quad \begin{matrix} j = 1, 2, \ldots, N \\ l = 26, 24, 22, 19. \end{matrix}$$

The basic solution strategy for this problem is the same as the strategy used for the growth case.

The first step will be to identify the first critical section on the route. In this case, the criterion for identifying critical sections will be a measure of the impact that congestion has on operating costs.

Given that the first critical section is identified, the allocations to the affected allocation areas are set. They are computed as part of the identification process and then removed from the problem. Since low growth implies that there will not be any future relief pairs to energize, only one allocation is determined for each allocation area. This process is then repeated until the section adjacent to the CO is identified as a critical section in the finest available gauge.

This procedure follows the same steps as the simple route, single-gauge algorithm presented in Ref. 4. That algorithm was shown to be optimal. Unfortunately, we cannot make the same statement for this one. However, experience with the algorithm has shown that it does provide good feasible solutions.

3.4.1 Identification of critical sections

The first critical section on a route is determined by solving the following problem:

Minimize
$$\sum_{i \in I(j,l)} g_i(x_i^1)$$

subject to
$$\sum_{i \in I(j,l)} x_i^1 < s_{j,l} \tag{10}$$

for each section and gauge. The solution vectors to these problems will contain elements

$$x_i^{1*} = \frac{\beta_i s_{j,l}}{\sum_{i \in I(j,l)} \beta_i}, \; i \in I(j, l), \tag{11}$$

where B_i is defined by

$$\beta_i \equiv x_i^0 \alpha_i . \tag{12}$$

These solution vectors are analogous to solutions presented in Ref. 4.

For each problem solution, an equalized marginal value of a pair $EMVP(j, l)$ is determined, where

$$EMVP(j, l) = -\left.\frac{\partial g_i(x_i^1)}{\partial x_1^1}\right|_{x_i^1 = x_i^{1*}}, \; i \in I(j, l). \tag{13}$$

Intuitively, $-(\partial g_i(x_i^1)/\partial x_i^1)$ is a measure of the relative change in the operating costs in allocation area a_i with respect to a change in x_i. Since additional pairs always result in nonincreasing operating costs, $-(\partial g_i(x_i^1)/\partial x_i^1)$ is always nonnegative. The greater the value of $-(\partial g_i(x_i^1)/\partial x_i^1)$ for a given x_i^1, the greater the relative benefit of an additional pair in allocation area a_i. When the values of $-(\partial g_i(x_i^1)/\partial x_i^1)$ are equal for a given set of allocation areas, no further reduction of operating costs can be achieved by reallocating pairs from one area to another. The solution to eq. (10) gives the set of allocations which equalize the values of $-(\partial g_i(x_i^1)/\partial x_i^1)$ among allocation areas and hence the name equalized marginal value of a pair.

The maximum $EMVP(j^1, l^1)$ identifies the critical section, j^1, and the problem gauge l^1. If there is more than one such critical section (equal maximums), we select the one nearest the CO. If there are equal maximums for a given j^1, the finest such gauge is selected as the problem gauge. The values x_i^*; $i \in I(j^1, l^1)$ are now fixed.

3.4.2 Algorithm interactions and stopping condition

If $j^1 = 1$ (i.e., is the section adjacent to the CO) and l^1 = the finest available gauge, the problem is solved. Otherwise, the next step is to eliminate all allocation areas fed by or through section j^1 and requiring gauge l^1 or coarser in section j^1. The pairs allocated to the aforementioned allocation areas must then be subtracted from the $s_{j,l}$ and a next critical section and set of allocations defined.

3.5 The general feeder route

The feeder route that is characterized by substantial growth in demand and the feeder route characterized by little or no growth are both special cases of the general feeder route. In the general case, there can be a little of each. Some branches on the route can be growing quite rapidly while others may not be growing at all. Another possibility is that all or part of the general route will be growing at a rate that is neither fast or slow. Relief may be required, but it may not be required for a number of years.

In such cases, we would like to insure that section relief, at least in the near term, will not be advanced due to a premature allocation area shortage. At the same time, however, we would like to be able to minimize operating expenses in those areas where relief will not be required.

To accomplish these ends, the two allocation algorithms described in the preceding sections are combined into a single heuristic algorithm. Pairs are first allocated to satisfy growth requirements up to some arbitrarily specified time T. This is accomplished using the first algorithm. The remaining facilities are then allocated to minimize operating expenses in the slow growing portions of the network using the second algorithm.

