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Diffeomorphisms and Newton-Direction Algorithms*

By I. W. SANDBERG

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This paper shows that an iterative process, with certain desirable convergence properties, can be used to compute the solution of an important general equation when certain conditions are met. More specifically, let f be a function from U into B, where B is a Banach space and U is a nonempty open subset of B. One main result reported on is a proof of the existence of a superlinearly convergent algorithm that globally converges to a solution x of $f(x) = a$ *for each* $a \in B$, *whenever f is a* C^1*-diffeomorphism of U onto B, and either* $B = R^n$ *or f satisfies certain other conditions that are frequently met in applications. For the case of an important class of monotone diffeomorphisms f in a Hilbert space H (examples arise, for example, in signal-theory studies), the "other conditions" reduce to simply the requirement that* f' *(the F-derivative of f) be uniformly continuous on closed bounded subsets of H.*

I. INTRODUCTION AND OUTLINE OF RESULTS

Let f be a function from U into B, where B is a Banach space with norm $|\cdot|$ and U is a nonempty open subset of B. We say that f is *differentiable* on a set $S \subset U$ if f has a Frechet derivative $f'(s)$ at each point s of S.† (If, for example, $B = R^n$ with the usual Euclidean norm, then f is differentiable on U if it is continuously differentiable on U in the usual sense.) By f a C^1-*diffeomorphism*, we mean that f is a homeomorphism of U onto B, and f' and $(f^{-1})'$ exist and are continuous on U and B, respectively. (We emphasize that here *continuity* refers to the dependence of the derivatives on the points at which they are evaluated, not to their boundedness as operators, which is assured by definition.‡)

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† In other words, f is *differentiable* on $S \subset U$ if for each $s \in S$, there is a bounded linear map $f'(s): B \to B$ such that $f(s + h) = f(s) + f'(s)h + o(|h|)$ for $(s + h) \in U$.

‡ And, of course, this continuity is with respect to the usual induced norm of a bounded linear map of B into B.

C^1-diffeomorphisms frequently arise in applications. One of the main purposes of this paper is to show the existence of a superlinearly convergent algorithm that globally converges to a solution x of $f(x) = a$ for each $a \in B$, whenever f is a C^1-diffeomorphism and either $B = R^n$ or f satisfies certain additional conditions that are frequently met in applications.

With f not necessarily a C^1-diffeomorphism, suppose that $a \in B$ and an initial point $x^0 \in U$ are specified. Let L denote $\{u \in U: |f(u) - a| \leq |f(x^0) - a|\}$, and assume that f' exists on L. Consider the following process (which, possibly, might not be able to be carried out) for generating a sequence $x^1, x^2, \cdots$ of points in U.

Process N: Define $s(v) = |f(v) - a|$ for $v \in U$. Choose real numbers κ and σ such that $0 < \kappa < ½ < \sigma < 1$. For $k = 0, 1, \cdots$ do the following.

If $f(x^k) = a$, set $x^{k+1} = x^k$.

If $f(x^k) \neq a$, determine $\phi_k \in B$ such that $f'(x^k)\phi_k = a - f(x^k)$. Then choose $\gamma_k > 0$ such that $(x^k + \gamma_k\phi_k) \in U$ and

$$\kappa \leq \frac{s^2(x^k) - s^2(x^k + \gamma_k\phi_k)}{2\gamma_k s^2(x^k)} \leq \sigma,$$

and take $\gamma_k = 1$ when possible. Set $x^{k+1} = x^k + \gamma_k\phi_k$.

Since $x^{k+1} - x^k = \gamma_k\phi_k$ when $f(x^k) \neq a$, with γ_k a scalar and ϕ_k as indicated, Process N is a *Newton-direction* process with a particular subprocess for determining each steplength $\gamma_k|\phi_k|$. Of course, if $\gamma_k = 1$ for all k, then the iterates $x^1, x^2, \cdots$ are the same as those produced by Newton's method, assuming that each $f'(x^k)^{-1}$ exists.†

All our results concern Process N and are given in Section II, where we first prove two preliminary theorems. The first of these, Theorem 1, provides conditions on f, a, and x^0, under which Process N can be carried out and any sequence $x^1, x^2, \cdots$ produced is a sequence of eventually arbitrarily good approximate solutions of $f(x) = a$, in the sense that $|f(x^k) - a| \to 0$ as $k \to \infty$. Under the assumptions that the process can be carried out and that we have $x^k \to x$ as $k \to \infty$, where x satisfies $f(x) = a$, Theorem 2 shows that $x^1, x^2, \cdots$ converges *superlinearly* in the usual sense that $|x^{k+1} - x| = c_k|x^k - x|$, $k \geq 0$, where $c_k \to 0$ as $k \to \infty$, provided merely that $(f')^{-1}$ exists and is continuous in some neighborhood N_x of x. Process N is similar to an algorithm studied in Ref. 1, pp. 43–44, in connection with mappings in R^n. Results related to Theorems 1 and 2 are given in Ref 1 where, however, the strong hypothesis that f is twice continuously differentiable plays a central role in the proofs, as does the use of the sequential

† With regard to the convergence properties of Newton's method and the results given in this paper, it is a simple matter to give an example of a diffeomorphism f of R^1 onto R^1 such that $f(0) = 0$ and for some x^0 the Newton iterates for $a = 0$ do not converge.

compactness of closed bounded sets in R^n. (For a general $B \neq R^n$, closed bounded subsets of B need not be sequentially compact.) Newton-direction algorithms, which are sometimes referred to as "damped Newton" methods, have been considered also by other writers (see, for example, Ref. 2, which contains a good bibliography).

