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An Overflow System in Which Queuing Takes Precedence

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When calls offered to a primary group of trunks find all of them busy, provisions are often made for these calls to overflow to other groups of trunks. Such traffic overflow systems have been of interest for a long time, but recently overflow systems that allow for some calls to be queued have been of importance. In this paper we analyze a traffic overflow system with queuing, which consists of a primary and a secondary group. The system which we consider here differs from the two systems we investigated earlier, in that no overflow from the primary to the secondary is permitted if there is a waiting space available in the primary queue. As with the earlier investigations, we adopt an analytical approach which considerably reduces the dimensions of the problem, and simplifies the calculation of various steady-state quantities of interest. Our results include expressions for the loss probabilities, the average waiting times in the queues, and the average number of demands in service in each group.

I. INTRODUCTION

In this paper a particular overflow system with queuing is analyzed. The system consists of two groups, a primary and a secondary, with n_k servers and q_k waiting spaces, which receive demands from independent Poisson sources S_k with arrival rates $\lambda_k > 0$, $k = 1$ and 2, respectively, as depicted in Fig. 1. The service times of the demands are independent, and exponentially distributed with mean service rate $\mu > 0$. If all n_2 servers in the secondary are busy when a demand from S_2 arrives, the demand is queued if one of the q_2 waiting spaces is

In Memoriam: Joanne B. Fry, Associate Editor of The Bell System Technical Journal since 1978, died in an automobile accident January 2, 1981.

available, otherwise it is lost (blocked and cleared from the system). Demands waiting in the secondary queue enter service (in some prescribed order) as servers in the secondary become free.

If all n_1 servers in the primary are busy when a demand from S_1 arrives, the demand is queued in the primary, if one of the q_1 waiting spaces is available. No overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free. If all n_1 servers in the primary are busy and all q_1 waiting spaces are occupied, when a demand from S_1 arrives, the demand is served in the secondary, if there is a free server and there are no demands waiting in the secondary queue, otherwise it is lost.

The overflow system described above differs from the two systems which we investigated earlier,[1,2] in that no overflow from the primary to the secondary is permitted if there is a waiting space available in the primary queue. This restriction was one invoked by Anderson.[3] In the two systems investigated earlier, arriving calls can overflow when the primary queue is not full. The system considered in this paper is a particular case of the one considered by Rath,[4] which was composed of two queues, one of which is allowed to overflow to the other under specified conditions involving the queue lengths. He obtained some numerical solutions using a Gauss–Seidel iteration technique, but none of these correspond to the particular system that we are considering. He also developed an approximate procedure for analyzing his system, based on the use of the Interrupted Poisson Process.

Here we analyze the overflow system using a technique analogous to the one introduced in the earlier paper.[1] Let p_{ij} denote the steady-state probability that there are i demands in the primary and j demands in the secondary, either in service or waiting. These probabilities satisfy a set of generalized birth-and-death equations, which take the form of partial difference equations connecting neighboring states. We carry out an analysis that reduces the dimensions of the problem, which may be considerable in cases of interest. The basic technique is to separate variables in the region away from a certain boundary of state space. This leads to an eigenvalue problem for the separation constant. The eigenvalues are roots of a polynomial equation. The probabilities p_{ij} are then represented in terms of the corresponding eigenfunctions. The constant coefficients in these representations are determined from the boundary conditions and the normalization condition that the sum of the probabilities is unity.

Various steady-state quantities are of interest, which may be expressed in terms of the probabilities p_{ij}. The quantities include the loss (or blocking) probabilities, the average waiting times in the queues, and the average number of demands in service in each group. These quantities may be expressed directly in terms of the constant coefficients which occur in the representations for the probabilities p_{ij}. Thus the steady-state quantities of interest may be calculated directly, without having to calculate the probabilities p_{ij}. Here again the reduction in the dimensions of the problem is valuable.

Only the theoretical results are presented in this paper. Numerical results will be presented in a forthcoming paper by Kaufman, Seery, and Morrison.[5] Results will be given there for the two overflow systems considered previously, based on the earlier analysis,[1] as well as for the system considered in this paper.