3.5.1 Growth route phase

The first step is to pick the time T. A good value for T would be the length of the district's construction program planning interval. In the example presented in the following section, a value of T equal to 4 years was selected. Critical sections are then identified. Those with shortage times greater than T are ignored.

If there are no sections with an expected shortage before T, only the low-growth algorithm will be used. If there are sections that will exhaust before T *and* if the section adjacent to the CO ($j = 1$) will have a shortage before T in the finest available gauge, then only the growth route algorithm will be used. If there will be a shortage before T, but not in the first section, then both will be used.

Assume that section fs_j, with problem gauge l, is the last section on the route with a shortage time before T. Also assume that this is the $(k - 1)$th critical section identified on the route. Then the section adjacent to the CO is identified as the kth critical section with the finest available gauge as its problem gauge.

After all the growth critical sections have been identified, the next step is to determine allocations for all allocation areas $a_i \notin A(k, 1)$, as in Section 3.1.1. Allocations for the allocation area whose most critical section is the one adjacent to the co (i.e. all $a_i \in A(k, 1)$) will be determined using the low-growth algorithm.

After the allocation area allocations are determined, section allocations are computed for those sections that feed the $k - 1$ critical sections. This will insure that when these sections are relieved, they will have sufficient facilities to last exactly T years.

3.5.2 Low growth route phase

The above allocation areas and the pairs allocated to them are now removed from the problem. A new capacity matrix $S_{n\times4}$ is defined with elements $s_{j,l}$ that reflect the fact that all pairs allocated during the growth route phase are not available for further allocation. Note that this implies that all the $s_{j,l}$ for the sections fed by and including the $(k - 1)$th critical section will be zero if the problem gauge in that section is the finest available gauge.

The low-growth algorithm is now used to develop allocations for the $a_i \in A(k, 1)$.

IV. SAMPLE ALLOCATION

We now apply the general allocation algorithm to a typical suburban feeder route. For this example, sections and allocation areas will not be numbered consecutively as in the previous discussion. Rather, Bell System standard four-digit numbering will be employed.

Figure 5 is a schematic diagram for this route. In the figure, the individual paths to the co are identified by numbered arrows. It should be noted that the three allocation areas fed by section 1411 are actually components of a single geographic area which is fed by all three paths. The decimal values concatenated to the four-digit allocation area numbers correspond to the feeding paths.

The section capacity matrix for the route is presented in Table I. Recall that the element, $s_{j,l}$, equals the number of cable pairs that are available for service in fs_j of gauge l or coarser. Only three columns are presented in Table I since there is no 19-gauge cable required or available on this route.

Projected year-end facility requirements for each allocation area over a four-year planning horizon—beginning January 1, 1979—are presented in Table II. It is assumed that demand grows linearly within each one-year growth period.

Allocation area gauge requirements are summarized in Table III. Recall that $i \in I(j, l)$ implies that a_i requires facilities of gauge l or coarser in fs_j.