Theorems 1 and 2 are used in Section II to prove three results concerning C^1-diffeomorphism and Process N. Theorem 3 asserts that, whenever f is a C^1-diffeomorphism with certain properties and a and x^0 are given, Process N can be carried out, and any sequence x^1, x^2, $\cdots$ generated converges superlinearly to the unique solution x of $f(x) = a$. Theorem 4 shows that the conclusion of Theorem 3 holds for an important class of "monotone operators" which is of interest, for example, in connection with studies of nonlinear elliptic partial differential equations, demodulation of very wide band frequency-modulated signals, recovery of distorted signals, and nonlinear networks.

Our final result, Theorem 5, is a characterization of C^1-diffeomorphisms in R^n; it asserts that a continuously differentiable map $f: U \rightarrow R^n$ is a C^1-diffeomorphism if and only if Process N can always be carried out, and for each a always produces a sequence of iterates that converges superlinearly to a solution x of $f(x) = a$ that does not depend on the initial point x^0. A similar characterization in terms of a *steepest descent* process is given in Ref. 3.†,‡

Sections 2.2, 2.5, and 2.6.2 contain comments concerning Theorems 1, 3, and 4, respectively. An example of an f that meets the hypotheses of Theorem 4, and for which B is not finite dimensional is given in Section 2.6.3.

II. THEOREMS CONCERNING PROCESS *N*

Recall that f, U, B, the set L, Process N, and the terms "differentiable" and "C^1-diffeomorphism" are defined in Section I. Throughout this section, θ denotes the zero element of B, and if $A: B \rightarrow B$ is a bounded linear map, then $|A|$ denotes the usual induced norm of A. By f or f' *uniformly continuous* on a set $S \subset U$ is meant, as usual, that for each $\delta_1 > 0$ there is a $\delta_2 > 0$ such that $u, v \in S$ with $|u - v| < \delta_2$ implies that $|f(u) - f(v)| < \delta_1$ or $|f'(u) - f'(v)| < \delta_1$, respectively.

Theorem 1: Let $a \in B$ and $x^0 \in U$ be given. Assume that L is bounded, that f is differentiable on L, that f and f' are uniformly continuous on L, that $f'(u)^{-1}$ exists for each $u \in L$, and that there is a constant K

† A related early result concerning the use of Newton-direction algorithms in R^n, in which f is assumed to be *twice* continuously differentiable, is contained in the appendix of Ref. 4, and some recent peripherally related material on Newton-direction algorithms in R^n is given in Ref. 5.

‡ This writer is indebted to his colleague D. J. Rose, for discussions concerning general aspects of numerical analysis, and for reawakening this writer's interest in Newton-direction algorithms.

such that $|f'(u)^{-1}| \le K$ for $u \in L$. Then Process N can be carried out, and any sequence $x_1, x_2, \cdots$ generated by the process satisfies $|f(x^k) - a| \to 0$ as $k \to \infty$.

2.1 Proof of Theorem 1

Assume, for the purpose of the following inductive argument, that k is an integer for which the process can be used to determine x^j for $0 \le j \le k$ and that $x^k \in L$, which is certainly true for $k = 0$. If $f(x^k) = a$, then obviously x^{k+1} can be determined and $x^{k+1} \in L$.

Suppose now that $f(x^k) \ne a$. We have $x^k + \gamma\phi_k \in U$ for sufficiently small $\gamma > 0$, because U is open. Let $g(x^k, \gamma)$ denote

$$\frac{s^2(x^k) - s^2(x^k + \gamma\phi_k)}{2\gamma s^2(x^k)}$$

whenever the latter is defined. Since f is differentiable at x^k, $s(x^k + h) = |f(x^k) - a + f'(x^k)h + \delta|$ for $(x^k + h) \in U$, in which $|\delta|/|h| \to 0$ as $|h| \to 0$. Let $h = -\gamma f'(x^k)^{-1}[f(x^k) - a]$, with $\gamma > 0$ such that $(x^k + h) \in U$. We have $s(x^k + \gamma\phi_k) = |(1 - \gamma)[f(x^k) - a] + \delta|$. Since $|(1 - \gamma)[f(x^k) - a]| - |\delta| \le |(1 - \gamma)[f(x^k) - a] + \delta| \le |(1 - \gamma)[f(x^k) - a]| + |\delta|$, clearly $|(1 - \gamma)[f(x^k) - a] + \delta| = |(1 - \gamma)[f(x^k) - a]| + \eta|\delta|$ where $\eta \in [-1, 1]$ depends on γ. Thus, for sufficiently small $\gamma > 0$,

$$g(x^k, \gamma) - 1 = \frac{-\gamma^2|f(x^k) - a|^2 - 2(1 - \gamma)\eta|f(x^k) - a| \cdot |\delta| - \eta^2|\delta|^2}{2\gamma|f(x^k) - a|^2}. \tag{1}$$