Section II discusses the representation of the probabilities p_{ij} in terms of the eigenfunctions, and the boundary and normalization conditions. Various steady-state quantities of interest are calculated in Section III. The appendix gives properties of the eigenfunctions.

We assume throughout the analysis that $q_1 \geq 1$, since the system considered in this paper, and the two systems analyzed earlier, are identical if $q_1 = 0$, i.e., if there is no primary queue. However, the results of this paper reduce to those obtained earlier[1] if $q_1 = 0$. If q_2 is large, or even infinite, an alternate analysis, analogous to that presented for the other two systems,[2] may be carried out for the present system, but we do not pursue that here.

II. REPRESENTATION AND BOUNDARY CONDITIONS

We let p_{ij} denote the steady-state probability that there are i demands in the primary and j demands in the secondary, either in service or waiting. These probabilities satisfy a set of generalized birth-and-death equations,[6] which may be derived in a straightforward manner. We define the traffic intensities

$$a_1 = \lambda_1/\mu, \qquad a_2 = \lambda_2/\mu, \tag{1}$$

and let the total number of servers and waiting spaces in each group be

$$k_1 = n_1 + q_1, \qquad k_2 = n_2 + q_2. \tag{2}$$

It is convenient to introduce the function

$$\chi_l = \begin{cases} 1, & l \geq 0, \\ 0, & l < 0, \end{cases} \tag{3}$$

as well as the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{4}$$

Then the birth-and-death equations are

$$[a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij}$$

$$= a_1(1 - \delta_{i0})p_{i-1,j} + (1 - \delta_{j0})(a_1\delta_{ik_1}\chi_{n_2-j} + a_2)p_{i,j-1}$$

$$+ (1 - \delta_{ik_1})\min(i+1, n_1)p_{i+1,j} + (1 - \delta_{jk_2})\min(j+1, n_2)p_{i,j+1}, \quad (5)$$

for $0 \le i \le k_1$, $0 \le j \le k_2$. The normalization condition is

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} p_{ij} = 1. \quad (6)$$

For $i \ne k_1$, the variables in (5) may be separated, and there are solutions of the form $\alpha_i\beta_j$, where

$$[a_1 + \min(i, n_1) + \rho]\alpha_i = a_1(1 - \delta_{i0})\alpha_{i-1} + \min(i+1, n_1)\alpha_{i+1}, \quad (7)$$

for $0 \le i \le k_1 - 1$, and

$$[a_2(1 - \delta_{jk_2}) + \min(j, n_2) - \rho]\beta_j$$
$$= a_2(1 - \delta_{j0})\beta_{j-1} + (1 - \delta_{jk_2})\min(j+1, n_2)\beta_{j+1}, \quad (8)$$

for $0 \le j \le k_2$, and ρ is a separation constant. Hence, from (7),

$$(a_1 + i + \rho)\alpha_i = a_1(1 - \delta_{i0})\alpha_{i-1} + (i+1)\alpha_{i+1}, \quad (9)$$

for $0 \le i \le n_1 - 1$. The solution of (9) may be expressed in terms of Poisson–Charlier[7,8] polynomials. We here denote the solution of (9) for which $\alpha_0 = 1$ by $s_i(\rho, a_1)$. The properties of $s_i(\rho, a)$ which we will need are given in the appendix.

We assume that $q_1 \ge 1$. Then, from (7),

$$(a_1 + n_1 + \rho)\alpha_i = a_1\alpha_{i-1} + n_1\alpha_{i+1}, \quad (10)$$

for $n_1 \le i \le k_1 - 1$. The solution of (10) may be expressed in terms of Chebyshev polynomials of the second kind,[9] $U_l(x)$. It is convenient to define

$$\Omega_l(\rho) = \left(\frac{n_1}{a_1}\right)^{l/2} U_l\left(\frac{a_1 + n_1 + \rho}{2\sqrt{a_1 n_1}}\right). \quad (11)$$

