THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 60, No. 10, December 1981
Printed in U.S.A.

On The Performance of Phase-Shift-Keying Systems

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(Manuscript received June 18, 1981)

Coherent phase-shift keying (CPSK) and differential phase-shift keying (DPSK) are widely used modulation methods in digital communications. Bandwidth efficiency, good noise immunity, constant envelope, and simplicity of implementation make these schemes particularly attractive for use over the satellite, terrestrial radio and voiceband telephone channels. While system analyses abound in the literature, treatment is usually restricted to the additive Gaussian channel. Important issues determining ultimate performance, such as the joint effect of intersymbol interference and the acquisition of carrier phase have not been adequately addressed. The main purpose of this paper is to develop analytical tools that can be used to assess system performance under practical operating conditions. Pure coherent demodulation schemes such as CPSK are ideals which are rarely achieved in practice, and carrier phase must be estimated prior to and/or during data transmission. This requires start-up time, as well as added equipment, and the fidelity of the phase estimate ultimately determines performance. In contrast, DPSK is independent of carrier phase, since decisions are made on phase differences. However, this comes at a price, and it is known that ideal multiphase DPSK suffers an asymptotic performance penalty of 3 dB in signal-to-noise ratio (s/n) over ideal CPSK. We develop a new rigorous method for calculating the error rates of both CPSK and DPSK, under a variety of operating conditions. In particular, we find that the intersymbol interference penalty for quaternary DPSK is about 1 dB worse in s/n than for CPSK. We demonstrate that the detection efficiency of CPSK approaches the ideal, provided that the s/n of the phase-recovery circuit is about 10 dB more than that at the receiver input. Alternatively, for the same s/n, a 10-baud phase-locked loop integration time is required to achieve near-ideal performance.

I. INTRODUCTION

Coherent phase-shift keying (CPSK) and differential phase-shift keying (DPSK) are two techniques often used in digital communications over channels such as satellite, terrestrial radio, and voiceband telephone. The literature abounds in analyses of their performance under a variety of conditions. A sample collection of some of this literature may be found in Ref. 1. The chief reasons for the widespread use of these techniques are simplicity of implementation, superior performance over the additive Gaussian noise channel, minimal bandwidth occupancy, and minimal envelope variation.

The relative performance of CPSK and DPSK systems is well understood only in the presence of additive Gaussian noise. In this case, the detection efficiency of DPSK is known to be about 1 dB (in s/n) below that of CPSK for binary modulation and this degradation approaches 3 dB for multilevel systems. In applications where a 3-dB loss in s/n is important, such as in down-link satellite, space communications, and terrestrial radio under deep fading conditions, CPSK is the preferred method. In CPSK, however, the generation and extraction of a local carrier-phase reference at the receiver is required. A coherent phase estimate is usually obtained by using phase-locked loop (PLL) techniques, and because of frequency instabilities and phase jitter inherent in transmitter and receiver systems, carrier recovery loop bandwidths cannot be made arbitrarily small. Consequently, in practice a noisy phase estimate is obtained and only partial coherent reception can be claimed. The reason for using DPSK is its immunity from slow carrier-phase fluctuations; therefore, the phase recovery problem inherent in CPSK is avoided. However, the detection efficiency of DPSK may approach that of CPSK under noisy phase estimation conditions and intersymbol interference (ISI). The need to understand this phenomenon on a fundamental level is the principal objective of this paper.

As bandwidth occupancy is always important, the effects of ISI generated by the use of band-limiting filters must be taken into account in any analysis of these systems. Because of the linear nature of the demodulation process in CPSK, the effect of ISI has been treated in great detail. Since DPSK demodulation is inherently nonlinear, the analysis of performance is very difficult and no adequate analytical methods are currently available. Also, the combined effects of imperfect phase estimation and ISI on CPSK must be determined so that the relative detection efficiencies of band-limited DPSK and CPSK can be fairly assessed.

In Section II of this paper, we describe a technique for determining the degradation in M-ary CPSK operating in the presence of ISI, additive Gaussian noise, and imperfect carrier phase. In Section III, we consider the performance of M-ary DPSK subject to ISI and additive Gaussian noise.

II. COHERENT DETECTION

2.1 System description of CPSK

Figure 1 shows the M-ary CPSK system that we consider. The signal, $s(t)$, before the transmit filter can be represented as

$$s(t) = \text{Re}\{Ax(t)\exp[i(2\pi f_c t + \mu)]\},\ i = \sqrt{-1}, \tag{1}$$

where the baseband modulation signal is

$$x(t) = \sum_{k=-\infty}^{\infty} \exp(i\alpha_k)\text{rect}[(t - kT)/T], \tag{2}$$

and the constants A, f_c, and μ are the carrier amplitude, frequency (in Hz), and phase, respectively. Also, rect$(\cdot)$ is the rectangular window function, T the signaling interval, and the sequence of discrete phases $[\alpha_k]$ corresponds to the data sequence to be transmitted. Without loss of generality, we assume that the M phase values of α_k are uniformly distributed with equal probability between $(-\pi, \pi]$. So, α_k takes on value in the set $\alpha_k \epsilon \Lambda$,

$$\Lambda \triangleq \left[\frac{\pi}{M}, \frac{3\pi}{M}, \cdots, \frac{2M-1}{M}\pi\right].$$

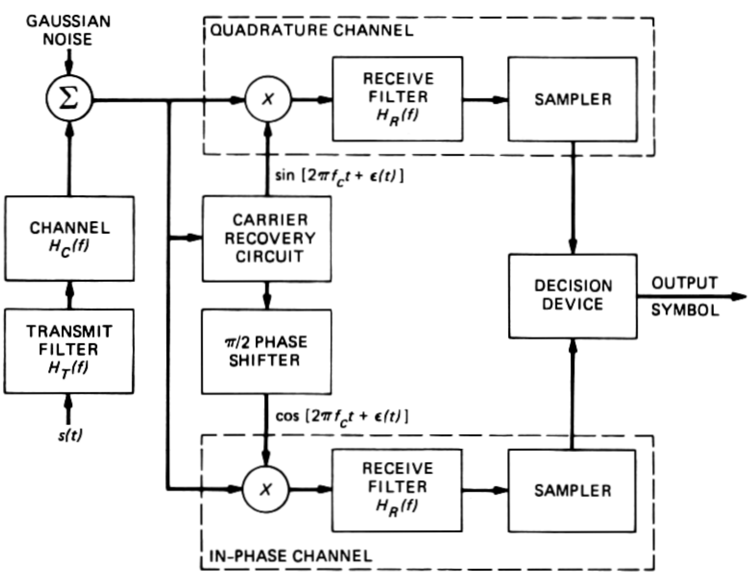

Fig. 1—M-ary CPSK receiver, $M > 2$.

We also assume that the data phases in different time slots are statistically independent.

In our model, the transmit filter, transmission channel, and receive filter are linear and time invariant. Therefore, the complex envelope, $y(t)$, at the output of the receive filter may be written as

$$y(t) = x(t) \odot h(t) + n(t) + i\hat{n}(t)$$

$$h(t) = h_T(t) \odot h_C(t) \odot h_R(t),$$

where $h_T(t)$, $h_C(t)$, and $h_R(t)$ are, respectively, the impulse response of the transmit filter, the channel, and the receive filter. The symbol $\odot$ denotes convolution. Also, $n(t) + i\hat{n}(t)$ is the complex envelope of the Gaussian noise passed through the receive filter. For symmetrical filters, $n(t)$ and $\hat{n}(t)$ are independently and identically distributed (iid) Gaussian random variables with mean zero, and variance

$$\sigma^2 = N_0 \int_{-\infty}^{\infty} |H_R(f)|^2 df,$$

where N_0 is the double-sided spectral density of the original white noise and $H_R(f)$ is the baseband equivalent transfer function of the receive filter.

2.1.1 Detection in CPSK

Assuming that the recovered carrier is $\exp[i(2\pi f_c t + \hat{\mu})]$, where $\hat{\mu}$ is an estimate of μ in eq. (1), the detector operates on the signal, $w(t)$, represented as

$$w(t) = \sum_{k=-\infty}^{\infty} z(t - kT)\exp[i(\alpha_k + \epsilon)] + \xi + i\eta, \tag{3}$$

where ξ and η are iid gaussian random variables with mean zero, and variance σ^2,

$$\epsilon = \mu - \hat{\mu},$$

is the phase error, and

$$z(t) = h_T(t) \odot h_C(t) \odot h_R(t) \odot \text{rect}\left(\frac{t}{T}\right).$$

To estimate the transmitted phase, $\alpha_0 = \Phi \epsilon \Lambda$ at $t = 0$, an ideal CPSK detector measures the phase θ of $w(t)$ at $t = t_0$, and a correct decision results when

$$\Phi - \frac{\pi}{M} < \theta < \Phi + \frac{\pi}{M},$$

$$\theta = \text{phase angle of } w(t_0),$$

$$w(t_0) = w(t)\Big|_{\substack{t=t_0 \\ \alpha_0=\Phi}}. \tag{4}$$

2.1.2 Error rate for M-ary CPSK

Here, we briefly review some known results for CPSK and then develop new results applicable to our more general model.

Error-rate calculations for ideal CPSK in added Gaussian noise can be found in Refs. 2 to 7. References 8 and 9 provide numerical methods for calculating the probability of error in the presence of ISI. Reference 10 takes into account ISI and demodulation phase error, but the results are restricted to only binary and quaternary systems. We now generalize these results.

Using the union bound and the representation of the received signal, eq. (3), it follows from eq. (4) that the probability of error, Pe(|Φ), given that the phase Φ is transmitted, is

$$\max(P_1, P_2) \leq \mathrm{Pe}(|\Phi) \leq P_1 + P_2,$$

where

$$\begin{aligned} P_1 &= \Pr\left[\sin\left(\theta - \Phi + \frac{\pi}{M}\right) < 0\right] \\ &= \Pr\left\{\mathrm{Im}\; w(t_0)\exp\left[-i\left(\Phi - \frac{\pi}{M}\right)\right] < 0\right\}, \\ P_2 &= \Pr\left[\sin\left(\theta - \Phi - \frac{\pi}{M}\right) > 0\right] \\ &= \Pr\left\{\mathrm{Im}\; w(t_0)\exp\left[-i\left(\Phi + \frac{\pi}{M}\right)\right] > 0\right\}. \end{aligned} \tag{5}$$

Note that the average symbol probability of error Pe is

$$\mathrm{Pe} = \frac{1}{M} \sum_{\Phi \in \Lambda} \mathrm{Pe}(|\Phi).$$

But, since the signal constellation is assumed to be circularly symmetric, Pe(|Φ) is independent of Φ.

For convenience, we shall now assume that $\Phi = \pi/M$. Hence,

$$\begin{aligned} P_1 = \Pr\Bigg(\mathrm{Im}\bigg\{r_0\exp\left[i\left(\beta_0 + \frac{\pi}{M} + \epsilon\right)\right] \\ + \textstyle\sum' r_k\exp\left[i\left(\beta_k + \alpha_k + \epsilon\right)\right]\bigg\} + \eta < 0\Bigg), \end{aligned} \tag{6}$$

where

$$r_k\exp(i\beta_k) = z(t_0 - kT)$$

and $\sum'$ denotes the exclusion of the term $k = 0$. A similar expression can be written for P_2.