Since

$$\frac{|\delta|}{\gamma|f(x^k) - a|} = \frac{|\delta|}{|h|} \cdot \frac{|h|}{\gamma|f(x^k) - a|}, \tag{2}$$

and $|h|\gamma^{-1}|f(x^k) - a|^{-1} \le K$, we see that the left side of (2) approaches zero as $\gamma \to 0$. Similarly,

$$\frac{|\delta|^2}{\gamma|f(x^k) - a|^2} = \frac{|\delta|^2}{|h|^2} \cdot \frac{|h|^2}{\gamma|f(x^k) - a|^2} \le \gamma K^2 \frac{|\delta|^2}{|h|^2}, \tag{3}$$

which shows that the extreme left side of (3) approaches zero as $\gamma \to 0$. Thus, using $|\eta| \le 1$, $g(x^k, \gamma) \to 1$ as $\gamma \to 0$. We use this fact as follows.

Let S denote $\{\gamma > 0 : x^k + \alpha\phi_k \in L \text{ for } \alpha \in [0, \gamma]\}$. Notice that S is not empty, because $g(x^k, \gamma) > 0$ for sufficiently small $\gamma > 0$. Let $\gamma^* = \sup S$, and observe that γ^* is finite, since L is bounded and $\phi_k \ne \theta$. Also, $x^k + \gamma^*\phi_k \in L$ because, by the continuity of f, L is closed. It follows that $\gamma^* \in S$, which requires that $g(x^k, \gamma^*) \le 0$ (because if $g(x^k, \gamma^*) > 0$, we arrive at the contradiction that for some $\lambda > 0$, $x^k + \alpha\phi_k \in L$ for $\alpha \in [\gamma^*, \gamma^* + \lambda]$). Since $g(x^k, \gamma)$ is defined and continuous for $\gamma \in (0, \gamma^*]$, and using $g(x^k, \gamma^*) \le 0$ as well as $g(x^k, \gamma) \to 1$ as $\gamma \to 0$, there is a $\gamma_k \in (0, \gamma^*)$ such that $\kappa \le g(x^k, \gamma_k) \le \sigma$, and, of course, for that γ_k, we have $x^{k+1} \in L$.

Thus, we can determine x^{k+1} as required and $x^{k+1} \in L$. This shows that the process can be carried out.

Now let $x^1, x^2, \cdots$ be any sequence produced by Process N. Suppose, for the purpose of obtaining a contradiction, that $|f(x^k) - a| \not\to 0$ as $k \to \infty$. We have $|f(x^k) - a| \neq 0$ for all k. Observe that, since $(1 - 2\kappa\gamma_k)s^2(x^k) \ge s^2(x^{k+1})$ for $k \ge 0$, the existence of a positive constant c such that $\gamma_k \ge c$ for all k implies that $|f(x^k) - a| \to 0$ as $k \to \infty$.

For all k we have $x^k \in L$, $|f(x^{k+1}) - a| < |f(x^k) - a|$, and $g(x^k, \gamma_k) \le \sigma$. Let $\{x^{k_i}\}$ be a subsequence of $\{x^k\}$ such that $\gamma_{k_i} \to 0$ as $i \to \infty$. Since $|f(x^k) - a|$ is monotone decreasing for increasing k and $|f(x^k) - a| \not\to 0$ as $k \to \infty$, there is a constant $\rho > 0$ such that $|f(x^{k_i}) - a| \ge \rho$ for all i.

Let h_i denote $-\gamma_{k_i} f'(x^{k_i})^{-1}[f(x^{k_i}) - a]$, and notice that $|h_i| \le \gamma_{k_i} K |f(x^0) - a|$ for all i. Let $\delta_i = f(x^{k_i} + h_i) - f(x^{k_i}) - f'(x^{k_i}) h_i$ for each i.

Lemma: Under the hypotheses of Theorem 1, $\gamma_{k_i}{}^{-1}|\delta_i| \to 0$ *as* $i \to \infty$.

Proof: In the following, let $i \ge 2$, notice that $|f(x^{k_i}) - a| < |f(x^1) - a| < |f(x^0) - a|$, and let ν denote $\frac{1}{2}(|f(x^0) - a| - |f(x^1) - a|)$. By the uniform continuity of f, choose $\delta > 0$ so that, if x^a and x^b are in L and $|x^a - x^b| < \delta$, then $|f(x^a) - f(x^b)| < \nu$.