The appendix gives the properties of these functions that we need. We note here, however, that $U_0(x) \equiv 1$ and $U_{-1}(x) \equiv 0$. From (9), (10), (53), and (64), with a suitable normalization, it follows that

$$\alpha_i(\rho) = \begin{cases} \left(\frac{n_1}{a_1}\right)^{q_1} s_i(\rho, a_1), & 0 \le i \le n_1, \\ \left(\frac{n_1}{a_1}\right)^{k_1 - i} [s_{n_1}(\rho, a_1)\Omega_{i-n_1}(\rho) - s_{n_1-1}(\rho, a_1)\Omega_{i-n_1-1}(\rho)], & n_1 \le i \le k_1. \end{cases} \quad (12)$$

Next, from (8),

$$(a_2 + j - \rho)\beta_j = a_2(1 - \delta_{j0})\beta_{j-1} + (j+1)\beta_{j+1}, \quad (13)$$

for $0 \le j \le n_2 - 1$. It follows from (53) that β_j is proportional to $s_j(-\rho, a_2)$ for $0 \le j \le n_2$. Also,

$$[a_2(1 - \delta_{jk_2}) + n_2 - \rho]\beta_j = a_2\beta_{j-1} + n_2(1 - \delta_{jk_2})\beta_{j+1}, \tag{14}$$

for $n_2 \le j \le k_2$. Corresponding to (11), we define

$$\Psi_l(\rho) = \left(\frac{n_2}{a_2}\right)^{l/2} U_l\left(\frac{a_2 + n_2 - \rho}{2\sqrt{a_2 n_2}}\right). \tag{15}$$

We also define

$$\phi_j(\rho) = \Psi_{k_2-j}(\rho) - \Psi_{k_2-j-1}(\rho). \tag{16}$$

The appendix gives the properties of these functions that we need. It follows from (14) and (62) that β_j is proportional to $\phi_j(\rho)$ for $n_2 - 1 \le j \le k_2$.

Consequently, we take

$$\beta_j(\rho) = \begin{cases} s_j(-\rho, a_2)\phi_{n_2}(\rho), & 0 \le j \le n_2, \\ s_{n_2}(-\rho, a_2)\phi_j(\rho), & n_2 - 1 \le j \le k_2, \end{cases} \tag{17}$$

where

$$s_{n_2-1}(-\rho, a_2)\phi_{n_2}(\rho) = s_{n_2}(-\rho, a_2)\phi_{n_2-1}(\rho). \tag{18}$$

This equation may be written in the form

$$\rho[s_{n_2}(1 - \rho, a_2)\Psi_{q_2}(\rho) - s_{n_2-1}(1 - \rho, a_2)\Psi_{q_2-1}(\rho)] = 0. \tag{19}$$

The expression in the square brackets in (19) is a polynomial in ρ of degree $k_2 = n_2 + q_2$. It was shown[1] that its zeros are positive and distinct, and we denote them by ρ_m, $m = 1, \cdots, k_2$. We also let $\rho_0 = 0$. It follows that we may represent the probabilities p_{ij} in the form

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} c_m\alpha_i(\rho_m)s_j(-\rho_m, a_2)\phi_{n_2}(\rho_m), & 0 \le j \le n_2, \\ \sum_{m=0}^{k_2} c_m\alpha_i(\rho_m)s_{n_2}(-\rho_m, a_2)\phi_j(\rho_m), & n_2 \le j \le k_2, \end{cases} \tag{20}$$

for $0 \le i \le k_1$, where $\alpha_i(\rho)$ is defined in (12), and the constants c_m are to be determined.

It remains to satisfy the boundary conditions corresponding to $i = k_1$ in (5), as well as the normalization condition (6). If we set $i = k_1$ in (5), we obtain

$$(a_1 + a_2 + n_1 + j)p_{k_1,j} = a_1 p_{k_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{k_1,j-1} + (j + 1)p_{k_1,j+1}, \tag{21}$$

for $0 \le j \le n_2 - 1$,

$$[a_2(1 - \delta_{q_2,0}) + n_1 + n_2]p_{k_1,n_2} = a_1 p_{k_1-1,n_2} + (a_1 + a_2)p_{k_1,n_2-1} + n_2(1 - \delta_{q_2,0})p_{k_1,n_2+1}, \tag{22}$$

and, if $q_2 \geq 1$,

$$[a_2(1 - \delta_{jk_2}) + n_1 + n_2]p_{k_1,j} = a_1 p_{k_1-1,j} + a_2 p_{k_1,j-1} + n_2(1 - \delta_{jk_2})p_{k_1,j+1}, \quad (23)$$

for $n_2 + 1 \leq j \leq k_2$.