Accurate estimations of P_1 and P_2 are easy to obtain in the presence of only Gaussian noise, but are more difficult when ISI is added and are even more tedious when the distribution of carrier-phase error, ϵ, must be taken into account.

In the next section we derive an exponentially tight upper bound on these quantities for a fixed carrier-phase error and then perform asymptotic [large signal-to-noise ratio (s/n)] analyses on these upper bounds for a given distribution of carrier-phase error.

2.1.3 Bounds on the error rate

We begin by writing eq. (6) as

$$P_1 = \left\langle \Pr\left[I(\epsilon) + \eta < -r_0 \sin\left(\frac{\pi}{M} + \beta_0 + \epsilon\right) \Big|_{\epsilon} \right] \right\rangle_{\epsilon}, \tag{7}$$

where $\langle\ \rangle_\epsilon$ denotes expectation with respect to ϵ, and where,

$$I(\epsilon) = \sum{}' r_k \sin(\beta_k + \alpha_k + \epsilon). \tag{8}$$

Before we can proceed with eq. (7), we need specific information on the probability density function (pdf) of the demodulating phase error ϵ. We shall assume that the phase reference is derived from a pure tone by a first-order PLL. It is well known[1] that the resulting pdf for the phase error, ϵ, is

$$p_\epsilon(\epsilon) = \frac{\exp(\lambda \cos \epsilon)}{2\pi I_0(\lambda)}, \; |\epsilon| \le \pi, \tag{9}$$

where λ is the s/n at the input to the PLL multiplied by the reciprocal of the PLL bandwidth,

$$\lambda = \frac{G}{N_p B_L}. \tag{10}$$

In eq. (10), G is the average power in the carrier, N_p is the double-sided noise spectral density, and B_L is the noise bandwidth of the linearized PLL. Also, in eq. (9), $I_0(x)$ represents the modified Bessel function of the first kind and of order 0. For a second-order PLL, the pdf of ϵ is also approximately given by eq. (9). We shall use this density to obtain bounds on P_1.

Since ϵ is a symmetric random variable, eq. (7) yields

$$P_1 = \frac{1}{2} \langle V(\epsilon) + V(-\epsilon) \rangle_\epsilon,$$

where

$$V(\epsilon) \triangleq \frac{1}{2} \operatorname{erfc}\left[\frac{r_0 \sin[(\pi/M) + \beta_0 + \epsilon] + \sum' r_k \sin(\alpha_k + \beta_k + \epsilon)}{\sqrt{2}\, \sigma} \right]. \tag{11}$$

Using upper bounding techniques and Laplace's method,[11] we show in Appendix A that

$$P_1 \leq J_{1a} + J_{2a},$$

where

$$J_{1a} \approx \frac{1}{2}\left[\frac{\exp\{-\rho^2[\sin^2[(\pi/M) + \beta_0 - \epsilon_0] + D(1 - \cos \epsilon_0)]\}}{\{\cos \epsilon_0 + (2/D) \cos 2[(\pi/M) + \beta_0 - \epsilon_0]\}^{1/2}}\right.$$
$$\left. + \frac{\exp\{-\rho^2 \sin^2[(\pi/M) + \beta_0]\}}{\{1 + (2/D) \cos 2[(\pi/M) + \beta_0]\}^{1/2}}\right], \rho^2 = \frac{r_0^2}{2(\sigma^2 + \sigma_I^2)} \gg 1,$$
$$\sigma_I^2 = \sum' r_k^2,\ D = \frac{\lambda}{\rho^2},$$
$$\epsilon_0 = \frac{1}{D}\sin 2\left(\frac{\pi}{M} + \beta_0\right)\left[1 - \frac{2}{D}\cos 2\left(\frac{\pi}{M} + \beta_0\right) + \cdots\right],\ D \gg 1.$$

and

$$J_{2a} \approx \left(1 - \frac{\delta}{\pi}\right)\sqrt{2\pi D\rho^2}\exp[-D\rho^2(1 - \cos \delta)],\ \delta = \frac{\pi}{M} + \beta_0.$$

Note that ρ^2 is the s/n of the system. Also, D can be regarded as the ratio of s/n in the phase recovery circuit to that in the PSK system or the integration time in bauds.

Similarly, we can show that

$$P_2 \leq J_{1a} + J_{2a}.$$

In summary, the average symbol probability of error, Pe, for M-ary CPSK system can be upper bounded by

$$\text{Pe} \leq \frac{\exp\{-\rho^2[\sin^2[(\pi/M) + \beta_0 - \epsilon_0] + D(1 - \cos \epsilon_0)]\}}{\{\cos \epsilon_0 + (2/D) \cos 2[(\pi/M) + \beta_0 - \epsilon]\}^{1/2}}$$
$$+ \frac{\exp\{-\rho^2 \sin^2[(\pi/M) + \beta_0]\}}{\{1 + 2/D \cos 2(\pi/M + \beta_0)\}^{1/2}}$$
$$+ 2\left[1 - \frac{(\pi/M) + \beta_0}{\pi}\right]\sqrt{2\pi D\rho^2}$$
$$\times \exp\left\{-D\rho^2\left[1 - \cos\left(\frac{\pi}{M} + \beta_0\right)\right]\right\},\ \rho^2 \gg 1,$$
$$\epsilon_0 = \frac{1}{D}\sin 2\left(\frac{\pi}{M} + \beta_0\right)\left[1 - \frac{2}{D}\cos 2\left(\frac{\pi}{M} + \beta_0\right) + \cdots\right],\ D \gg 1.$$

This upper bound becomes

$$P \leq 2\exp\left[-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0\right)\right], \tag{12}$$

when phase estimation is perfect, $D \to \infty$. Equation (12) is the well-known Chernoff bound for M-ary CPSK.[12]

If the observation interval of the PLL is large, $D \gg 1$, and if $M \gg 1$,

$$\epsilon_0 \approx \frac{1}{D} \sin 2\left(\frac{\pi}{M} + \beta_0\right)\left[1 - \frac{2}{D} \cos 2\left(\frac{\pi}{M} + \beta_0\right)\right],$$

and

$$\begin{aligned} P \leq & \exp\left(-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0\right)\right. \\ & \left.\times\left\{1 - \frac{2\cos^2[(\pi/M) + \beta_0]}{D}\left[1 - \frac{2}{D}\cos 2\left(\frac{\pi}{M} + \beta_0\right)\right]\right\}\right) \\ & + \exp\left[-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0\right)\right] \\ & \sim \exp\left(-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0\right)\right. \\ & \left.\times\left\{1 - \frac{2\cos^2[(\pi/M) + \beta_0]}{D}\left[1 - \frac{2}{D}\cos 2\left(\frac{\pi}{M} + \beta_0\right)\right]\right\}\right), \end{aligned}$$

$$\rho^2 = \frac{r_0^2}{2(\sigma^2 + \sigma_I^2)}, \ \rho^2 \gg 1, \ M > 2, \ D \gg 1. \tag{13}$$

Comparing eqs. (12) and (13), we see that the degradation in s/n because of imperfect phase estimate for multiphase CPSK systems is asymptotically given by

$$G = \left[\left\{1 - \frac{2}{D} \cos^2\left(\frac{\pi}{M} + \beta_0\right)\right\}\left\{1 - \frac{2}{D} \cos 2\left(\frac{\pi}{M} + \beta_0\right)\right\}\right]^{-1},$$

where $G \to 1$ as $D \to \infty$ as it should.

2.2 Example of quaternary ($M = 4$) CPSK system

Let us consider a quaternary ($M = 4$) CPSK system and assume that the channel is ideal.

If 4-pole Butterworth transmit and receive filters are used, the resulting average symbol probability of error is plotted in Fig. 2. Note that the bound is fairly tight and when the s/n of the phase recovery circuit is about 10 dB more than at the receiver input, the detection efficiency of CPSK is essentially determined by ISI alone. Alternatively, we can say that, for the same s/n, a 10-baud PLL integration time is required to achieve this ISI-limited performance. For this filter, the ISI penalty is about 1 dB.

If $M > 2$, it is well known that the penalty in s/n because of Gaussian noise alone is asymptotically given by $1/[\sin^2(\pi/M)]$.

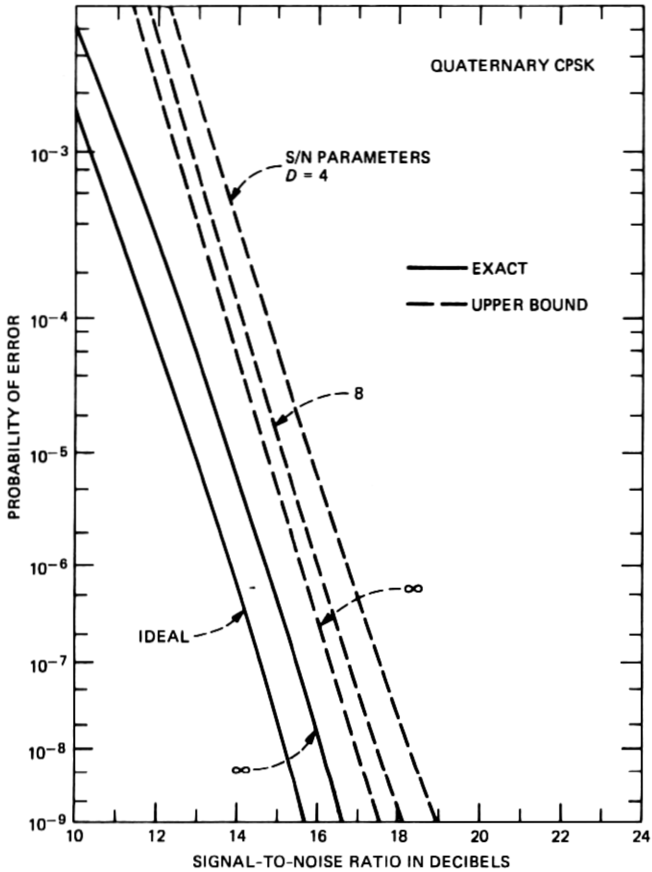

Fig. 2—Probability of error for quaternary phase-shift keying (QPSK) with rectangular signaling, noisy carrier-phase recovery, and 4-pole Butterworth transmit and receive filters. The s/n in decibels is defined as 10 $\log_{10}[T/2N_0]$, where N_0 is the double-sided noise spectral density and the ideal received signal power has been normalized to unity. Parameter D is the ratio of s/n in the phase recovery loop to that in the PSK system. The double-sided 3-dB bandwidth of the transmit filter is $2/T$ and that of the receive filter is $1.06/T$. Sampling time is $1.74T$.