Assume that $|h_i| < \delta$, which is satisfied for all sufficiently large i. To justify the use of a certain integral representation given below, we first establish that $x^{k_i} + \beta h_i \in L$ for $\beta \in [0, 1]$: If it were not true that $x^{k_i} + \beta h_i \in L$ for $\beta \in [0, 1]$, and since x^{k_i} is an interior point of L, there would be a $\beta_i \in (0, 1)$ such that $x^{k_i} + \beta_i h_i \in L$ and $|f(x^{k_i} + \beta_i h_i) - a| = |f(x^0) - a|$. On the other hand, $|f(x^{k_i} + \beta_i h_i) - a| \le |f(x^{k_i}) - a| + |f(x^{k_i} + \beta_i h_i) - f(x^{k_i})| \le |f(x^1) - a| + \nu < |f(x^0) - a|$, which is a contradiction.

Therefore, we have

$$f(x^{k_i} + h_i) - f(x^{k_i}) = \int_0^1 f'(x^{k_i} + \beta h_i)\, d\beta \cdot h_i,$$

and so

$$|\delta_i| = \left| \int_0^1 [f'(x^{k_i} + \beta h_i) - f'(x^{k_i})]\, d\beta \cdot h_i \right|$$

$$\le \int_0^1 |f'(x^{k_i} + \beta h_i) - f'(x^{k_i})|\, d\beta \cdot |h_i|$$

$$\leq \gamma_{k_i} K |f(x^0) - a| \sup_{\beta \in [0,1]} | f'(x^{k_i} + \beta h_i) - f'(x^{k_i}) | .$$

By the uniform continuity of f', and $|h_i| \to 0$ as $i \to \infty$, we have $\gamma_{k_i}^{-1} |\delta_i| \to 0$ as $i \to \infty$, which proves the lemma.

With $g(x^k, \gamma)$ as defined earlier, we have (see the steps leading to (1)),

$$g(x^{k_i}, \gamma_{k_i}) - 1 = -\frac{1}{2}\gamma_{k_i} - \frac{(1-\gamma_{k_i})\eta_i |\delta_i|}{\gamma_{k_i} |f(x^{k_i}) - a|} - \frac{\eta_i^2 |\delta_i|^2}{2\gamma_{k_i} |f(x^{k_i}) - a|^2},$$

in which $|\eta_i| \leq 1$, for all i. By the lemma, using the bound $|f(x^{k_i}) - a| \geq \rho$, we have $g(x^{k_i}, \gamma_{k_i}) \to 1$ as $i \to \infty$. This contradicts the inequality $g(x^{k_i}, \gamma_{k_i}) \leq \sigma$ for all i, and thus $|f(x^k) - a| \to 0$ as $k \to \infty$.

2.2 Comments

With regard to the hypotheses of Theorem 1, a sometimes useful sufficient condition for f and f' to be uniformly continuous on L is simply that f and f' satisfy uniform Lipschitz conditions on closed bounded subsets of U.

For the important special case in which $B = R^n$ and $|\cdot|$ is the usual Euclidean norm, a relatively short proof can be given of the proposition that, under the hypotheses of Theorem 1, we have $|f(x^k) - a| \to 0$ as $k \to \infty$ for any sequence generated by Process N: Suppose that $|f(x^k) - a| \not\to 0$ as $k \to \infty$. We have $|f(x^k) - a| < |f(x^0) - a|$ for all $k > 0$. Since $\{x^k\}$ is contained in the compact set L, we can construct a convergent subsequence $\{x^{k_i}\}$ with limit $x^* \in L$ such that $\lim_{i\to\infty} \gamma_{k_i} = 0$ and $f(x^*) \neq a$. Since x^* satisfies $|f(x^*) - a| < |f(x^0) - a|$, L contains an open neighborhood of x^*, and we may therefore assume without loss of generality that $\{x^{k_i}\}$ is contained in a convex subset of L.

By the mean-value theorem,

$$g(x^{k_i}, \gamma_{k_i}) = -\langle \nabla s^2(\eta_{k_i}), \phi_{k_i} \rangle / 2s^2(x^{k_i})$$

in which $\langle \cdot, \cdot \rangle$ denotes the usual inner product, ∇ the gradient, and η_{k_i} is a point on the open-line segment between x^{k_i} and $x^{k_i} + \gamma_{k_i}\phi_{k_i}$. Thus, with "tr" denoting the transpose,

$$g(x^{k_i}, \gamma_{k_i}) = \frac{2[f(\eta_{k_i}) - a]^{\mathrm{tr}} f'(\eta_{k_i}) f'(x_{k_i})^{-1} [f(x_{k_i}) - a]}{2|f(x^{k_i}) - a|^2},$$

from which it is clear that $g(x^{k_i}, \gamma_{k_i}) \to 1$ as $i \to \infty$, and this contradicts $g(x^{k_i}, \gamma_{k_i}) \leq \sigma$ for all i, which completes the proof.