If we substitute (20) into (21), after reduction with the help of the recurrence relations in the appendix, we find that

$$\sum_{m=0}^{k_2} c_m\{\rho_m[s_{n_1}(1 + \rho_m, a_1)\Omega_{q_1}(\rho_m) - s_{n_1-1}(1 + \rho_m, a_1)\Omega_{q_1-1}(\rho_m)]s_j(-\rho_m, a_2) + a_1[s_{n_1}(\rho_m, a_1)\Omega_{q_1}(\rho_m) - s_{n_1-1}(\rho_m, a_1)\Omega_{q_1-1}(\rho_m)]s_j(-1 - \rho_m, a_2)\} \cdot \phi_{n_2}(\rho_m) = 0, \quad (24)$$

for $0 \leq j \leq n_2 - 1$. Also, from (23), it is found that

$$\sum_{m=0}^{k_2} c_m \rho_m s_{n_2}(-\rho_m, a_2)[s_{n_1}(1 + \rho_m, a_1)\Omega_{q_1}(\rho_m) - s_{n_1-1}(1 + \rho_m, a_1)\Omega_{q_1-1}(\rho_m)]\phi_j(\rho_m) = 0, \quad (25)$$

for $n_2 + 1 \leq j \leq k_2$. It may be shown that the boundary condition (22) is redundant, as is to be expected. Thus the constants c_m are determined by (24) and (25) only to within a multiplicative constant, which is determined from the normalization condition (6).

From (20), with the help of (16), (19), (57), and (58), it is found that

$$\sum_{j=0}^{k_2} p_{ij} = c_0 \alpha_i(0)[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)], \quad (26)$$

for $0 \leq i \leq k_1$. But, from (12) and (66),

$$\alpha_i(0) = \begin{cases} \left(\dfrac{n_1}{a_1}\right)^{q_1} s_i(0, a_1), & 0 \leq i \leq n_1, \\[2ex] \left(\dfrac{n_1}{a_1}\right)^{k_1 - i} s_{n_1}(0, a_1), & n_1 \leq i \leq k_1. \end{cases} \quad (27)$$

Hence, from (26) and (27), with the help of (57), (58), and (65), the normalization condition (6) implies that

$$c_0[s_{n_1}(1, a_1)\Omega_{q_1}(0) - s_{n_1-1}(1, a_1)\Omega_{q_1-1}(0)] \cdot [s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)] = 1. \quad (28)$$

Once the constants c_m have been determined, the steady-state probabilities p_{ij} may be calculated from (20). We remark that the number of constants to be determined is only $k_2 + 1$, whereas the number of probabilities p_{ij} is $(k_1 + 1)(k_2 + 1)$.

III. SOME STEADY-STATE QUANTITIES

We proceed now to the calculation of various steady-state quantities of interest. These quantities are depicted in the diagram of Fig. 1, which indicates the mean flow rates. The loss probabilities L_1 and L_2 are given by

$$L_1 = \sum_{j=n_2}^{k_2} p_{k_1,j}, \qquad L_2 = \sum_{i=0}^{k_1} p_{i,k_2}, \tag{29}$$

and the probabilities that a demand from the primary, or secondary, source is queued on arrival are

$$Q_1 = \sum_{i=n_1}^{k_1-1} \sum_{j=0}^{k_2} p_{ij}, \qquad Q_2 = (1 - \delta_{q_2,0}) \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2-1} p_{ij}. \tag{30}$$

