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Criteria for the Response of Nonlinear Systems to be L-Asymptotically Periodic

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We consider the behavior of a general type of system governed by an input-output operator G that maps each excitation x into a corresponding response r. Here excitations and responses are R^n-valued functions defined on a set T. To accommodate both continuous time and discrete time cases, T is allowed to be either $[0, \infty)$ or $\{0, 1, 2, \cdots\}$. We address the following question. Under what conditions on G and x is it true that the response r is L-asymptotically periodic in the sense that $r = p + q$, where p is periodic with a given period τ, and q has finite energy (i.e., is square summable)? This type of question arises naturally in many applications. The main results given (which include a necessary and sufficient condition) are basically "tool theorems." To illustrate how they can be used, an example is discussed involving an integral equation that is often encountered in the theory of feedback systems.

I. INTRODUCTION

In this paper we consider the behavior of a general type of system governed by an input-output operator G that maps each excitation x into a corresponding response r. Here excitations and responses are R^n-valued functions defined on a set T. (As usual, n is an arbitrary positive integer.) To accommodate both continuous time and discrete time cases, we allow T to be either $[0, \infty)$ or $\{0, 1, 2, \cdots\}$. As in L_2-stability theory,[1-6] each x is drawn from a family $E(L)$ of functions whose truncations belong to a set L of finite-energy (i.e., square summable) functions (the details are given in Section 3.1), and G is assumed to map $E(L)$ into $E(L)$.

We address, and give in Section 3.2 results concerning, the following question. Under what conditions on G and x is it true that the response r is *L-asymptotically periodic* in the sense that $r = p + q$, where p is periodic with a given period τ, and q has finite energy?† This type of question arises naturally in many applications. Our results are basically "tool theorems" which appear to be widely applicable. An example is given in Section 3.4.

II. MOTIVATION AND BACKGROUND MATERIAL

To provide motivation for considering an abstract input-output operator G, and also to describe some earlier results related to those of Section III, we begin by recalling that an important example of a type of equation that arises in the study of physical systems (such as feedback systems or networks containing linear lumped and/or distributed elements, as well as memoryless, possibly time-varying, nonlinear elements) is the integral equation

$$x(t) = r(t) + \int_0^t k(t-\sigma)\psi[r(\sigma), \sigma]d\sigma, \qquad t \geq 0 \tag{1}$$

in which x and r take values in R^n (whose elements we take to be column vectors), k is an $n \times n$ matrix-valued function, and ψ maps $R^n \times [0, \infty)$ into R^n. In eq. (1), typically x takes into account initial conditions as well as inputs, and r is the output (i.e., is the intermediate or final output) corresponding to x (see, for instance, Ref. 2, pp. 872–4 for a specific application). Discrete-time counterparts of eq. (1) (see Ref. 7, pp. 449–51, for example) also arise often in system studies.

Much is known about the properties of eq. (1), e.g., see Refs. 2, 8, and 9. In particular, if $n = 1$ and the "circle criterion" of Ref. 2, together with certain associated conditions concerning k, ψ, x, and r described in the reference, are met; if $\psi(\cdot, t)$ is periodic in t with some period τ; and if $x = x_1 + x_0$ with x_1 bounded and τ-periodic and with x_0 bounded and such that $x_0(t) \to 0$ as $t \to \infty$, then we have $r = p + q$ in which p is periodic with period τ, and $q(t) \to 0$ as $t \to \infty$ (see Ref. 2, Theorem 4).‡

This result generalized to arbitrary n is proved in Ref. 6 by first showing that there is a τ-periodic p defined on $(-\infty, \infty)$ such that, with x_1 extended periodically for negative values of t, the *auxiliary equation*

$$x_1(t) = p(t) + \int_{-\infty}^t k(t-\sigma)\psi[p(\sigma), \sigma]d\sigma, \qquad t > -\infty \tag{2}$$

† In contrast, it is standard (see, for example, Ref. 6, p. 195) to mean by r is *asymptotically periodic* that $r = p + q$ with p continuous and as indicated, and with q continuous and such that its values go to zero as time approaches infinity. It is often possible to show without much difficulty that r is asymptotically periodic if r is L-asymptotically periodic and some natural additional hypotheses are met (for an example, see Section 3.4).