Theorem 2: Let $a \in B$ and $x^0 \in U$ be given. Assume that there is an $x \in U$ such that $f(x) = a$, that Process N can be carried out, and that it generates a sequence $x^1, x^2, \cdots$ such that $f(x^k) \neq a$ for all k, and

$x^k \to x$ as $k \to \infty$. Assume also that f' and $(f')^{-1}$ exist in a neighborhood N_x of x, and that $(f')^{-1}$ is continuous at the point x. Then there is an integer M such that $\gamma_k = 1$ for $k \geq M$, and we have $|x^{k+1} - x| (|x^k - x|)^{-1} \to 0$ as $k \to \infty$.

2.3 Proof of Theorem 2

We assume, without loss of generality, that $x^k \in N_x$ for all k. Let J and J_k denote $f'(x)$ and $f'(x_k)$, respectively. Since $\phi_k = J_k^{-1}(a - f(x^k)) = (J_k^{-1} - J^{-1})(a - f(x^k)) + J^{-1}(a - f(x^k))$, we have $|\phi_k| \leq |J_k^{-1} - J^{-1}| \cdot |a - f(x^k)| + |J^{-1}| \cdot |a - f(x^k)|$ for all k. By the continuity of $(f')^{-1}$ at x and $f(x^k) \to a$ as $k \to \infty$, it is clear that $\phi_k \to \theta$ as $k \to \infty$. Therefore, there is an integer m such that $x^k + \phi_k \in U$ for $k \geq m$. With regard to the existence of M, observe that, since $\kappa < ½ < \sigma$, it suffices to show that $g(x^k, 1) \to ½$ as $k \to \infty$ (g is defined in the proof of Theorem 1), or, equivalently, that $t(x^k, 1)$ given by $t(x^k, 1) = s(x^k + \phi_k)/s(x^k)$ for $k \geq m$, satisfies $t(x^k, 1) \to 0$ as $k \to \infty$.

Let $k \geq m$. Since f' exists at the point x, $f(x^k) - a = J(x^k - x) + \delta$ and $f(x^k + \phi_k) - a = J(x^k + \phi_k - x) + \Delta$, with $\delta = o(|x^k - x|)$ and $\Delta = o(|x^k + \phi_k - x|)$, and of course $\phi_k = -J_k^{-1}[J(x^k - x) + \delta]$. Since J is an invertible bounded linear operator from B onto B, there are positive constants β_1 and β_2 such that $\beta_1 |u| \leq |Ju| \leq \beta_2 |u|$ for $u \in B$.†

Thus, for some $\bar{m} > m$, $|J(x^k - x)| \geq 2|\delta|$ and $|J(x^k + \phi_k - x)| \geq |\Delta|$ for $k \geq \bar{m}$. We have, for $k \geq \bar{m}$, $s(x^k, 1) \leq 4|J(x^k + \phi_k - x)| \cdot |J(x^k - x)|^{-1} \leq 4\beta_2\beta_1^{-1}|x^k + \phi_k - x| \cdot |x^k - x|^{-1}$. Therefore, for $k \geq \bar{m}$,

$$\begin{aligned}
¼\beta_1\beta_2^{-1}s(x^k, 1) &\leq |x^k - x|^{-1}(|x^k - x - (x^k - x) \\
&\qquad - J^{-1}\delta + (J^{-1} - J_k^{-1})[J(x^k - x) + \delta]|) \\
&\leq |x^k - x|^{-1}(|J^{-1}\delta| + |J^{-1} - J_k^{-1}| \\
&\qquad \cdot |J(x^k - x)| + |J^{-1} - J_k^{-1}| \cdot |\delta|). \qquad (4)
\end{aligned}$$

Using $|J^{-1}\delta| \leq \beta_1^{-1}|\delta|$, as well as $\delta = o(|x^k - x|)$ and $|J^{-1} - J_k^{-1}| \to 0$ as $k \to \infty$, we see that $t(x^k, 1) \to 0$ as $k \to \infty$, which proves the existence of M such that $x^{k+1} = x^k + \phi_k$ for $k \geq M$. Finally, notice that the right side of (4) is $|x^{k+1} - x| \cdot |x^k - x|^{-1}$ for $k \geq M$. Thus, our proof shows that $|x^{k+1} - x| \cdot |x^k - x|^{-1} \to 0$ as $k \to \infty$, which completes the proof of Theorem 2.

Theorem 3: Assume that f is a C^1-diffeomorphism. Let f and f' be uniformly continuous on closed bounded subsets of U, and let $|(f^{-1})'|$ be bounded on closed bounded subsets of B. Then for each $a \in B$ and each $x^0 \in U$, Process N can be carried out; any sequence $x^1, x^2, \cdots$

† The left inequality is a consequence of the proposition, due to Banach, that if $A : B \to B$ is a bounded linear invertible operator, then A^{-1} is bounded.

generated by the process converges to the unique solution x of $f(x) = a$, and is such that we have $|x^{k+1} - x| = c_k |x^k - x|$ for $k \geq 0$, in which $c_k \to 0$ as $k \to \infty$.