The probability that a demand arriving from the primary source

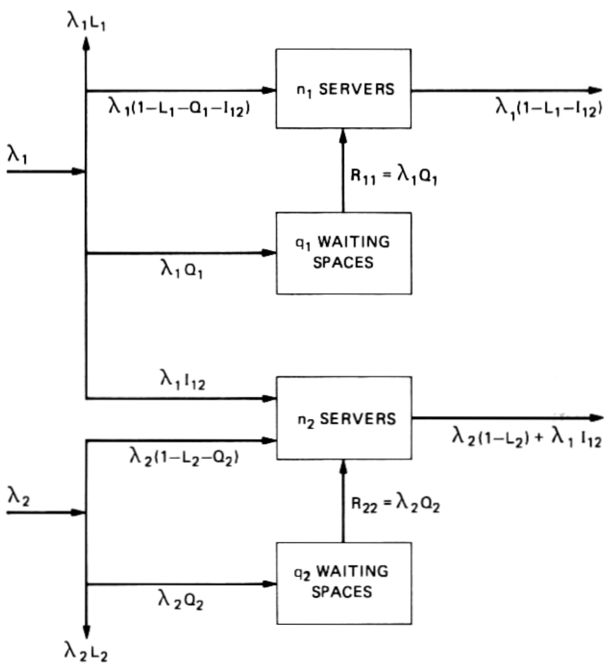

Fig. 1—Mean flow rates for an overflow system with queuing; Poisson arrival rates λ_1 and λ_2, loss probabilities L_1 and L_2, queuing probabilities Q_1 and Q_2, and overflow probability I_{12}.

overflows (immediately) is

$$I_{12} = \sum_{j=0}^{n_2-1} p_{k_1,j}. \tag{31}$$

Since the mean service rate is μ, the mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1\mu \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} p_{ij}, \tag{32}$$

while the mean departure rate from the secondary queue to the secondary servers is

$$R_{22} = n_2\mu(1 - \delta_{q_2,0}) \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} p_{ij}. \tag{33}$$

The average number of demands in the primary and secondary queues are

$$V_1 = \sum_{i=n_1}^{k_1} \sum_{j=0}^{k_2} (i - n_1)p_{ij}, \qquad V_2 = \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2} (j - n_2)p_{ij}. \tag{34}$$

Also, the average number of demands in service in the two groups are

$$X_1 = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \min(i, n_1)p_{ij}, \qquad X_2 = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \min(j, n_2)p_{ij}. \tag{35}$$

Now, according to Little's theorem,[6] the average number of demands in a queuing system is equal to the average rate of arrival of demands to that system times the average time spent in that system. If we apply this result to the primary and secondary queues, we find that the average waiting times of the demands which are queued in the primary or in the secondary are given by

$$W_1 = \frac{V_1}{\lambda_1 Q_1}, \qquad W_2 = \frac{V_2}{\lambda_2 Q_2} \qquad (q_2 \geq 1), \tag{36}$$

respectively, independently of the order of service within each queue. Also, if we apply Little's theorem to the primary and secondary groups of servers, we obtain

$$\lambda_1(1 - L_1 - I_{12}) = \mu X_1, \qquad \lambda_2(1 - L_2) + \lambda_1 I_{12} = \mu X_2. \tag{37}$$

The steady-state quantities of interest may be expressed in terms of the constants c_m with the help of the representations in (20). From (29) it is found, with the help of (12) and (16), that

$$L_1 = \sum_{m=0}^{k_2} c_m[s_{n_1}(\rho_m, a_1)\Omega_{q_1}(\rho_m) - s_{n_1-1}(\rho_m, a_1)\Omega_{q_1-1}(\rho_m)]$$
$$\cdot s_{n_2}(-\rho_m, a_2)\Psi_{q_2}(\rho_m). \tag{38}$$

We define

$$d_0 = c_0[s_{n_2}(1, a_2)\Psi_{q_2}(0) - s_{n_2-1}(1, a_2)\Psi_{q_2-1}(0)]$$
$$= [s_{n_1}(1, a_1)\Omega_{q_1}(0) - s_{n_1-1}(1, a_1)\Omega_{q_1-1}(0)]^{-1}, \quad (39)$$

from (28). Then, from (30) it is found, with the help of (26), (27), and (65), that

$$Q_1 = d_0 s_{n_1}(0, a_1)[\Omega_{q_1}(0) - 1]. \quad (40)$$

Moreover, from (29) and (31), it follows that

$$L_1 + I_{12} = d_0 s_{n_1}(0, a_1), \quad (41)$$

and from (32) it follows that $R_{11} = \lambda_1 Q_1$, as is to be expected, since in the steady state the mean departure rate from the queue is equal to the mean arrival rate to it.