The upper bound in eq. (13) indicates that if the definition of s/n is modified to take into account the ISI power σ_I^2, the additional penalty, because of imperfect phase estimation, is

$$G_t = \left[\sin^2\left(\frac{\pi}{M} + \beta_0\right) \times \left\{1 - \frac{2}{D}\cos^2\left(\frac{\pi}{M} + \beta_0\right)\left[1 - \frac{2}{D}\cos 2\left(\frac{\pi}{M} + \beta_0\right)\right]\right\}\right]^{-1}.$$

This quantity is plotted in Fig. 3. We observe that the s/n penalty

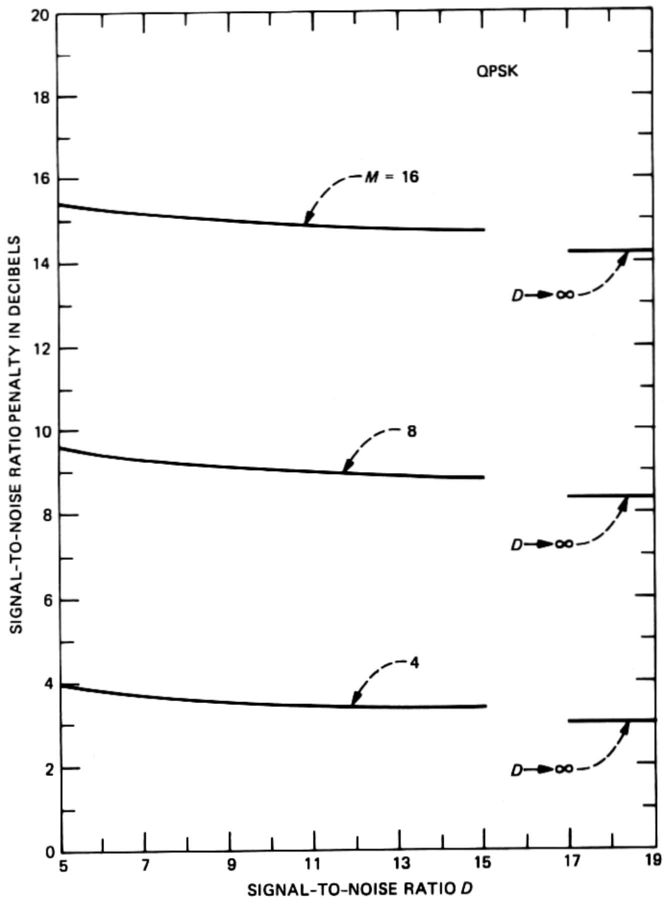

Fig. 3—Signal-to-noise ratio penalty for M-ary CPSK with imperfect phase estimation. Parameter D is the ratio of s/n in the phase recovery loop to that in the PSK system.

because of ISI is independent of M. In Fig. 3, also note that $G_t \to 1/[\sin^2(\pi/M)]$ as $D \to \infty$.

III. DIFFERENTIAL DETECTION

3.1 System description of DPSK

The M-ary DPSK system is shown in Fig. 4. As before, the baseband modulated DPSK signal can be represented as

$$x(t) = \sum_{k=-\infty}^{\infty} \exp(i\alpha_k)\mathrm{rect}[(t - kT)/T]. \qquad (2)$$

Here, however, the sequence of phases $[\beta_k] = [\alpha_{k+1} - \alpha_k]$ corresponds to the data sequence to be transmitted. Again, we assume that M

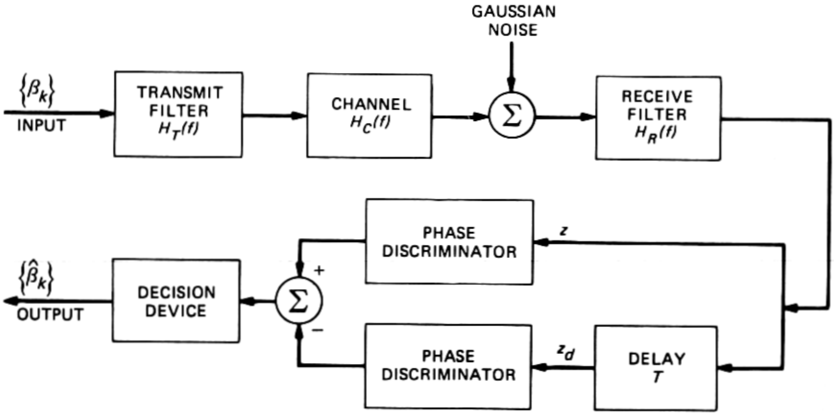

Fig. 4—M-ary DPSK system.

phase values of β_k are equally distributed over the interval $[0, 2\pi)$ and choose

$$\beta_k = (2l - 1)\frac{\pi}{M}, \; 1 \le l \le M, \text{ modulo } 2\pi.$$

As in CPSK, we represent the set

$$\left[\frac{\pi}{M}, \frac{3\pi}{M}, \cdots, (2M-1)\frac{\pi}{M}\right]$$

by $[\Lambda]$. Also, we shall assume that the phase symbols, β_k's, in different time slots are statistically iid.

If the received phasor at time t_0 is indicated by z and the one in the succeeding interval is indicated by z_d, the detected phase difference measured by an ideal differential detector is

$$\theta = \text{angle of } w, \; w \triangleq z^* z_d,$$

where $*$ represents the complex conjugate. For the system shown in Fig. 4,

$$z = \sum_{k=-\infty}^{\infty} (g_k + ip_k)\exp(i\alpha_k) + n_c + in_s, \tag{14}$$

and

$$z_d = \sum_{k=-\infty}^{\infty} (g_{k-1} + ip_{k-1})\exp(i\alpha_k) + n_{c_d} + in_{s_d}, \tag{15}$$

where g_k and p_k are real,

$$g_k + ip_k = \int_{-\infty}^{\infty} h(t_0 - \mu)\text{rect}\left(\frac{\mu - kT}{T}\right) d\mu. \tag{16}$$

As before, $h(t)$ is the overall impulse response of the system with transfer characteristic

$$H(f) = H_T(f)H_C(f)H_R(f). \tag{17}$$

In eqs. 14 and 15, n_c, n_s, n_{c_d}, and n_{s_d} are iid real Gaussian random variables with mean zero and variance

$$\sigma^2 = N_0 \int_{-\infty}^{\infty} |H_R(f)|^2 df,$$

where N_0, is the double-sided spectral density of the added white Gaussian noise. In eq. (17), H_T is the transfer function of the transmit filter, H_R, of the receive filter, and $H_C(f)$, the transfer function of the channel. The assumption that the Gaussian noise at t_0 is independent of the noise at $t_0 - T$ can be justified if the receive filter bandwidth is small compared with $1/T$. Most of our analysis can be extended if these two noise samples are correlated.

3.1.1 Probability of error for M-ary systems

If the transmitted symbol associated with the time index $k = 0$ is $\Phi \epsilon \Lambda$, a correct decision is made when the received phase difference θ is such that

$$\Phi - \frac{\pi}{M} < \theta < \Phi + \frac{\pi}{M}.$$

As before, the following bounds apply:

$$\max(P_1, P_2) \leq \text{Pe}(|\Phi) \leq P_1 + P_2,$$

where

$$P_1 = \Pr\left[\sin\left(\theta - \Phi + \frac{\pi}{M}\right) < 0\right],$$

$$P_2 = \Pr\left[\sin\left(\theta - \Phi - \frac{\pi}{M}\right) > 0\right]. \tag{18}$$

These statements are identical to the ones that apply to CPSK, but here θ represents a "differential phase" and, therefore, the estimation of these probabilities becomes extremely involved. Good estimates are only available when Gaussian noise is the sole source of impairment.

We proceed to analyze P_1 and observe that a bound on P_1 also provides a bound on P_2. Since the calculations are extremely tedious, we relegate the details to appendices and strive to develop only the main ideas here. Therefore, from eq. (18), we get

$$
\begin{aligned}
P_1(|\Phi) &= \Pr\left[\sin\left(\theta - \Phi + \frac{\pi}{M}\right) < 0\right] \\
&= \Pr\left\{\operatorname{Im} w \exp\left[-i\left(\Phi - \frac{\pi}{M}\right)\right] < 0\right\} \\
&= \Pr\left\{\operatorname{Im} z^* z_d \exp\left[-i\left(\Phi - \frac{\pi}{M}\right)\right] < 0\right\} \\
&= \Pr\left\{\operatorname{Re} (z)^* z_d \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right] < 0\right\} \\
&= \Pr(\operatorname{Re} z_1^* z_2 < 0),
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
z_1 &= z \\
z_2 &= z_d \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right],
\end{aligned}
$$

and z and z_d are given in eqs. (14) and (15). Since

$$
\operatorname{Re} z_1^* z_2 = \left|\frac{z_1 + z_2}{2}\right|^2 - \left|\frac{z_1 - z_2}{2}\right|^2, \tag{20}
$$

eqs. (19) and (20) yield

$$
P_1(|\Phi) = \Pr(|w_1| < |w_2|), \tag{21}
$$

where

$$
\begin{aligned}
w_1 &= \frac{z_1 + z_2}{2} = \frac{z + z_d \exp - i[\Phi - (\pi/M) + (\pi/2)]}{2} \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2}\left\{(g_k + ip_k) + (g_{k-1} + ip_{k-1})\right. \\
&\quad \left.\times \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]\right\}\exp(i\alpha_k) + \xi_+ + i\eta_+ \\
w_2 &= \frac{z_1 - z_2}{2} = \frac{z - z_d \exp - i[\Phi - (\pi/M) + (\pi/2)]}{2} \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{2}\left\{(g_k + ip_k) - (g_{k-1} + ip_{k-1})\right. \\
&\quad \left.\times \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]\right\}\exp(i\alpha_k) + \xi_- + i\eta_-,
\end{aligned}
$$

and where ξ_+, η_+, ξ_-, and η_- are Gaussian noise terms, given by

$$\xi_+ = \frac{n_c + \mathrm{Re}(n_{cd} + in_{sd}) \exp - i[\Phi - (\pi/M) + (\pi/2)]}{2}$$

$$\eta_+ = \frac{n_s + \mathrm{Im}(n_{cd} + in_{sd}) \exp - i[\Phi - (\pi/M) + (\pi/2)]}{2}$$

$$\xi_- = \frac{n_c - \mathrm{Re}(n_{cd} + in_{sd}) \exp - i[\Phi - (\pi/M) + (\pi/2)]}{2}$$

and

$$\eta_- = \frac{n_s - \mathrm{Im}(n_{cd} + in_{sd}) \exp - i[\Phi - (\pi/M) + (\pi/2)]}{2}.$$

It can be verified that the above Gaussian random variables are iid with mean zero and variance $\sigma^2/2$.

3.1.2 Exact computation of probability of error

For a given symbol sequence, the conditional probability of error is seen from eq. (21) to be given by the probability that a particular Gaussian quadratic form exceeds another. This is a well-known problem and the answer can conveniently be expressed in terms of the tabulated Marcum Q function.[13] Thus, after some algebra, eq. (21) can be shown as[14]

$$\begin{aligned} P_1\,(|\,\Phi, \text{symbol sequence}) &= \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\left[1 - Q\left(\frac{a_+}{\sqrt{\sigma_1^2 + \sigma_2^2}}, \frac{a_-}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right)\right] \\ &\quad + \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} Q\left(\frac{a_-}{\sqrt{\sigma_1^2 + \sigma_2^2}}, \frac{a_+}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right), \end{aligned} \tag{22}$$

where

$$Q(a, b) = \int_b^\infty \exp\left(-\frac{a^2 + x^2}{2}\right) I_0(ax)x\,dx,$$

and $I_n(\cdot)$ is the modified Bessel function of the first kind and of order n,

$$a_+ = |\langle w_1 \rangle_{\xi_+, \eta_+}|,$$

$$a_- = |\langle w_2 \rangle_{\xi_-, \eta_-}|,$$

$$\sigma_1^2 = \langle \xi_+^2 \rangle = \langle \eta_+^2 \rangle = \sigma^2/2$$

and

$$\sigma_2^2 = \langle \xi_-^2 \rangle = \langle \eta_-^2 \rangle = \sigma^2/2.$$

The major difficulty at this point is clearly carrying out the averages in eq. (22) over all possible symbol sequences. In general, a_- and a_+ contain an infinite number of ISI terms and the averaging process is difficult. Clearly, for a small number of ISI terms, it can be carried out by enumeration. But, in general, the number of terms in computing the average explodes exponentially and enumeration becomes intractable. For example, for 10 ISI terms and a quaternary DPSK system, the number of terms is about a million! So, we obviously need more efficient methods of estimating these averages.[15]

In this paper, we assume that the number of dominant ISI terms contained in a_+ and a_- is not large and that they become insignificant when ISI samples are far away from the desired sample. Assuming that the same number N indicates the number of dominant preceding and succeeding ISI samples (total significant ISI terms is $2N$), our approach, then, is to obtain upper and lower bounds on P_1 as a function of N and demonstrate that these bounds coincide with $N \to \infty$.