‡ The earlier related circle criteria in Refs. 10 and 11 address different issues.

is satisfied. Then, using eq. (1) and the fact that eq. (2) gives

$$x_1(t) - \int_{-\infty}^{0} k(t-\sigma)\psi[p(\sigma), \sigma]d\sigma$$

$$= p(t) + \int_{0}^{t} k(t-\sigma)\psi[p(\sigma), \sigma]d\sigma, \qquad t \geq 0,$$

it is proved that when $x = x_1 + x_0$, we have $r(t) - p(t) \to \theta$ as $t \to \infty$, in which θ is the zero element of R^n. A similar proof shows that if $n = 1$ (for the sake of simplifying a statement of a result) and both x_0 and s, where $s(t) = \int_t^\infty |k(\sigma)| d\sigma$ for $t \geq 0$, have finite energy, then under the conditions indicated above, $r - p$ has finite energy [see Ref. 2, Corollary 1(a)]. The proofs in Ref. 2 are of a functional analytic nature. For material related in a general sense concerning systems of differential equations, and in which a Lyapunov-function approach is used, see Ref. 12, pp. 210–23. Concerning more recent material, a result along the same lines as the one described in the preceding paragraph for interconnected systems[6] governed by a somewhat different class of integral equations, is proved in Ref. 13. There, too, an auxiliary-equation approach is used.†

Under reasonable conditions on k and ψ (see Section 3.4), the set $E(L)$, previously described (and defined in Section III), contains exactly one solution r of eq. (1) for each $x \in E(L)$. We now introduce the typically trivial restriction that only solutions r of eq. (1) contained in $E(L)$ are of interest to us. Thus, under reasonable conditions, there is associated in a natural way with eq. (1) a map $G{:}E(L) \to E(L)$ such that $r = Gx$ for each $x \in E(L)$. Of course, many other examples can be given in which such a map G arises.

Assuming that $\psi(\cdot, \sigma)$ in eq. (1) is independent of σ and that $\psi(\theta, 0) = \theta$, notice that the G associated with eq. (1) has the property that it is *time invariant* in the usual sense that the response to a delayed input is the delayed response to the original input. (For a precise definition of time invariance, see Section 3.3.) This type of property of G, rather than the concept of an auxiliary equation, plays a central role in our approach in Section III.

III. L-ASYMPTOTIC PERIODICITY, TIME INVARIANCE, AND PERIODICALLY-VARYING SYSTEMS

3.1 Preliminary notation and definitions

Throughout the remainder of the paper the following notation and definitions are used.

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† The method used in Ref. 13 to show the existence of a periodic solution of the auxiliary equation is very different from that in Ref. 2.

The symbol T denotes either $[0, \infty)$ or $\{0, 1, 2, \cdots\}$. Elements of R^n are taken to be column vectors, v' denotes the transpose of an arbitrary $v \in R^n$, and θ stands for the zero element of R^n.‡

If $T = [0, \infty)$, then L denotes the set of Lebesgue measurable functions v from T into R^n such that

$$\int_0^\infty v'(t)v(t)dt < \infty.$$

Alternatively, when $T = \{0, 1, 2, \cdots\}$, L stands for the set of maps v from T into R^n such that

$$\sum_{t=0}^{\infty} v'(t)v(t) < \infty.$$

The norm $\|v\|$ of an arbitrary element v of L is defined by

$$\|v\| = \left(\int_0^\infty v'(t)v(t)dt\right)^{1/2} \quad \text{if} \quad T = [0, \infty),$$

and

$$\|v\| = \left(\sum_{t=0}^{\infty} v'(t)v(t)\right)^{1/2} \quad \text{if} \quad T = \{0, 1, 2, \cdots\}.$$

With this norm, L is a Banach space of finite energy (i.e., square summable) functions.

For $v:T \to R^n$ and $\omega \in T$, $v_{(\omega)}$ denotes the map from T into R^n defined by $v_{(\omega)}(t) = v(t)$ for $t \in T$ with $t \leq \omega$, and $v_{(\omega)}(t) = \theta$ for $t \in T$ such that $t > \omega$. We use $E(L)$ to denote the "extended set" $\{v:T \to R^n \,|\, v_{(\omega)} \in L \text{ for } \omega \in T\}$, and θ_E stands for the zero element of $E(L)$. [Note that $E(L)$ is the set of *all* maps $v:T \to R^n$ when $T = \{0, 1, 2, \cdots\}$.]

We say that a map $H:E(L) \to E(L)$ is *causal* (see Ref. 2, p. 888) if we have $(Hv)_{(\omega)} = [Hv_{(\omega)}]_{(\omega)}$ for each $v \in E(L)$ and each $\omega \in T$.

For any $v \in E(L)$ and each $\tau_0 \in T$, $v(\cdot + \tau_0)$ denotes the element w of $E(L)$ defined by $w(t) = v(t + \tau_0)$, $t \in T$.

The symbol τ denotes a fixed positive element of T, and P stands for the set of periodic functions $\{v \in E(L) \,|\, v(t + \tau) = v(t) \text{ for } t \in T\}$.

A central role is played by the set S defined by $S = \{v \in L \,|$ there is a $v^* \in L$ with the property that

$$\sum_{k=1}^{K} v(\cdot + k\tau) \to v^*$$

as $K \to \infty\}$, where $\to v^*$ means convergence in norm to v^*.

‡ We have repeated the definitions of T and θ for the reader's convenience.

Finally, for each $\omega \in T$, the "delay map" $D_\omega: E(L) \to E(L)$ is defined by $(D_\omega v)(t) = v(t-\omega)$ for $t \geq \omega$, and $(D_\omega v)(t) = \theta$ for $t < \omega$.