We define

$$\Delta_q(\xi) = \sum_{l=1}^{q} l\xi^{q-l} = \begin{cases} [q - (q+1)\xi + \xi^{q+1}]/(1-\xi)^2, & \xi \neq 1, \\ \tfrac{1}{2}\, q(q+1), & \xi = 1. \end{cases} \quad (42)$$

Then, from (34), with the help of (26), (27), and (39), it is found that

$$V_1 = d_0 s_{n_1}(0, a_1)\Delta_{q_1}\left(\frac{n_1}{a_1}\right). \quad (43)$$

Also, from (35), with the help of (56), (58), (59), and (65), it follows that

$$X_1 = d_0 a_1 \left(\frac{n_1}{a_1}\right)^{q_1} s_{n_1-1}(1, a_1) + s_{n_1}(0, a_1)[\Omega_{q_1}(0) - 1]\Big\}. \quad (44)$$

It may be verified, with the help of (39), (57), and (65), that (41) and (44) are consistent with (37). The explicitness of the expressions for the quantities in (40), (41), (43), and (44) is due to the fact that these quantities are not affected by the secondary. This, of course, is not the case for the loss probability L_1, which is given by (38).

Next, from (29), since $\phi_{k_2}(\rho) \equiv 1$, it is found, with the help of (20) and (68), that

$$L_2 = \sum_{m=0}^{k_2} c_m s_{n_2}(-\rho_m, a_2)[s_{n_1}(1 + \rho_m, a_1)\Omega_{q_1}(\rho_m)$$
$$- s_{n_1-1}(1 + \rho_m, a_1)\Omega_{q_1-1}(\rho_m)]. \quad (45)$$

Also, from (35), with the help of (6) and (59), it follows that

$$X_2 = n_2 - \sum_{m=0}^{k_2} c_m s_{n_2-1}(2 - \rho_m, a_2)\phi_{n_2}(\rho_m)$$
$$\cdot[s_{n_1}(1 + \rho_m, a_1)\Omega_{q_1}(\rho_m) - s_{n-1}(1 + \rho_m, a_1)\Omega_{q_1-1}(\rho_m)]. \quad (46)$$

In view of (38), (41), (45), and (46), the second relationship in (37) provides a useful numerical check.

We now define

$$r_j = \sum_{i=0}^{k_1} p_{ij}, \qquad n_2 \le j \le k_2. \tag{47}$$

If we sum on i in (5), we obtain

$$[a_2(1 - \delta_{jk_2}) + n_2]r_j = a_2 r_{j-1} + n_2(1 - \delta_{jk_2})r_{j+1}, \tag{48}$$

for $n_2 + 1 \le j \le k_2$. It follows that

$$n_2 r_j = a_2 r_{j-1}, \qquad n_2 + 1 \le j \le k_2. \tag{49}$$

Hence, since $L_2 = r_{k_2}$, from (29) and (47),

$$r_j = \left(\frac{n_2}{a_2}\right)^{k_2 - j} L_2, \qquad n_2 \le j \le k_2. \tag{50}$$

Then, from (30) and (33), with the help of (63), we obtain

$$Q_2 = [\Psi_{q_2}(0) - 1]L_2, \tag{51}$$

and $R_{22} = \lambda_2 Q_2$, as is to be expected. Also, from (34) and (42), it follows that

$$V_2 = \Delta_{q_2}\left(\frac{n_2}{a_2}\right)L_2. \tag{52}$$

This completes the calculation of expressions for the steady-state quantities of interest.

IV. ACKNOWLEDGMENT

The author is grateful to G. M. Anderson for bringing this problem to his attention.