For any N, the evaluation of these bounds requires M^{2N} computations. This can be carried out with modest effort on a high-speed digital computer. The error becomes smaller when N is increased.

We show in Appendix B that the error probability can be bounded as

$$\chi_1(N) \le P_1(|\Phi) \le \chi_2(N),$$

where

$$\chi_1(N) = \frac{1}{1+(1-\Delta)^2} \times \left\{1 - \left\langle Q\left[\frac{\sqrt{2}a_+}{\sigma\sqrt{1+(1-\Delta)^2)}}, \frac{\sqrt{2}a_-(1-\Delta)}{\sigma\sqrt{1+(1-\Delta)^2}}\right]\right\rangle\right\}$$
$$+ \frac{(1-\Delta)^2}{1+(1-\Delta)^2}\left\langle Q\left[\frac{\sqrt{2}a_-(1-\Delta)}{\sigma\sqrt{1+(1-\Delta)^2}}, \frac{\sqrt{2}a_+}{\sigma\sqrt{1+(1-\Delta)^2}}\right]\right\rangle$$
$$- \left[1 - \left\langle Q\left(\frac{\sqrt{2}a_-}{\sigma}, \frac{\sqrt{2}p}{\sigma}\right)\right\rangle\right] - 4\exp\left(-\frac{\Delta^2 p^2}{2\sum_{\substack{k<-N\\k>N}} G_k^2}\right)$$
$$- 4\exp\left(-\frac{\Delta^2 p^2}{2\sum_{\substack{k<-N\\k>N}} H_k^2}\right) \tag{23}$$

and

$$\chi_2(N) = \frac{1}{1 + (1 + \Delta)^2}$$

$$\times\left\{1 - \left\langle Q\left[\frac{\sqrt{2}a_+}{\sigma\sqrt{1 + (1 + \Delta)^2}}, \frac{\sqrt{2}a_-(1 + \Delta)}{\sigma\sqrt{1 + (1 + \Delta)^2}}\right]\right\rangle\right\}$$

$$+ \frac{(1 + \Delta)^2}{1 + (1 + \Delta)^2}\left\langle Q\left[\frac{\sqrt{2}a_-(1 + \Delta)}{\sigma\sqrt{1 + (1 + \Delta)^2}}, \frac{\sqrt{2}a_-}{\sigma\sqrt{1 + (1 + \Delta)^2}}\right\rangle\right]$$

$$+ \left[1 - \left\langle Q\left(\frac{\sqrt{2}a_-}{\sigma}, \frac{\sqrt{2}p}{\sigma}\right)\right\rangle\right] + 4\exp\left(-\frac{\Delta^2 p^2}{2\sum_{\substack{k<-N\\k>N}} G_k^2}\right)$$

$$+ 4\exp\left(-\frac{\Delta^2 p^2}{2\sum_{\substack{k<-N\\k>N}} H_k^2}\right). \tag{24}$$

In eqs. (23) and (24), Δ and p are arbitrary, $0 \le \Delta \le 1$, and

$$G_k = \left|\frac{1}{2}\left\{(g_k + ip_k) + (g_{k-1} + ip_{k-1})\exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]\right\}\right|,$$

$$H_k = \left|\frac{1}{2}\left\{(g_k + ip_k) - (g_{k-1} + ip_{k-1})\exp\left[-i\left(\Phi - \frac{\pi}{m} + \frac{\pi}{2}\right)\right]\right\}\right|.$$

For any N, we can choose Δ and p by trial and error so that the difference between the upper and lower bounds is a minimum. Since this optimization is not critical to our method, we choose

$$p = \left[\frac{2\sigma_R^2}{\Delta}\left(\frac{\langle a_-^2\rangle}{\sigma^2} + \ln\frac{\sigma^2}{\sigma_R^2}\Delta\right)\right]^{1/2},$$

where

$$\sigma_R^2 = \max\left(\sum_{\substack{k<-N\\k>N}} G_k^2, \sum_{\substack{k<-N\\k>N}} H_k^2\right).$$

For $\Delta \ll 1$, the difference, Z, between the upper and the lower bounds can be shown as

$$Z = 2\Delta\left[1 + \frac{a_{+\max}^2}{\sigma^2} - \frac{a_{-\min}^2}{\sigma^2}\right]P_1(|\Phi, N)$$

$$+ 8\exp\left(\frac{-\langle a_-^2\rangle}{\sigma^2}\right)\frac{\sigma_R^2}{\sigma^2\Delta}\left(\frac{\langle a_-^2\rangle}{\sigma^2} + \ln\frac{\sigma^2}{\sigma_R^2}\Delta\right)$$

$$+ 8\exp\left\{-\frac{\Delta}{\sum_{\substack{k<-N\\k>N}} G_k^2}\left[\sigma_R^2\left(\frac{\langle a_-^2\rangle}{\sigma^2} + \ln\frac{\sigma^2}{\sigma_R^2}\Delta\right)\right]\right\}$$

$$+ 8\exp\left\{-\frac{\Delta}{\sum_{\substack{k<-N\\k>N}} H_k^2}\left[\sigma_R^2\left(\frac{\langle a_-^2\rangle}{\sigma^2} + \ln\frac{\sigma^2}{\sigma_R^2}\Delta\right)\right]\right\}, \tag{25}$$

where

$$P_1(|\Phi, N) = \frac{1}{2}\left[1 - \left\langle Q\left(\frac{a_+}{\sigma}, \frac{a_-}{\sigma}\right)\right\rangle\right] + \frac{1}{2}\left\langle Q\left(\frac{a_-}{\sigma}, \frac{a_+}{\sigma}\right)\right\rangle,$$

and a_+ and a_- contain only the first $2N$ significant terms. When $N \to \infty$, Z in eq. (25) can be seen to approach zero.

Since a_+ and a_- in eqs. (23) and (24) contain a finite number of ISI terms, we can use the direct method to evaluate the averages and then compute the bounds. We choose the initial N so that $\sigma_R^2 < 1$, $\Delta = \sqrt{\sigma_R}$. We then increase N so that the desired accuracy of computation is achieved.

3.1.3 Upper bound on the probability of error

Since the exact evaluation of $P_1(|\Phi)$ is difficult—though we have developed in the last section numerical techniques which can be used to compute P_1 with any desired accuracy—we attempt to derive an upper bound on P_1.

Although our bounding approach seems reasonable, the final bound that we obtain turns out to be loose. Our purpose in including this section is to alert readers about this approach and to emphasize the importance of the tedious, but necessary, computations outlined in Section 3.1.2.

To facilitate our bounding techniques, we need the following relations. For any two random variables x and y and any two real numbers a and Δ, we can show (see Fig. 5) that

$$\begin{aligned}\Pr(x > a + \Delta) - \Pr(y < -\Delta) &\le \Pr(x + y > a) \\ &\le \Pr(x > a - \Delta) + \Pr(y > \Delta).\end{aligned} \tag{26}$$

Equations (21) and (26) yield

$$\begin{aligned}P_1(|\Phi) &= \Pr(|w_2| - |w_1| > 0) \\ &\le \Pr(|w_2| > A) + \Pr(-|w_1| > -A) \\ &= \Pr(|w_2| > A) + \Pr(|w_1| < A),\end{aligned} \tag{27}$$

where A is arbitrary. We choose $A > 0$ so that the upper bound in eq. (27) is a minimum. The method of choosing A will be discussed later.

Now, for any complex random variable $z = x + iy$, we can show (see Fig. 6) that

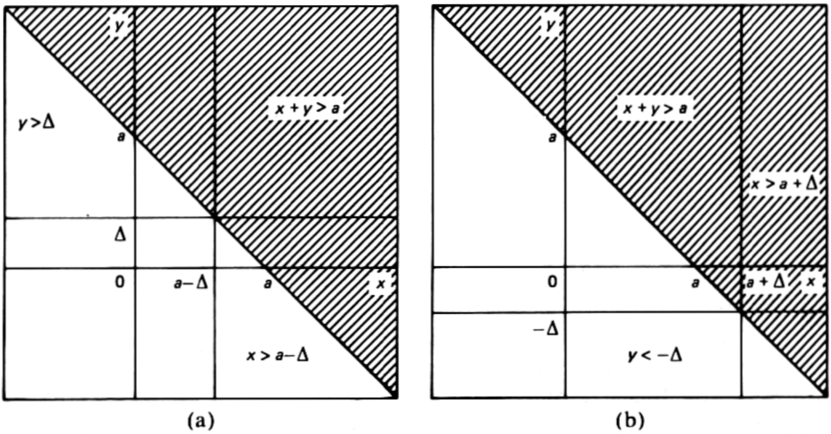

Fig. 5—(a) Upper bound on $\Pr(x + y > a)$, x and y, any two random variables and Δ is arbitrary. (b) Lower bound on $\Pr(x + y > a)$, x and y, any two random variables and Δ is arbitrary.

$$\Pr(|z| > a) \le \Pr(|\mathrm{Re}\, z| > a_1) + \Pr(|\mathrm{Im}\, z| > \sqrt{a^2 - a_1^2}).$$

Hence,

$$\Pr(|w_2| > A) \le \Pr(|\mathrm{Re}\, w_2| > A_1) + \Pr(|\mathrm{Im}\, w_2| > A_2), \qquad A_1^2 + A_2^2 = A^2, \tag{28}$$

where

$$\mathrm{Re}\, w_2 = \xi_- + \sum_{k=-\infty}^{\infty} C_k \cos(\alpha_k + \lambda_k)$$

$$\mathrm{Im}\, w_2 = \eta_- + \sum_{k=-\infty}^{\infty} C_k \sin(\alpha_k + \lambda_k)$$

$$C_k = \left| \frac{1}{2} \left\{ (g_k + ip_k) - (g_{k-1} + ip_{k-1}) \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2} \right) \right] \right\} \right|$$

and

$$C_k \exp(i\lambda_k) = \frac{1}{2} \left\{ (g_k + ip_k) - (g_{k-1} + ip_{k-1}) \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2} \right) \right] \right\}.$$

Since the real and imaginary parts of w_2 are the sum of a Gaussian random variable and a set of interference terms, various methods given in Refs. 16 to 18 and 19 to 27 can be used to bound $\Pr(|\mathrm{Re}\, w_2| > A_1)$ and $\Pr(|\mathrm{Im}\, w_2| > A_2)$, $A_1^2 + A_2^2 = A^2$. Even though the other bounds are sometimes claimed to be tighter, we shall use the simpler Chernoff bounds.