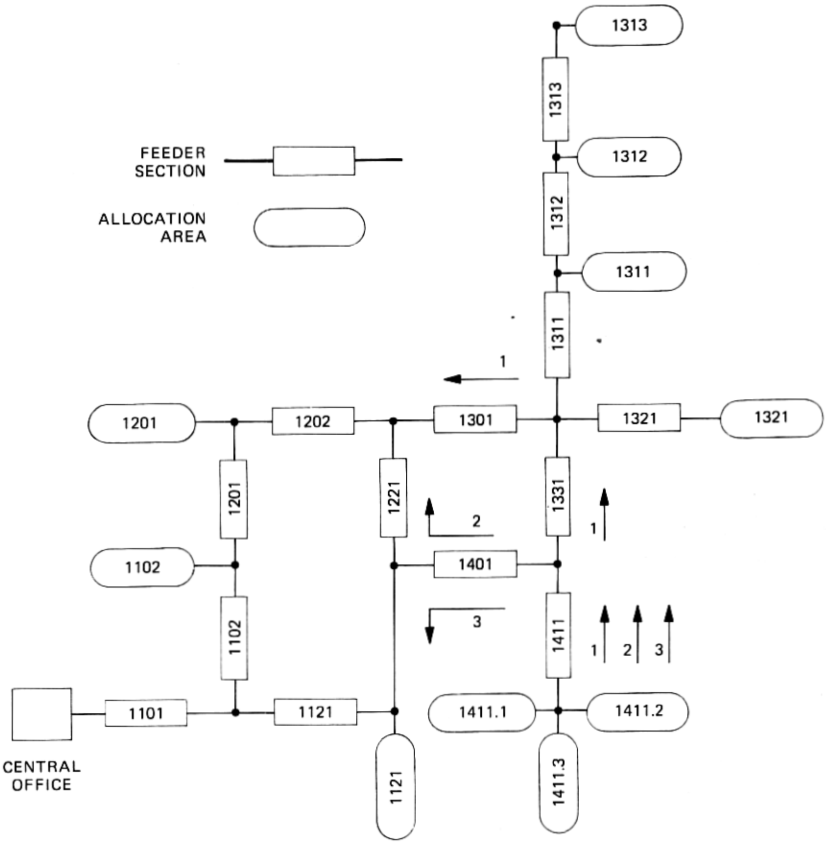

Fig. 5—Schematic diagram for sample route.

Table I—Section capacity matrix

Section	Gauge $l = 26$	$l = 24$	$l = 22$
1101	13000	3975	0
1102	12100	3975	0
1121	1100	0	0
1201	11200	5025	100
1202	8925	3700	100
1221	900	900	0
1301	8500	3275	150
1311	4325	4325	125
1312	3500	3500	1725
1313	1725	1725	1725
1321	2700	2700	0
1331	2650	900	0
1401	1800	1800	0
1411	1800	1800	0

Table II—Projected year-end allocation area facility requirements

Allocation Area	Facility Requirements 1978	1979	1980	1981	1982
1102	965	972	981	989	1001
1121	172	181	190	199	212
1201	1626	1652	1697	1742	1813
1311	986	997	1019	1040	1073
1312	1265	1301	1347	1393	1462
1313	1283	1349	1410	1467	1556
1321	2114	2139	2183	2225	2288
1411.1	77	83	97	110	130
1411.2	582	596	618	638	666
1411.3	536	539	543	546	552

Table III—Allocation area section requirements

Section	Gauge	Allocation Areas
1101	26	1102, 1121, 1201, 1311, 1321, 1411.1, 1411.2, 1411.3
	24	1312, 1313
1102	26	1102, 1201, 1311, 1321, 1411.1, 1411.2
	24	1312, 1313
1121	26	1121, 1411.3
1201	26	1201, 1311, 1321, 1411.1, 1411.2
	24	1312, 1313
1202	26	1311, 1321, 1411.1, 1411.2
	24	1312, 1313
1221	26	1411.2
1301	26	1311, 1321, 1411.1
	24	1312, 1313
1311	24	1311, 1312, 1313
1312	24	1312
	22	1313
1313	22	1313
1321	26	1321
1331	24	1411.1
1401	26	1411.3
	24	1411.2
1411	26	1411.3
	24	1411.1, 1411.2

4.1 Growth route phase

A value of four years was selected for the study interval (0, T). With a common fill at relief factor, $f_j = 0.85$ for all j, only two sections, 1311 and 1301, have shortage times in the (0, T) interval. Section 1311 is identified as the first critical section with a projected shortage in 24-gauge facilities at $t_1 = 1.22$ years, and section 1301 is identified as the second critical section with a shortage in 24 gauge at $t_2 = 2.25$ years. Note the latter has ample 26-gauge facilities which more than satisfies all projected fine gauge demand.

As in Section 3.3.1, the section adjacent to the co, section 1101, is identified as the third critical section. By convention, t_3 is set equal to T (= 4 years) and the problem gauge is defined to be $l = 26$.

Elements of the sets $A(k, 1)$ and $A(k, 2)$, $k = 1, 3$ are presented in Table IV. Section 1311 is shown to be the firstmost critical section for allocation areas 1311, 1312, and 1313. Section 1301 is the firstmost critical section for no allocation area and is the secondmost critical section for allocation areas 1312 and 1313. Allocation area 1311 only requires facilities of 26 gauge in section 1301.

The set of allocations x_i^n for the three allocation areas are shown in Table V. These are accumulated into the sets

$$X(1) = 1179, 1543, 1603$$

$$X(2) = 1599, 1676$$

$$X(3) = 1262, 1720, 1831.$$

They imply the allocation area and section allocations summarized in Table VI. The gauge shown in the table is the second (coarser) gauge required for two-gauge resistance design. The break section is the

Table IV—Elements of sets $A(k, 1)$ and $A(k, 22)$

k	Critical Section	Elements of $A(k, 1)$	Elements of $A(k, 2)$
1	1311	1311 1312 1313	
2	1301		1312 1313
3	1101	1101. 1121 1201 1321 1411.1 1411.2 1411.3	1311 1312 1313

Table V—Sets of allocation area allocations

Allocation Area	Individual Allocations		
	x_i^1	x_i^2	x_i^3
1311	1179	1262	
1312	1543	1599	1720
1313	1603	1676	1831

Table VI—Results of growth phase algorithm

Allocation Area		Gauge	Break Section	Theoretical Allocation
1311		24	1301	1179
1312		24	—	1543
1313		22	1311	1603
Section	**Energize Relief in Section**			
1301	1311	26	—	83
		24	—	129
1202	1301	24	—	276

section in which the design switches to the next finer gauge. A blank in this column implies a single gauge requirement.