2.4 Proof of Theorem 3

Let a and x^0 be given. Let ρ be a constant such that $|(f^{-1})'(u)| \leq \rho$ for $u \in \{u \in B : |u| \leq |a| + |f(x^0) - a|\}$. If $v \in L$, then $f(v) = a + b$ with $|b| \leq |f(x^0) - a|$. For any such b,

$$f^{-1}(a + b) = f^{-1}(a) + \int_0^1 (f^{-1})'(a + \beta b)\, d\beta \cdot b$$

and thus $|f^{-1}(a + b)| \leq |f^{-1}(a)| + \rho |f(x^0) - a|$, which shows that L is bounded.

Since f is a C^1-diffeomorphism, $u = f^{-1}[f(u)]$ for $u \in U$ and $v = f[f^{-1}(v)]$ for $v \in B$. By the continuous differentiability of f and f', we have, with I the identity operator on B, $I = (f^{-1})'(f(u)) \cdot f'(u)$ for $u \in U$, and $I = f'[f^{-1}(v)] \cdot (f^{-1})'(v)$ for $v \in B$. Now set $v = f(u)$, and observe that, for $u \in U$, $I = (f^{-1})'(f(u)) \cdot f'(u)$ and $I = f'(u) \cdot (f^{-1})'(f(u))$. Thus $f'(u)^{-1}$ exists, because $f'(u)$ possesses both a left inverse and a right inverse, and it satisfies $f'(u)^{-1} = (f^{-1})'(f(u))$ for $u \in U$. Let K satisfy $|(f^{-1})'(w)| \leq K$ for $w \in \{w \in B : |w| \leq |a| + |f(x^0) - a|\}$. Choose $v \in L$, and notice that $|f'(v)^{-1}| \leq K$ because $|f(v)| \leq |a| + |f(x^0) - a|$.

By Theorem 1, Process N can be carried out, and any sequence that it generates satisfies $|f(x^k) - a| \to 0$ as $k \to \infty$. Let $x^1, x^2, \cdots$ be any such sequence.

Clearly, $x^k = f^{-1}(a + \delta_k)$ with $|\delta_k| \to 0$ as $k \to \infty$. Since f^{-1} is continuous, $x^k \to x$ as $k \to \infty$, where x is the unique solution of $f(x) = a$.

If $x^0 \neq x$, then x is an interior point of L, and is therefore contained in some neighborhood N_x of x such that $N_x \subset L$. For $w \in N_x$, we have $|f'(w)^{-1} - f'(x)^{-1}| = |f'(w)^{-1}[f'(x) - f'(w)]f'(x)^{-1}| \leq K^2 |f'(x) - f'(w)|$, which shows that $(f')^{-1}$ is continuous at x. Thus, by Theorem 2 and the observation that if $x^j = x$ for some j, then $x^k = x$ for $k > j$, there is a sequence $c_0, c_1, \cdots$ with the properties stated in Theorem 3.† This completes the proof.

2.5 Comment

Here we give a related result concerning conditions under which f is a C^1-diffeomorphism, which is sometimes useful. Assume that f' exists and is continuous on U, and that U is convex. Let A_1 and A_2 be any two families of compact subsets of B and U, respectively, such

† In this connection, from Theorem 2 and known local properties of Newton's method, it follows that we have *quadratic* convergence (i.e., $|x^{k+1} - x| \leq \beta |x^k - x|^2$ for $k \geq 0$ and some constant β) if in addition f' is locally Lipschitzian on U.

that if C and D are compact sets in U and B, respectively, then there are $S \in A_1$ and $T \in A_2$ such that $C \subseteq T$ and $D \subseteq S$.

If f is a C^1-diffeomorphism, we have, as in the proof of Theorem 3, $I = (f^{-1})'(f(u)) \cdot f'(u)$ and $I = f'(u) \cdot (f^{-1})'(f(u))$ for $u \in U$, and thus then $f'(u)^{-1}$ exists for $u \in U$. This observation, together with Theorem 3 of Ref. 6, and the implicit function theorem in Ref. 7, p. 273 show that f is a C^1-diffeomorphism if and only if $f'(u)^{-1}$ exists for each $u \in U$, and for each $S \in A_1$, there is a $T \in A_2$ such that $f(s) \in S$ implies that $s \in T$. A corresponding result for $B = U = R^n$ is well known.[8]

2.6 Monotone diffeomorphisms in Hilbert space

Let $\psi: [0, \infty) \to [0, \infty)$ be continuous, strictly increasing, and such that $\psi(0) = 0$, $\psi(\alpha) \to \infty$ as $\alpha \to \infty$, and $\alpha^{-1}\psi(\alpha) \geq c$ for $\alpha \in (0, \bar{\alpha})$ for some positive constants c and $\bar{\alpha}$. Notice that, for example, $\psi(\alpha) = \alpha$ meets these conditions.