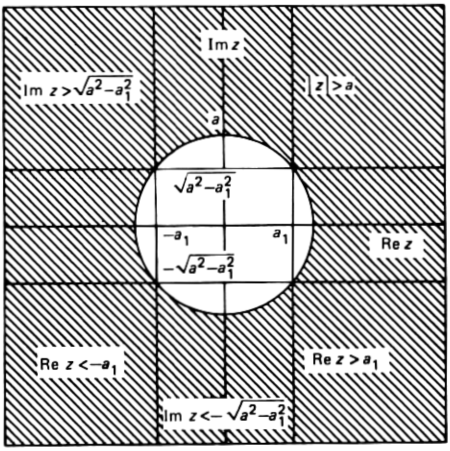

Fig. 6—Upper bound on $\Pr(|z| > a)$. Parameter Δ satisfies $0 \le \Delta \le a$.

Appendix C shows that

$$\Pr(|w_2| > A) \le 2 \exp\left[-\frac{(A_1 - C_{M1})^2}{\sigma^2 + \sigma_-^2}\right] + 2 \exp\left[-\frac{(A_2 - C_{M2})^2}{\sigma^2 + \sigma_-^2}\right],$$

where

$$A_1 = A \cos\left(\frac{\pi}{4} + \frac{\pi}{2M}\right)$$

$$A_2 = A \sin\left(\frac{\pi}{4} + \frac{\pi}{2M}\right)$$

$$C_{M1} = \max\{[C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)]\}$$

$$\alpha_1 - \alpha_0 = \frac{\pi}{M}$$

$$C_{M2} = \max\{[C_0\sin(\alpha_0 + \lambda_0) + C_1\sin(\alpha_1 + \lambda_1)]\}$$

$$\alpha_1 - \alpha_0 = \frac{\pi}{M}$$

$$\sigma_-^2 = \sum{}'' C_k^2$$

and

$$\sum{}'' \triangleq \sum_{\substack{k=-\infty \\ k \ne 0 \\ k \ne -1}}^{\infty} .$$

Also, for any complex random variable $z = x + iy$, we can show (see Fig. 7) that

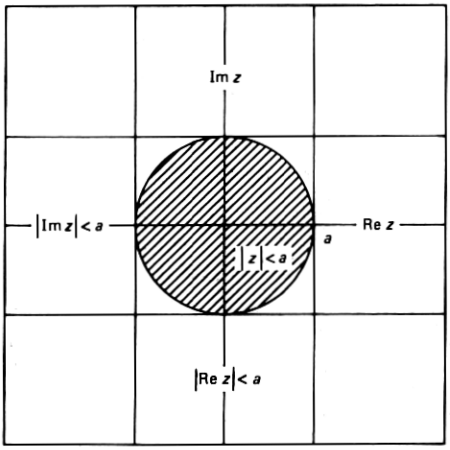

Fig. 7—Upper bound on $\Pr(|z| < a)$.

$$\Pr(|z| < a) \le \Pr(|\text{Re } z| < a, |\text{Im } z| < a)$$

$$\Pr(|z| < a) \le \Pr(|\text{Re } z| < a)$$

$$\Pr(|z| < a) \le \Pr(|\text{Im } z| < a).$$

Hence,

$$\Pr(|w_1| < A) \le \Pr(|\text{Re } w_1| < A).$$

Since

$$\text{Re } w_1 \le |\text{Re } w_1|,$$

$$\Pr(|\text{Re } w_1| < A) \le \Pr(\text{Re } w_1 < A),$$

and

$$\Pr(|w_1| < A) \le \Pr(\text{Re } w_1 < A).$$

We write

$$\text{Re } w_1 = \xi_+ + \sum_{k=-\infty}^{\infty} D_k \cos(\alpha_k + \delta_k)$$

$$\text{Im } w_1 = \eta_+ + \sum_{k=-\infty}^{\infty} D_k \sin(\alpha_k + \delta_k),$$

$$D_k = \left| \frac{1}{2} \left\{ (g_k + ip_k) + (g_{k-1} + ip_{k-1}) \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2} \right) \right] \right\} \right|,$$

$$D_k \exp(i\delta_k) = \frac{1}{2} \left\{ (g_k + ip_k) + (g_{k-1} + ip_{k-1}) \exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2} \right) \right] \right\},$$

and

$$\sum_{k=-\infty}^{\infty} D_k\cos(\alpha_k + \delta_k)$$

$$= D_0\cos(\alpha_0 + \delta_0) + D_1\cos(\alpha_1 + \delta_1) + \sum{}'' D_k\cos(\alpha_k + \delta_k).$$

Using Chernoff bounding techniques, it can be shown that

$$\Pr(|w_1| < A) \leq \exp\left[-\frac{(D_M - A)^2}{\sigma^2 + \sigma_+^2}\right],$$

where

$$D_M = \min[D_0\cos(\alpha_0 + \delta_0) + D_1\cos(\alpha_1 + \delta_1)]$$

$$\alpha_1 - \alpha_0 = \frac{\pi}{M}$$

$$\sigma_+^2 = \sum{}'' D_k^2.$$

The upper bound on $P_1(|\beta_0 = \pi/M)$ can, therefore, be written as

$$P_1\left(\middle|\beta_0 = \frac{\pi}{M}\right) \leq 2\exp\left[-\frac{(A_1 - C_{M1})^2}{\sigma^2 + \sigma_-^2}\right] + 2\exp\left[-\frac{(A_2 - C_{M2})^2}{\sigma^2 + \sigma_-^2}\right] + \exp\left[-\frac{(D_M - A)^2}{\sigma^2 + \sigma_+^2}\right],$$

$$A_1 - C_{m1} \geq 0,\ A_2 - C_{M2} \geq 0,\ D_M - A \geq 0,\ A_1^2 + A_2^2 = A^2. \quad (29)$$

The bound is minimum when A, A_1, and A_2 are chosen so that the derivative of eq. (29) is zero. This can be found by using well-known numerical methods.

3.2 Example of quaternary ($M = 4$) DPSK system

Let us consider a quaternary ($M = 4$) DPSK system and assume that the channel is ideal.

If 4-pole Butterworth transmit and receive filters are used, the bound given by eq. (29) is plotted in Fig. 8. The bound with zero ISI is plotted in Fig. 9. The exact probability of error with ISI is plotted in Fig. 10. With or without ISI, the bound is unfortunately not very tight. Actually, one can show that the penalty as predicted by the bound with zero ISI is about 4.6 dB worse than the actual penalty for a binary system, and 8.3 dB worse for a quaternary system. This is inherent in our techniques and not the result of using Chernoff bounding methods. In our opinion, obtaining tighter bounds is still an open problem. Comparing Figs. 2 and 10, we note that ISI penalty for quaternary DPSK is about 1 dB worse than for CPSK. We needed 9 ISI terms to compute Pe with 5 percent accuracy.

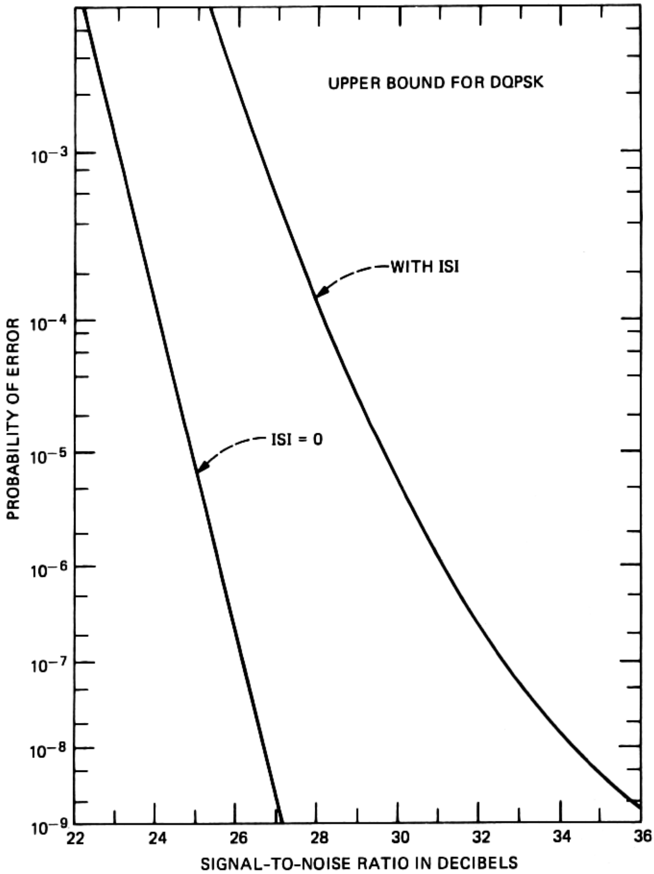

Fig. 8—Upper bound on the probability of error for differential QPSK (DQPSK) with the same transmit and receive filters as in Fig. 2. Other assumptions are as in Fig. 2.

IV. SUMMARY AND CONCLUSIONS

For multiphase M-ary CPSK, we develop an analytical procedure for determining detection efficiency when the system is subject to additive Gaussian noise, ISI, and imperfect carrier-phase estimation. For a large s/n, we provide a simple formula for calculating the combined penalty caused by ISI and noisy phase recovery. For multiphase DPSK, where the detection is inherently nonlinear, a rigorous method is developed for calculating the error rate in the presence of ISI and additive Gaussian noise. Using these analytical techniques, it is possible to compare the performance of CPSK and DPSK and examine various parameter trade-offs. Numerical examples are provided to illustrate our methods.

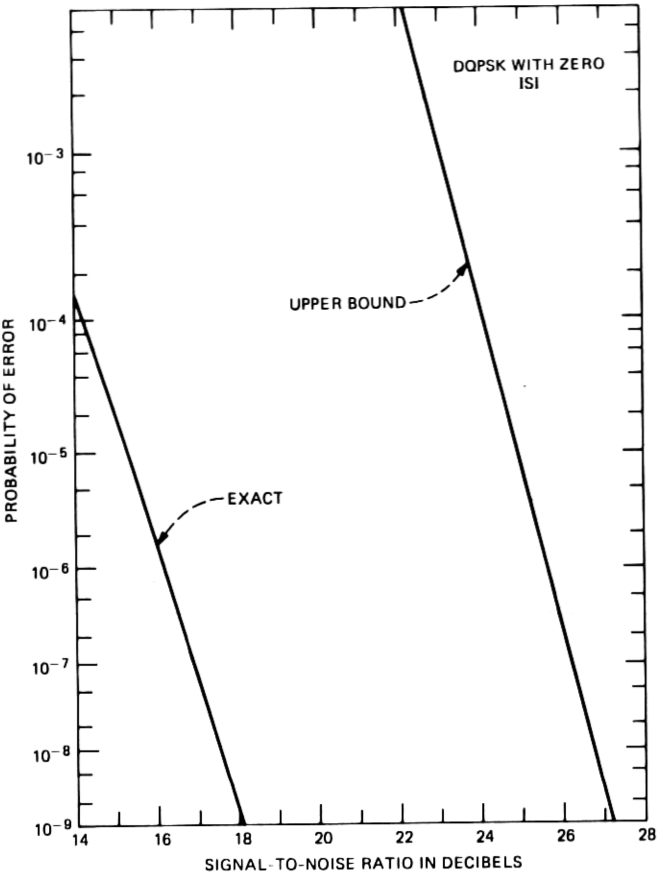

Fig. 9—Upper bound on the probability of error for DQPSK with zero ISI.