4.2 Low growth route phase

Sufficient facilities have now been allocated to satisfy the growth requirements for allocation areas 1311, 1312, and 1313 over the designated four-year interval. These allocation areas and their allocated facilities are now removed from the problem. The modified $S_{n\times 3}$ matrix is presented in Table VII.

The low-growth route algorithm can now be used to allocate facilities to the remaining allocation areas to minimize operating costs within these allocation areas over the four-year study interval. The current numbers of pairs allocated, x_i^0, and values of β_i and γ_i for each allocation area, assuming $\lambda_i = 10$ for all i, and $r = 0.1$, are given in Table VIII.

Recall that critical sections are identified for the low-growth phase by solving the set of problems defined by eq. (10). For each solution set, we then determine values for

$$EMVP(j, l) = \left. \frac{-\partial g_i(x_i)}{\partial x_i} \right|_{x_i = x_i^*}$$

$$= \frac{\lambda_i \left(\sum_{i \in I(j,l)} \beta_i \right)^{\lambda_i}}{s_{j,l}^{\lambda_i}}, \qquad (14)$$

where eq. (14) is obtained by combining eqs. (2), (3), (11), (12), and (13). The results of this computation are presented in Table IX.

Table VII—Modified section capacity matrix

Section	Gauge $l = 26$	$l = 24$	$l = 22$
1101	8187	424	0
1102	7287	424	0
1121	1100	0	0
1201	6387	1474	0
1202	4112	149	0
1221	900	900	0
1301	3963	0	0
1311	0	0	0
1312	350	354	122
1313	122	122	122
1321	2700	2700	0
1331	2650	900	0
1401	1800	1800	0
1411	1800	1800	0

The maximum $EMVP(j, l)$ for the route is 20.74, the value for section 1102 ($l = 26$). The allocation areas requiring 26-gauge facilities or coarser in this section are allocation areas 1102, 1201, 1321, 1411.1, 1411.2, and 1411.3. The corresponding allocations x_i^* for these areas are presented in Table X.

These five allocation areas are now eliminated from the problem and the pairs allocated to them are removed from the section capacity matrix. The algorithm is then repeated on the remainder of the route until allocations are determined for each allocation area. This required two more iterations; the resulting allocations are also given in Table X.

Table VIII—Allocation area parameters for low-growth algorithm

Allocation Area	Pairs Available (x_i^0)	β_i	γ_i
1102	1700	1530.0	132.53
1121	225	321.8	948.41
1201	2100	2310.0	1046.49
1321	2500	2850.0	1002.59
1411.1	100	141.0	3512.86
1411.2	650	903.5	1275.48
1411.3	675	810.0	2135.25

Table IX—Equalized marginal values of a pair for each section

	EMVP (j,l)	
Section	$l = 26$	$l = 24$
1101	17.24	0
1102	20.74	0
1121	13.60	0
1201	8.01	0
1202	6.40	0
1221	10.67	0
1301	0.50	0
1321	18.18	18.18
1331	0	0
1401	5.87	0.01
1411	18.75	0.04

Table X—Theoretical allocations developed during low-growth route phase

Allocation Area	Allocations		
	Iteration No. 1	Iteration No. 2	Iteration No. 3
1102	1432		
1201	2162		
1321	2668		
1411.1	179		
1411.2	846		
1411.3		775	
1121			325

V. SUMMARY

New heuristic allocation algorithms for the loop plant network have been presented in this paper. One algorithm is designed to defer the need for new facilities in critical sections on routes with growing customer demand. The second is designed to minimize the present worth of operating expenses on routes with little or no growth in demand. Both extend previously published algorithms to the practical case of multi-gauge routes with complex topologies.

A general-case algorithm that combines the high-growth and low-growth algorithms into a single two-phase algorithm is also described. This algorithm is designed to defer the need for new facilities plus reduce operating expenses over a specified study interval.

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