Theorem 4: Let f map a real Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$, into itself such that $\langle f(u) - f(v), u - v \rangle \geq |u - v| \psi(|u - v|)$ for all $u, v \in H$. Assume that f' exists and is uniformly continuous on closed bounded subsets of H. Then f is a C^1-diffeomorphism of H onto H, and the conclusion of Theorem 3 (*with U and B replaced with H*) *holds.*

2.6.1 Proof of Theorem 4

Since f is continuous and satisfies the indicated inequality with ψ : $[0, \infty) \to [0, \infty)$ continuous, strictly increasing, and such that $\psi(0) = 0$ and $\psi(\alpha) \to \infty$ as $\alpha \to \infty$, it can be shown that f is a homeomorphism of H onto H[9] (see also Refs. 10 and 11 for early results along similar lines).

Now let $h \in H$ with $h \neq \theta$ be arbitrary, and let v be any point in H. Since $\langle f(v + th) - f(v), th \rangle \geq |th| \psi(|th|)$ for each real number t, it follows from the existence of f', and the inequality $\alpha^{-1}\psi(\alpha) \geq c$ for $\alpha \in (0, \bar{\alpha})$, that $|h|^{-2}\langle f'(v)h, h \rangle \geq c$. This implies that $f'(v)^{-1}$ exists and that $|f'(v)^{-1}| \leq c^{-1}$ for $v \in H$.[12] Since f is onto, by the inverse function theorem in Ref. 7, p. 273, f^{-1} is continuously differentiable on H. Therefore, f is a C^1-diffeomorphism of H onto H.

As in the proof of Theorem 3, $(f^{-1})'(f(u)) = f'(u)^{-1}$ for $u \in H$, which shows that $|(f^{-1})'|$ is bounded on H. Finally, by the uniform continuity of f', $|f'|$ is bounded on convex closed bounded sets, and since any bounded set in H is contained in some convex closed bounded subset of H, it easily follows that f is uniformly continuous on closed bounded sets.† Therefore, by Theorem 3, the proof is complete.

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† This type of argument shows also that the uniform continuity requirement on f in Theorem 3 can be dropped when $U = B$.

2.6.2 Notes

An early related result is the following. If H is a real or complex Hilbert space, and $f: H \to H$ is Lipschitz as well as *uniformly monotone* in the sense that for some constant $c > 0$, we have

$$\mathrm{Re}\langle f(u) - f(v), u - v\rangle \geq c\,|u - v|^2 \qquad \text{for} \quad u, v \in H,$$

then f is a homeomorphism of H onto H, and, with λ the Lipschitz constant, given $a \in H$, and any starting point $x^0 \in H$, the iterates x^1, x^2, $\cdots$ defined by

$$x^{k+1} = x^k + c\lambda^{-2}(a - f(x^k)), \quad k \geq 0$$

converge to the solution x of $f(x) = a$.[11]† A similar proposition is given in Ref. 13, p. 131, and a modification of the proposition described above, of use in Banach spaces, is given in Ref. 14.

2.6.3 An example

Here we give a simple example of a uniformly monotone operator for which H is not finite dimensional and the hypotheses of Theorem 4 are met. The example arises in connection with results concerning the recovery of distorted signals (see Ref. 11 and the work of Beurling, Landau and Miranker, and Zames cited there).

Let L_2 denote the linear space of square-integrable functions from R^1 into R^1, let the inner product $\langle \cdot, \cdot \rangle$ in L_2 be given, as usual, by

$$\langle u, v\rangle = \int_{-\infty}^{\infty} u(t)v(t)\, dt,$$

and let $|\cdot|$ denote the corresponding norm. Let H be the closed subspace of elements v of L_2 for which $V(\omega)$, the Fourier transform of v, given by

$$V(\omega) = \text{l.i.m.} \int_{-\infty}^{\infty} v(t)e^{-i\omega t}\, dt,$$

vanishes for almost all $\omega \notin \Omega$, where Ω is a fixed bounded interval $[-\omega_0, \omega_0]$. Of course, for any $v \in H$ we have

$$v(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} V(\omega)e^{i\omega t}\, d\omega, \quad \text{a.e.} \tag{5}$$

and we shall use the fact that

$$\text{ess}_t\text{sup}\,|v(t)| \leq \left(\frac{\omega_0}{\pi}\right)^{1/2} |v|, \quad v \in H, \tag{6}$$

† For a particularly interesting application, see Ref. 15.

which easily follows from (5), the Schwarz inequality, and Plancherel's identity.

Let $P: L_2 \to L_2$ denote the projection operator that projects L_2 onto H, and let $Q: H \to L_2$ be defined by the condition that for each $v \in H$, $(Qv)(t) = q[v(t)]$ for almost all t, where $q: R^1 \to R^1$ is continuously differentiable, such that $q(0) = 0$, and such that there are positive constants c and λ with the property that $c \leq q'(\alpha) \leq \lambda$ for all α. Finally, let $f: H \to H$ be given by $f = PQ$.

It can easily be verified that $\langle Q(u) - Q(v), u - v \rangle \geq c|u - v|^2$ for $u, v \in H$. Since P is self-adjoint, $\langle f(u) - f(v), u - v \rangle \geq c|u - v|^2$ for u and v in H. Similarly, using $|P| = 1$, we have $|f(u) - f(v)| \leq \lambda |u - v|$ for $u, v \in H$.