APPENDIX A

Chernoff Bound on the Probability of Error

From Section 2.1.3,

$$P_1 = \frac{1}{2}\int_{-\pi}^{\pi} [V(\epsilon) + V(-\epsilon)]p_\epsilon(\epsilon)d\epsilon$$

$$= \int_{0}^{\pi} [V(\epsilon) + V(-\epsilon)]p_\epsilon(\epsilon)d\epsilon$$

$$= J_1 + J_2,$$

where

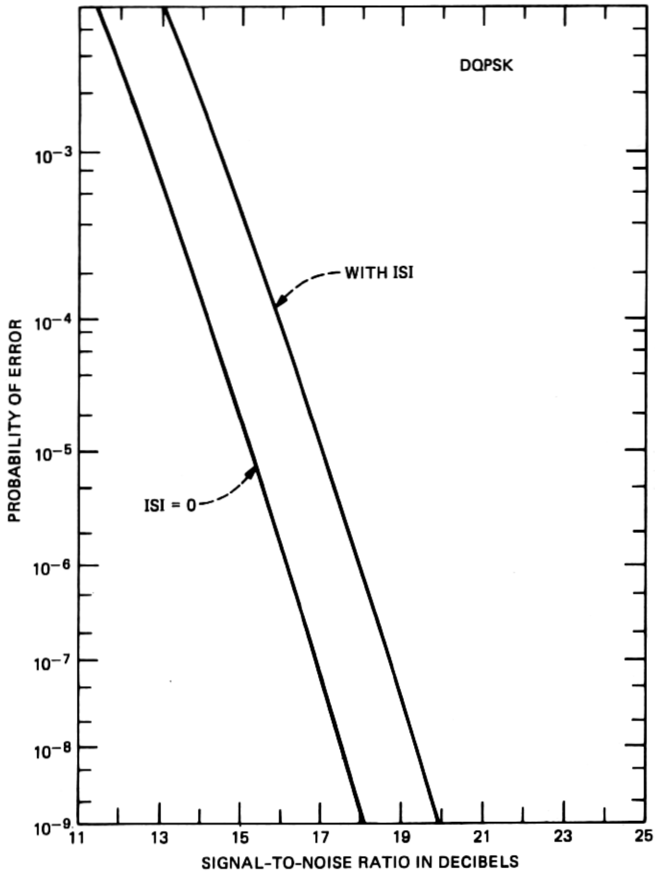

Fig. 10—Probability of error for DQPSK with the same transmit and receive filters as in Fig. 2. Other assumptions are as in Fig. 2. Note that 9 ISI terms were needed to compute Pe with 5 percent accuracy.

$$J_1 = \int_0^{\delta} [V(\epsilon) + V(-\epsilon)] p_\epsilon(\epsilon) d\epsilon$$

$$J_2 = \int_{\delta}^{\pi} [V(\epsilon) + V(-\epsilon)] p_\epsilon(\epsilon) d\epsilon, \tag{30}$$

and where $V(\epsilon)$ is given in eq. (11). Note that $\sin(\pi/M + \beta_0 - \epsilon) > 0$ for $0 \le \epsilon < \pi/M + \beta_0$; also, $\sin(\pi/M + \beta_0 + \epsilon) > 0$ for $0 \le \epsilon < \pi - (\pi/M + \beta_0)$. Hence, $\sin(\pi/M + \beta_0 + \epsilon) > 0$ for $0 \le \epsilon < \delta$, where

$$\delta = \min\left[\frac{\pi}{M} + \beta_0, \pi - \left(\frac{\pi}{M} + \beta_0\right)\right].$$

Since

$$0 \leq \mathrm{erfc}(x) \leq 2,$$

$$0 \leq [V(\epsilon) + V(-\epsilon)] \leq 2,$$

and, therefore,

$$J_2 \leq 2 \int_\delta^\pi p_\epsilon(\epsilon) d\epsilon = \Pr(|\epsilon| > \delta).$$

Now, from eq. (11),

$$V(\epsilon) = \Pr\left[-\eta - I(\epsilon) > r_0 \sin\left(\frac{\pi}{M} + \beta_0 + \epsilon\right)\right], \tag{31}$$

where η is a zero mean Gaussian random variable with variance σ^2 and $I(\epsilon)$ is given in eq. (8).

Using the Chernoff bound

$$\Pr[x > a] \leq \exp(-\mu a)\langle \exp(\mu x)\rangle, \qquad \mu \geq 0,$$

eq. (31) yields

$$V(\epsilon) \leq \exp\left[-\lambda r_0 \sin\left(\frac{\pi}{M} + \beta_0 + \epsilon\right)\right] \exp\{-\lambda[\eta + I(\epsilon)]\}. \tag{32}$$

For a given ϵ, I and η are independent, and since the data phases α_k in different time slots are iid,

$$\exp\{-\lambda[\eta + I(\epsilon)]\} = \exp \frac{\lambda^2 \sigma^2}{2} \prod{}' \langle \exp[-\lambda r_k \sin(\beta_k + \alpha_k + \epsilon)]\rangle_{\alpha_k}. \tag{33}$$

We shall now assume that M is an even number so that if $\Phi \epsilon \Lambda$, $(\pi + \Phi)\epsilon\Lambda$. Hence,

$$\begin{aligned} &\langle \exp[-\lambda r_k \sin(\beta_k + \alpha_k + \epsilon)]\rangle_{\alpha_k}, \qquad 0 \leq \alpha_k < 2\pi \\ &= \frac{1}{2}\langle \exp[-\lambda r_k \sin(\beta_k + \alpha_k + \epsilon)] \\ &+ \exp[-\lambda r_k \sin(\beta_k + \pi + \alpha_k + \epsilon)]\rangle_{\alpha_k}, \qquad 0 \leq \alpha_k < \pi \\ &= \langle \cosh \lambda r_k \sin(\beta_k + \alpha_k + \epsilon)\rangle_{\alpha_k}, \qquad 0 \leq \alpha_k < \pi. \end{aligned}$$

Since

$$\begin{aligned} &\cosh x \leq \exp(x^2/2), \\ &\langle \exp[-\lambda r_k \sin(\beta_k + \alpha_k + \epsilon)]\rangle_{\alpha_k}, \qquad 0 \leq \alpha_k < 2\pi \\ &\leq \langle \exp\left[\frac{\lambda^2 r_k^2}{2} \sin^2(\beta_k + \alpha_k + \epsilon)\right]\rangle_{\alpha_k}, \qquad 0 \leq \alpha_k < \pi \\ &\leq \exp\left(\frac{\lambda^2 r_k^2}{2}\right). \end{aligned} \tag{34}$$

From eqs. (32), (33), and (34),

$$V(\epsilon) \le \exp\left[-\lambda r_0 \sin\left(\frac{\pi}{M} + \beta_0 + \epsilon\right)\right]\exp\left[\frac{\lambda^2}{2}(\sigma^2 + \sigma_I^2)\right], \qquad \lambda \ge 0,$$

where

$$\sigma_I^2 = \sum{}' r_k^2.$$

Similarly,

$$V(-\epsilon) \le \exp\left[-\lambda r_0 \sin\left(\frac{\pi}{M} + \beta_0 - \epsilon\right)\right]\exp\left[\frac{\lambda^2}{2}(\sigma^2 + \sigma_I^2)\right].$$

Hence, for $0 \le \epsilon < \pi/M + \beta_0$, $\sin(\pi/M + \beta_0 - \epsilon) > 0$, and

$$V(-\epsilon) \le \exp\left[-\frac{r_0^2 \sin^2[(\pi/M) + \beta_0 - \epsilon]}{2(\sigma^2 + \sigma_I^2)}\right].$$

Also, for $0 \le \epsilon < \pi - (\pi/M + \beta_0)$, $\sin(\pi/M + \beta_0 + \epsilon) > 0$, and

$$V(\epsilon) \le \exp\left[-\frac{r_0^2 \sin^2[(\pi/M) + \beta_0 + \epsilon]}{2(\sigma^2 + \sigma_I^2)}\right].$$

Hence,

$$\begin{aligned} V(\epsilon) + V(-\epsilon) &\le \exp\left[-\frac{r_0^2 \sin^2[(\pi/M) + \beta_0 + \epsilon]}{2(\sigma^2 + \sigma_I^2)}\right] \\ &\quad + \exp\left[-\frac{r_0^2 \sin^2[(\pi/M) + \beta_0 - \epsilon]}{2(\sigma^2 + \sigma_I^2)}\right], \\ &= \exp\left[-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0 + \epsilon\right)\right] \\ &\quad + \exp\left[-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0 - \epsilon\right)\right], \end{aligned}$$

$$\rho^2 = \frac{r_0^2}{2(\sigma^2 + \sigma_I^2)}, \qquad \sigma_I^2 = \sum{}' r_k^2,$$

$$0 \le \epsilon < \delta = \min\left[\frac{\pi}{M} + \beta_0, \pi - \left(\frac{\pi}{M} + \beta_0\right)\right]. \tag{35}$$

The parameter ρ^2 is the s/n of the system.

Note that β_0 is the phase angle of the complex overall impulse response evaluated at $t = t_0$. In a well-designed system, it is usually small, and eq. (35) shows that the optimum value of β_0 is zero. Also, since β_0 is usually small and we are interested in $M > 2$, δ is usually $\pi/M + \beta_0$.

Thus, from eqs. (9), (30), and (35), we conclude that

$$J_1 \leq J_{1a} = \frac{1}{2\pi I_0(D\rho^2)} \int_0^{\delta} \left\{ \exp\left[-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0 - \epsilon \right) \right] + \exp\left[-\rho^2 \sin^2\left(\frac{\pi}{M} + \beta_0 + \epsilon \right) \right] \right\} \exp[D\rho^2 \cos \epsilon] d\epsilon, \quad (36)$$

where the quantity D can be regarded as the ratio of s/n in the phase recovery circuit to that in the PSK system or the integration time in bauds.