APPENDIX

We define $s_i(\lambda, a)$ by the recurrence relation

$$(a + i + \lambda)s_i(\lambda, a) = a(1 - \delta_{i0})s_{i-1}(\lambda, a) + (i + 1)s_{i+1}(\lambda, a);$$
$$s_0(\lambda, a) = 1, \tag{53}$$

for $i = 0, 1, \cdots$. Thus $s_n(\lambda, a)$ is a polynomial of degree n in both λ and a, and it may be related to a Poisson–Charlier polynomial.[7,8] However, we give here the properties of $s_n(\lambda, a)$ which we will need. An explicit formula is[1]

$$s_i(\lambda, a) = \sum_{r=0}^{i} \frac{(\lambda)_r a^{i-r}}{r!(i-r)!}, \tag{54}$$

where

$$(\lambda)_0 = 1, \qquad (\lambda)_r = \lambda(\lambda + 1) \cdots (\lambda + r - 1), \qquad r = 1, 2, \cdots. \tag{55}$$

It was also shown[1] that

$$(i + 1)s_{i+1}(\lambda, a) = as_i(\lambda, a) + \lambda s_i(\lambda + 1, a) \tag{56}$$

and

$$s_i(\lambda, a) = s_i(\lambda + 1, a) - (1 - \delta_{i0})s_{i-1}(\lambda + 1, a). \tag{57}$$

From (57) it follows that

$$\sum_{i=0}^{n} s_i(\lambda, a) = s_n(\lambda + 1, a), \tag{58}$$

and, from (56) and (58), we deduce that

$$\sum_{i=0}^{n} (n - i)s_i(\lambda, a) = (1 - \delta_{n0})s_{n-1}(\lambda + 2, a). \tag{59}$$

We now turn our attention to the Chebyshev polynomials of the second kind,[9] $U_l(x)$. They may be defined by the recurrence relation

$$2xU_l(x) = U_{l+1}(x) + U_{l-1}(x); \qquad U_{-1}(x) \equiv 0, \quad U_0(x) \equiv 1, \tag{60}$$

for $l = 0, 1, \cdots$. From (15) and (60) it follows that

$$(a_2 + n_2 - \rho)\Psi_l(\rho) = a_2\Psi_{l+1}(\rho) + n_2\Psi_{l-1}(\rho),$$
$$\Psi_{-1}(\rho) \equiv 0, \qquad \Psi_0(\rho) \equiv 1. \tag{61}$$

From (16) and (61) we deduce that

$$[a_2(1 - \delta_{jk_2}) + n_2 - \rho]\phi_j(\rho) = a_2\phi_{j-1}(\rho) + n_2(1 - \delta_{jk_2})\phi_{j+1}(\rho), \tag{62}$$

for $j \le k_2$. Also, from (61), it may be shown by induction that

$$\Psi_l(0) = \sum_{r=0}^{l} \left(\frac{n_2}{a_2}\right)^r. \tag{63}$$

Next, from (11) and (60) it follows that

$$(a_1 + n_1 + \rho)\Omega_l(\rho) = a_1\Omega_{l+1}(\rho) + n_1\Omega_{l-1}(\rho),$$
$$\Omega_{-1}(\rho) \equiv 0, \qquad \Omega_0(\rho) \equiv 1. \tag{64}$$

In particular, it is found by induction that

$$\Omega_l(0) = \sum_{r=0}^{l} \left(\frac{n_1}{a_1}\right)^r. \tag{65}$$

Since $s_i(0, a) = a^i/i!$, from (54), it follows that

$$s_{n_1}(0, a_1)\Omega_l(0) - s_{n_1-1}(0, a_1)\Omega_{l-1}(0) = s_{n_1}(0, a_1), \qquad l = 0, 1, \cdots. \tag{66}$$

Next, from (9) and (10), we deduce that

$$\rho \sum_{i=0}^{k_1} \alpha_i(\rho) = (n_1 + \rho)\alpha_{k_1}(\rho) - a_1\alpha_{k_1-1}(\rho). \tag{67}$$

Then, with the help of (12), (56), (57), and (64), it may be shown that

$$\sum_{i=0}^{k_1} \alpha_i(\rho) = s_{n_1}(1 + \rho, a_1)\Omega_{q_1}(\rho) - s_{n_1-1}(1 + \rho, a_1)\Omega_{q_1-1}(\rho), \tag{68}$$

for $\rho \neq 0$. Moreover, this result holds for $\rho = 0$ also, from continuity.

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