For each $v \in H$, let $R_v: H \to L_2$ be defined by $(R_v u)(t) = q'[v(t)]u(t)$ for almost all t and any $u \in H$. (The definition makes sense because $q'[v(\cdot)]$ is measurable, and we have $\text{ess}_t\sup |q'[v(t)]| < \infty$.)

Now let $v \in H$ be arbitrary but fixed, and let $\delta(h) \in H$ be given by $\delta(h) = (PQ)(v + h) - (PQ)(v) - (PR_v)h$ for each $h \in H$. Using $|P| = 1$, we have $|\delta(h)| \leq |Q(v + h) - Qv - R_v h|$, and thus $|\delta(h)| \leq \rho(h)|h|$, where

$$\rho(h) = \text{ess}_t\sup \int_0^1 |q'[v(t) + \beta h(t)] - q'[v(t)]| \, d\beta.$$

Since q' is uniformly continuous on compact sets, $\text{ess}_t\sup|v(t)| < \infty$, and $\text{ess}_t\sup|h(t)| \to 0$ *as* $|h| \to 0$ [see (6)], we have $\rho(h) \to 0$ as $|h| \to 0$. This shows that $f'(v)$ exists and that $f'(v) = PR_v$.

On the other hand, $|PR_{v_a} - PR_{v_b}| \leq \sup\{|(R_{v_a} - R_{v_b})w| : w \in H, |w| = 1\}$ for v_a and v_b in H, where $|PR_{v_a} - PR_{v_b}|$ denotes the induced norm $\sup\{|(PR_{v_a} - PR_{v_b})w| : w \in H, |w| = 1\}$ of $(PR_{v_a} - PR_{v_b}) : H \to H$. Thus,

$$|PR_{v_a} - PR_{v_b}| \leq \text{ess}_t\sup|q'[v_a(t)] - q'[v_b(t)]|$$

for $v_a, v_b \in H$. Now let γ be an arbitrary positive constant, and suppose that v_a and v_b in H satisfy $|v_a| \leq \gamma$ and $|v_b| \leq \gamma$. Let $\sigma > 0$ be another constant. Using (6), $\text{ess}_t\sup \max(|v_a(t)|, |v_b(t)|) \leq K$ where $K = (\omega_0/\pi)^{1/2}\gamma$. By the uniform continuity of $q'(\cdot)$ for $|\alpha| \leq K$, let δ be such that $|q'(\alpha_1) - q'(\alpha_2)| < \sigma$ when $|\alpha_1| \leq K$, $|\alpha_2| \leq K$, and $|\alpha_1 - \alpha_2| < \delta$. Thus, by (6), if $|v_a - v_b| \leq \delta(\omega_0/\pi)^{-1/2}$, then $|PR_{v_a} - PR_{v_b}| < \sigma$. Since σ is arbitrary, f' is uniformly continuous on bounded subsets of H. This shows that f satisfies the hypotheses of Theorem 4 (and it proves that there is a *superlinearly* convergent algorithm that recovers bandlimited signals that are nonlinearly distorted by Q and subsequently bandlimited by P).

Theorem 5: *Let* $B = R^n$, *and let* f *be a continuously differentiable* *map of U into R^n. Then f is a C^1-diffeomorphism of U onto R^n if and only if*

(*i*) *Process N can be carried out for each $a \in R^n$ and each $x^0 \in U$.*

(*ii*) *For each a, any sequence produced by the process converges superlinearly to a solution x of $f(x) = a$, and x does not depend on x^0.*

2.7 Proof

If f is a C^1-diffeomorphism, then f, f', and $(f^{-1})'$ are continuous, and then, since closed bounded subsets of R^n are compact, f, f', and $|(f^{-1})'|$ are uniformly continuous and bounded, respectively, on closed bounded sets. Thus, by Theorem 3, (*i*) and (*ii*) hold when f is a C^1-diffeomorphism.

On the other hand, suppose that (*i*) and (*ii*) are satisfied. Then $f'(u)^{-1}$ exists for each u, because otherwise there would be an x^0 and an a such that $a \neq f(x^0)$ and such that there is no solution ϕ_0 of $f'(x^0)\phi_0 = a - f(x^0)$. Also, f is one-to-one and onto. Since there is an x such that $f(x) = a$ for each $a \in R^n$, and $f'(u)^{-1}$ exists for all u, it follows from a standard inverse function theorem (see, for example, Ref. 7, p. 273) that $(f^{-1})'$ exists and is continuous on R^n. Thus, (*i*) and (*ii*) imply that f is a C^1-diffeomorphism.

2.8 Comments

Our concern throughout this paper is primarily with studies of what is possible. We do not claim that of all applicable Newton-direction algorithms, Process N is the most efficient. In fact, it is not difficult to modify Process N to improve its performance in certain specific cases. Also, results along the lines of this paper can be proved using Newton-direction algorithms that, unlike Process N, do not typically require, for the determination of the γ_k, a one-dimensional search procedure for a finite number of values of k. These results will be reported on in later papers.[16,17]

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