We have not been able to evaluate eq. (31) in closed form but, if desired, numerical techniques can be used. To obtain physical insight, we shall assume that $\rho^2 \gg 1$ and use Laplace's method to evaluate eq. (36); the technique is an application of the following theorem: *If $h(t)$ is a real function of a real variable t, has a unique maximum at $t = a$, $\alpha_1 \leq a \leq \alpha_2$, and if x is a large positive variable, it can be shown that*[11]

$$f(x) = \int_{\alpha_1}^{\alpha_2} g(t) \exp[xh(t)] dt \approx g(a) \exp[xh(a)] \left(\frac{-\pi}{2xh''(a)} \right)^{1/2}.$$

From eq. (36),

$$J_{1a} = \frac{1}{2\pi I_0(D\rho^2)} \{J_b + J_c\}, \quad (37)$$

$$J_b = \int_0^{\delta} \exp\left\{ \rho^2 \left[D \cos \epsilon - \sin^2\left(\epsilon - \frac{\pi}{M} - \beta_0 \right) \right] \right\} d\epsilon, \quad (38)$$

$$J_c = \int_0^{\delta} \exp\left\{ \rho^2 \left[D \cos \epsilon - \sin^2\left(\epsilon + \frac{\pi}{M} + \beta_0 \right) \right] \right\} d\epsilon. \quad (39)$$

The saddle point ϵ_0 at which the exponent in eq. (38) reaches its maximum in $(0, \delta)$ is given by the solution of

$$\frac{\sin \epsilon_0}{\sin 2[(\pi/M) + \beta_0 - \epsilon_0]} = \frac{1}{D}. \quad (40)$$

The transcendental eq. (39) can only be solved numerically. However, we can obtain a series solution for ϵ_0 by using Lagrange's reversion formula.[27] If a function $f(z)$ is regular in a neighborhood of z_0 and if $f(z_0) = w_0$, $f'(z_0) \neq 0$, then it can be shown[28] that

$$f(z) = w$$

has a unique solution

$$z = z_0 + \sum_{n=1}^{\infty} \frac{(w - w_0)^2}{n!} \left\{ \frac{d^{n-1}}{dz^{n-1}} [\Phi(z)]^n \right\}_{z=z_0}, \tag{41}$$

where

$$\Phi(z) = \frac{z - z_0}{f(z) - w_0}.$$

Choosing $z_0 = 0$, eqs. (40) and (41) yield

$$\epsilon_0 = \frac{1}{D} \sin 2\left(\frac{\pi}{M} + \beta_0\right) \times \left[1 - \frac{2}{D} \cos 2\left(\frac{\pi}{M} + \beta_0\right) + \cdots\right], \qquad D \gg 1, \qquad 0 \le \epsilon_0 < \delta.$$

From Laplace's formula and eq. (38)

$$J_b \approx \exp\left\{-\rho^2\left[\sin^2\left(\frac{\pi}{M} + \beta_0 - \epsilon_0\right) - D \cos \epsilon\right]\right\} \times \left\{\frac{\pi}{2\rho^2[D \cos \epsilon_0 + 2 \cos 2(\pi/M + \beta_0 - \epsilon_0)]}\right\}^{1/2}. \tag{42}$$

Similarly, it can be shown that the exponent in eq. (40) reaches its maximum in $(0, \delta)$ at $\epsilon = 0$ and

$$J_c \approx \exp\left\{-\rho^2\left[\sin^2\left(\frac{\pi}{M} + \beta_0\right) - D\right]\right\} \times \left[\frac{\pi}{2\rho^2[D + 2 \cos(\pi/M + \beta_0)]}\right]^{1/2}. \tag{43}$$

For $\rho^2 \gg 1$, $D > 1$,

$$I_0(D\rho^2) \approx \frac{\exp(D\rho^2)}{\sqrt{2\pi D\rho^2}}. \tag{44}$$

From eqs. (37) to (39) and (42) to (44)

$$J_{1a} \approx \frac{1}{2}\left(\frac{\exp\{-\rho^2[\sin^2[(\pi/M) + \beta_0 - \epsilon_0] - D(1 - \cos \epsilon_0)]\}}{[\cos \epsilon_0 + (2/D)\cos 2[(\pi/M) + \beta_0 - \epsilon_0]]^{1/2}} + \frac{\exp[-\rho^2 \sin^2[(\pi/M) + \beta_0)]}{[1 + (2/D)\cos 2[(\pi/M) + \beta_0]]^{1/2}}\right).$$

Also,

$$J_2 \leq J_{2a} = 2 \int_{\delta}^{\pi} \rho_{\epsilon}(\epsilon) dt$$

$$= \frac{1}{\pi I_0(D\rho^2)} \int_{\delta}^{\pi} \exp(D\rho^2 \cos \epsilon d\epsilon)$$

$$\leq \frac{\pi - \delta}{\pi} \frac{\exp(D\rho^2 \cos \delta)}{I_0(D\rho^2)}$$

$$\approx \left(1 - \frac{\delta}{\pi}\right) \sqrt{2\pi D\rho^2} \exp[-D\rho^2(1 - \cos \delta)]$$

$$\rho^2 \gg 1, \qquad D > 1.$$

APPENDIX B

Upper and Lower Bounds on the Probability of Error in DPSK

Let us write

$$z = z_N + n_c + in_s + z_R$$

$$z_d = z_{dN} + n_{c_d} + in_{s_d} + z_{dR},$$

where

$$z_N = \sum_{\substack{k \geq -N \\ k \leq N}} (g_k + ip_k) \exp(i\alpha_k)$$

$$z_R = \sum_{\substack{k < -N \\ k > N}} (g_k + ip_k) \exp(i\alpha_k)$$

$$z_{dN} = \sum_{\substack{k \geq -N \\ k \leq N}} (g_{k-1} + ip_{k-1}) \exp(i\alpha_k)$$

and

$$z_{dR} = \sum_{\substack{k < -N \\ k > N}} (g_{k-1} + ip_{k-1}) \exp(i\alpha_k).$$

Note that z_N and z_{dN} contain a finite number of ISI terms, whereas z_R and z_{dR} contain an infinite number. Without loss of generality, we shall also assume that g_k and p_k are monotonic decreasing functions of $|k|$. N is an arbitrary positive number.

Now

$$P_1(|\Phi) = \Pr(|w_{1N} + w_{1R}| < |w_{2N} + w_{2R}|), \tag{45}$$

where

$$w_{1N} = \frac{z_N + z_{dN}\exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]}{2} + \xi_+ + i\eta_+,$$

$$w_{1R} = \frac{z_R + z_{dR}\exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]}{2},$$

$$w_{2N} = \frac{z_N - z_{dN}\exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]}{2} + \xi_- + i\eta_-$$

and

$$w_{2R} = \frac{z_R - z_{dR}\exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]}{2}.$$

Note that w_{1N} and w_{2N} contain a finite number of ISI terms, and $\xi_+ + i\eta_+$ and $\xi_- + i\eta_-$ are independent zero means complex Gaussian processes.

For any two complex numbers s_1 and s_2,

$$|s_1| - |s_2| \le |s_1 + s_2| \le |s_1| + |s_2|. \tag{46}$$

Hence, eqs. (45) and (46) yield

$$\begin{aligned}\Pr(|w_{1N}| < |w_{2N}| - |w_{1R}| - |w_{2R}|) \le P_1(|\Phi) \\ = \Pr(|w_{1N} + w_{1R}| < |w_{2N} + w_{2R}|) \\ \le \Pr(|w_{1N}| < |w_{2N}| + |w_{1R}| + |w_{2R}|)\end{aligned}$$

or

$$\Pr\left(\frac{|w_{1N}|}{|w_{2N}|} < 1 - \frac{|w_{1R}| + |w_{2R}|}{|w_{2N}|}\right) \le P_1(|\Phi) \le \Pr\left(\frac{|w_{1N}|}{|w_{2N}|} < 1 + \frac{|w_{1R}| + |w_{2R}|}{|w_{2N}|}\right).$$

Using the bound in eq. (26), eq. (46) yields

$$\begin{aligned}\Pr\left(\frac{|w_{1N}|}{|w_{2N}|} < 1 - \Delta\right) - \Pr\left(\frac{|w_{1R}| + |w_{2R}|}{|w_{2N}|} > \Delta\right) \le P_1(|\Phi) \\ \le \Pr\left(\frac{|w_{1N}|}{|w_{2N}|} < 1 + \Delta\right) + \Pr\left(\frac{|w_{1R}| + |w_{2R}|}{|w_{2N}|} > \Delta\right),\end{aligned} \tag{47}$$

where we choose $0 \le \Delta < 1$.

For any two random variables x and y and any $a > 0$,[10]

$$\Pr(xy > a) \le \Pr\left(|x| > \frac{a}{p}\right) + \Pr(|y| > p), \tag{48}$$

where $p > 0$ is arbitrary. From eqs. (47) and (48),

$$\Pr\left(\frac{|w_{1N}|}{|w_{2N}|} < 1 - \Delta\right) - \Pr(|w_{2N}| < p) - \Pr(|w_{1R}| + |w_{2R}| > \Delta p) \le P_1(|\Phi) \le \Pr\left(\frac{|w_{1N}|}{|w_{2N}|} < 1 + \Delta\right) + \Pr(|w_{2N}| < p) + \Pr(|w_{1R}| + |w_{2R}| > \Delta p). \tag{49}$$

Since w_{1N} and w_{2N} are independent complex Gaussian processes, from Ref. 13,

$$\Pr\left(\frac{|w_{1N}|}{|w_{2N}|} < 1 \pm \Delta\right) = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2(1 \pm \Delta)^2} \times\left\{1 - \left\langle Q\left[\frac{a_+}{\sqrt{\sigma_1^2 + \sigma_2^2(1 \pm \Delta)^2}}, \frac{a_-(1 \pm \Delta)}{\sqrt{\sigma_1^2 + \sigma_2^2(1 \pm \Delta)^2}}\right]\right\rangle\right\} + \frac{\sigma_2^2(1 \pm \Delta)^2}{\sigma_1^2 + \sigma_2^2(1 \pm \Delta)^2} \times \left\langle Q\left[\frac{a_-(1 \pm \Delta)}{\sqrt{\sigma_1^2 + \sigma_2^2(1 \pm \Delta)^2}}, \frac{a_+}{\sqrt{\sigma_1^2 + \sigma_2^2(1 \pm \Delta)^2}}\right]\right\rangle, \quad \sigma_1^2 = \sigma_2^2 = \frac{\sigma^2}{2}, \tag{50}$$

where a_+ and a_- are the appropriate truncated values of a_+ and a_- defined in Section 3.1.2. Also, from Ref. 13,

$$\Pr(|w_{2N}| < p) = 1 - \left\langle Q\left(\frac{a_-}{\sigma_2}, \frac{p}{\sigma_2}\right)\right\rangle. \tag{51}$$

Using Chernoff bounding techniques, we can also show that

$$\Pr(|w_{1R}| > \Delta p) \le 4 \exp\left(-\frac{\Delta^2 p^2}{2 \sum\limits_{\substack{k<-N \\ k>N}} G_k^2}\right) \tag{52}$$

$$\Pr(|w_{2R}| > \Delta p) \le 4 \exp\left(-\frac{\Delta^2 p^2}{2 \sum\limits_{\substack{k<-N \\ k>N}} H_k^2}\right), \tag{53}$$

where

$$G_k = \left|\frac{1}{2}\left\{(g_k + ip_k) + (g_{k-1} + ip_{k-1})\exp\left[-i\left(\Phi - \frac{\pi}{M} + \frac{\pi}{2}\right)\right]\right\}\right|,$$

and

$$H_k = \left| \frac{1}{2} \left\{ (g_k + ip_k) - (g_{k-1} + ip_{k-1}) \exp\left[-i \left(\Phi - \frac{\pi}{M} + \frac{\pi}{2} \right) \right] \right\} \right|.$$

Equations (49) to (53) yield

$$\chi_1(N) \le P_1(|\Phi) \le \chi_2(N),$$

where

$$\begin{aligned}
\chi_1(N) = & \frac{1}{1 + (1 - \Delta)^2} \\
& \times \left\{ 1 - < Q \left[\frac{\sqrt{2} a_+}{\sigma \sqrt{1 + (1 - \Delta)^2}}, \frac{\sqrt{2} a_- (1 - \Delta)}{\sigma \sqrt{1 + (1 - \Delta)^2}} \right] > \right\} \\
& + \frac{(1 - \Delta)^2}{1 + (1 - \Delta)^2} < Q \left[\frac{\sqrt{2} a_- (1 - \Delta)}{\sigma \sqrt{1 + (1 - \Delta)^2}}, \frac{\sqrt{2} a_+}{\sigma \sqrt{1 + (1 - \Delta)^2}} \right] > \\
& - \left[1 - < Q \left(\frac{\sqrt{2} a_-}{\sigma}, \frac{\sqrt{2} p}{\sigma} \right) > \right] \\
& - 4 \exp \left(- \frac{\Delta^2 p^2}{2 \sum_{\substack{k < -N \\ k > N}} G_k^2} \right) - 4 \exp \left(- \frac{\Delta^2 p^2}{2 \sum_{\substack{k < -N \\ k > N}} H_k^2} \right)
\end{aligned}$$

and

$$\begin{aligned}
\chi_2(N) = & \frac{1}{1 + (1 + \Delta)^2} \\
& \times \left\{ 1 - < Q \left[\frac{\sqrt{2} a_+}{\sigma \sqrt{1 + (1 + \Delta)^2}}, \frac{\sqrt{2} a_- (1 + \Delta)}{\sigma \sqrt{1 + (1 + \Delta)^2}} \right] > \right\} \\
& + \frac{(1 + \Delta)^2}{1 + (1 + \Delta)^2} < Q \left[\frac{\sqrt{2} a_- (1 + \Delta)}{\sigma \sqrt{1 + (1 + \Delta)^2}}, \frac{\sqrt{2} a_+}{\sigma \sqrt{1 + (1 + \Delta)^2}} \right] > \\
& + \left\{ 1 - < Q \left(\frac{\sqrt{2} a_-}{\sigma}, \frac{\sqrt{2} p}{\sigma} \right) > \right\} \\
& + 4 \exp \left(- \frac{\Delta^2 p^2}{2 \sum_{\substack{k < -N \\ k > N}} G_k^2} \right) + 4 \exp \left(- \frac{\Delta^2 p^2}{2 \sum_{\substack{k < -N \\ k > N}} H_k^2} \right),
\end{aligned}$$

where Δ and p are arbitrary.

APPENDIX C

Upper Bound on the Probability of Error

Now,

$$\Pr(|\mathrm{Re}\ w_2| > A_1) = \Pr(\mathrm{Re}\ w_2 > A_1) + \Pr(-\mathrm{Re}\ w_2 > A_1)$$

and

$$\Pr(\mathrm{Re}\ w_2 \geq A_1) \leq \exp(-\mu A_1)\langle \exp(\mu\ \mathrm{Re}\ w_2)\rangle$$

$$= \exp(-\mu A_1)\langle \exp(\mu\ Re\ w_2)\rangle.$$

Since the Gaussian random variable ξ_- is assumed to be independent of the interference

$$\langle \exp(\mu\ \mathrm{Re}\ w_2)\rangle = \exp(\mu^2\sigma^2/4) \quad \exp\left[\mu \sum_{k=-\infty}^{\infty} C_k \cos(\alpha_k + \lambda_k)\right].$$

Since $\beta_k = \alpha_{k+1} - \alpha_k$, and we assume that β_k's are iid, we can assume that α_k's are independent, and

$$\alpha_k \in \Lambda, \qquad k \text{ even}$$

$$\alpha_k \in \left[0, \frac{2\pi}{M}, \frac{4\pi}{M}, \cdots, (2M-2)\frac{\pi}{M}\right] \triangleq \Lambda_s, \qquad k \text{ odd;}$$

that is, the signal constellation for odd (even) k can be obtained by a simple rotation of the constellation for even (odd) k.

Let us now assume that the transmitted symbol Φ is π/M so that

$$P_1(|\Phi) = P_1\left(|\beta_0 = \frac{\pi}{M}\right)$$

$$= \frac{1}{M}\sum P_1\left(|\alpha_1 - \alpha_0 = \frac{\pi}{M}\right).$$

Noting that

$$\sum_{k=-\infty}^{\infty} C_k\cos(\alpha_k + \lambda_k) = C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)$$

$$+ \sum'' C_k\cos(\alpha_k + \lambda_k),$$

$$\Pr\left(\mathrm{Re}\ w_2 > A_1 | \alpha_1 - \alpha_0 = \frac{\pi}{M}\right) \leq \exp\left\{-\mu A_1 + \frac{\mu^2\sigma^2}{4}\right.$$

$$\left. + \mu[C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)]\right\}$$

$$\times \langle \exp[\mu \sum'' C_k\cos(\alpha_k + \lambda_k)]\rangle.$$

We use the notation $''$ to indicate that $k = 0$, and $k = 1$ terms are not included.

Now,

$$\langle \exp[\mu \textstyle\sum'' C_k \cos(\alpha_k + \lambda_k)]\rangle = \prod'' \langle \exp[\mu C_k \cos(\alpha_k + \lambda_k)]\rangle$$

$$= \prod_{\text{even}}'' \langle \exp[\mu C_k \cos(\alpha_k + \lambda_k)]\rangle$$

$$\times \prod_{\text{odd}}'' \langle \exp[\mu C_k \cos(\alpha_k + \lambda_k)]\rangle.$$

Most often, $M = 2^L$, L an integer, and since this assumption simplifies our bound, we shall assume that M is an integer power of two. (A slightly more complex bound can be derived if $M \neq 2^L$.) Now,

$$\prod_{\text{even}}'' \langle \exp[\mu C_k \cos(\alpha_k + \lambda_k)]\rangle_{(\alpha_k | -\pi < \alpha_k \leq \pi)}$$

$$= \prod_{\text{even}}'' \langle \cosh[\mu C_k \cos(\alpha_k + \lambda_k)]\rangle_{(\alpha_k | 0 \leq \alpha_k < \pi)}$$

$$= \prod_{\text{even}}'' \frac{1}{2} < \cosh[\mu C_k \cos(\alpha_k + \lambda_k)] + \cosh\left[\mu C_k \cos\left(\frac{\pi}{2} + \alpha_k + \lambda_k\right)\right] >_{\left(\alpha_k | 0 < \alpha_k \leq \frac{\pi}{2}\right)}$$

$$= \prod_{\text{even}}'' < \cosh\left[\mu C_k \cos\frac{\pi}{4}\cos\left(\alpha_k + \lambda_k + \frac{\pi}{4}\right)\right] \times \cosh\left[\mu C_k \sin\frac{\pi}{4}\sin\left(\alpha_k + \lambda_k + \frac{\pi}{4}\right)\right] >_{\left(\alpha_k | 0 < \alpha_k \leq \frac{\pi}{2}\right)}.$$

Since

$$\cosh x \leq \exp(|x|),$$

and

$$\cosh x \leq \exp(x^2/2),$$

$$< \cosh\left[\mu C_k \cos\frac{\pi}{4}\cos\left(\alpha_k + \lambda_k + \frac{\pi}{4}\right)\right] \times \cosh\left[\mu C_k \cos\frac{\pi}{4}\sin\left(\alpha_k + \lambda_k + \frac{\pi}{4}\right)\right] >_{\left(\alpha_k | 0 < \alpha_k \leq \frac{\pi}{2}\right)} \leq \exp(\mu C_k)$$

and

$$< \cosh\left[\mu C_k \cos\frac{\pi}{4}\cos\left(\alpha_k + \lambda_k + \frac{\pi}{4}\right)\right] \times \cosh\left[\mu C_k \cos\frac{\pi}{4}\sin\left(\alpha_k + \lambda_k + \frac{\pi}{4}\right)\right] >_{(\alpha_k | 0 \leq \alpha_k \leq \pi/2)} \leq \exp\left(\frac{\mu^2 C_k^2}{4}\right).$$

Identical bounds can be derived when k is odd. Therefore, we have

$$\langle \exp[\mu \textstyle\sum'' C_k \cos(\alpha_k + \lambda_k)] \rangle \le \exp\left(\mu \sum''_{k \in \Omega_1} C_k + \frac{\mu^2}{4} \sum''_{k \in \Omega_1^c} C_k^2\right),$$

where Ω_1 is a subset of $[\cdots, -2, -1, 2, 3, \cdots]$. For simplicity, we choose Ω_1 to be the null set.

Choosing optimum μ,

$$\Pr\left(\text{Re } w_2 > A_1 \,|\, \alpha_1 - \alpha_0 = \frac{\pi}{M}\right)$$

$$\le \exp\left[-\frac{\{A_1 - [C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)]\}^2}{\{\sigma^2 + \sum'' C_k^2\}}\right],$$

$$A_1 - [C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)] > 0.$$

Similarly, we can show that

$$\Pr\left(-\text{Re } w_2 > A_1 \,|\, \alpha_1 - \alpha_0 = \frac{\pi}{M}\right)$$

$$\le \exp\left(-\frac{\{A_1 - [C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)]\}^2}{(\sigma^2 + \sum'' C_k^2)}\right),$$

$$A_1 + [C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)] \ge 0,$$

$$\Pr\left(|\text{Im } w_1| > A_2 \,|\, \alpha_1 - \alpha_0 = \frac{\pi}{M}\right)$$

$$\le \exp\left(-\frac{\{A_2 - [C_0\sin(\alpha_0 + \lambda_0) + C_1\sin(\alpha_1 + \lambda_1)]\}^2}{(\sigma^2 + \sum'' C_k^2)}\right)$$

$$+ \exp\left(-\frac{\{A_2 + [C_0\sin(\alpha_0 + \lambda_0) + C_1\sin(\alpha_1 + \lambda_1)]\}^2}{(\sigma^2 + \sum'' C_k^2)}\right),$$

$$A_2 \pm [C_0\sin(\alpha_0 + \lambda_0) + C_1\sin(\alpha_1 + \lambda_1)] \ge 0, \qquad A_1^2 + A_2^2 = A^2,$$

and

$$\Pr(|w_2| > A) \le 2\exp\left(-\frac{(A_1 - C_{M1})^2}{\sigma^2 + \sigma_-^2}\right) + 2\exp\left(-\frac{(A_2 - C_{M2})^2}{\sigma^2 + \sigma_-^2}\right),$$

where

$$C_{M1} = \max\{[C_0\cos(\alpha_0 + \lambda_0) + C_1\cos(\alpha_1 + \lambda_1)]\}$$

$$\alpha_1 - \alpha_0 = \frac{\pi}{M},$$

$$C_{M2} = \max\{[C_0\sin(\alpha_0 + \lambda_0) + C_1\sin(\alpha_1 + \lambda_1)]\}$$

$$\alpha_1 - \alpha_0 = \frac{\pi}{M},$$

$$\sigma_-^2 = \sum{}'' C_k^2.$$

For zero ISI, it can be shown that optimum values of A_1 and A_2 are

$$A_2 = A \cos\left(\frac{\pi}{4} + \frac{\pi}{2M}\right),$$

$$A_2 = A \sin\left(\frac{\pi}{4} + \frac{\pi}{2M}\right).$$

Even when there is ISI, we shall use these values of A_1 and A_2.

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