3.2 L-Asymptotic periodicity

We shall use the following hypothesis:

H.1: G is a map from $E(L)$ into $E(L)$ such that for any $v \in E(L)$, we have $(GD_\tau v)(t) = (D_\tau Gv)(t)$ for $t \in T$ with $t \geq \tau$.

This hypothesis is satisfied whenever G is a causal map of $E(L)$ into itself that is either time invariant or periodically varying with period τ (see Section 3.3). Our main result is the following:

Theorem 1: Assume that H.1 holds. Let $x \in E(L)$, and let r denote Gx. Then r has the form $p + q$ with $p \in P$ and $q \in L$ if and only if $(Gx - GD_\tau x) \in S$.

Proof: Suppose first that $(Gx - GD_\tau x) = v$ for some $v \in S$. Let $v^* \in L$ be such that

$$\sum_{k=1}^{K} v(\cdot + k\tau) \to v^*$$

as $K \to \infty$. We shall use the proposition that

$$v^*(\cdot + \tau) + v(\cdot + \tau) = v^*(\cdot), \tag{3}$$

which follows from the inequality

$$\|v^*(\cdot + \tau) + v(\cdot + \tau) - v^*(\cdot)\| \leq \left\| v^*(\cdot + \tau) - \sum_{k=2}^{K} v(\cdot + k\tau) \right\| + \left\| \sum_{k=1}^{K} v(\cdot + k\tau) - v^*(\cdot) \right\| \tag{4}$$

for $K \geq 2$, and the fact that the right side of inequality (4) approaches zero as $K \to \infty$.

Let p_0 denote $r + v^*$, which is clearly an element of $E(L)$. Since $r(t) = (GD_\tau x)(t) + v(t)$ for $t \in T$, by H.1, we have $r(t) = r(t-\tau) + v(t)$ for $t \geq \tau$. Therefore, for $t \in T$, $p_0(t+\tau) = r(t+\tau) + v^*(t+\tau) = r(t) + v(t+\tau) + v^*(t+\tau)$. On the other hand, using eq. (3), $r(t) + v(t+\tau) + v^*(t+\tau) = r(t) + v^*(t) = p_0(t)$ for all $t \in T$ if $T = \{0, 1, 2, \cdots\}$, and for almost all $t \in T$ if $T = [0, \infty)$. Therefore, with p the element of P defined by $p(t) = p_0(t)$ for $t \in [0, \tau) \cap T$, we have $p_0(t) - p(t) = \theta$ for all $t \in T$ if $T = \{0, 1, 2, \cdots\}$ and for almost all $t \in T$ if $T = [0, \infty)$, and clearly $r = p + (p_0 - p - v^*)$ in which $(p_0 - p - v^*) \in L$.

Suppose now that $r = p + q$ with $p \in P$ and $q \in L$, and let $u = (Gx - GD_\tau x)$. For $t \geq \tau$, $u(t) = r(t) - r(t-\tau) = p(t) + q(t) - p(t-\tau) - q(t-\tau) = q(t) - q(t-\tau)$ which, together with $u \in E(L)$, shows that $u \in L$.

Let $u^{(K)}(\cdot)$ in L be defined by

$$u^{(K)}(t) = \sum_{k=1}^{K} u(t + k\tau)$$

for $t \in T$ and any positive integer K, and let J be an integer such that $J > K$. Using $u(t + k\tau) = q(t + k\tau) - q[t + (k-1)\tau]$ for $k \geq 1$ and $t \in T$, we have, for $t \in T$,

$$\begin{aligned} u^{(J)}(t) - u^{(K)}(t) &= \sum_{k=1}^{J} u(t + k\tau) - \sum_{k=1}^{K} u(t + k\tau) \\ &= \sum_{k=(K+1)}^{J} u(t + k\tau) \\ &= q(t + J\tau) - q(t + K\tau). \end{aligned}$$

Thus, $\| u^{(J)} - u^{(K)} \| \leq \| q(\cdot + J\tau) \| + \| q(\cdot + K\tau) \|$. Since $\| q(\cdot + K\tau) \| \to 0$ as $K \to \infty$, $\{u^{(K)}\}_1^\infty \subset L$ is a Cauchy sequence, and, by the completeness of L, there is a $u^* \in L$ such that $\| u^{(K)} - u^* \| \to 0$ as $K \to \infty$. This concludes the proof.

3.2.1 Comments

The following example shows that S is a *proper* subset of L. Let G be the identity operator on $E(L)$, take $n = 1$, and let $x \in E(L)$ be defined by $x(t) = \ln 2$ for $t \in [0, 2] \cap T$, and $x(t) = \ln t$ for $t \in (2, \infty) \cap T$. Let $\tau = 1$. Then, $(Gx - GD_\tau x)(t) = r(t) - r(t-1) = \ln[t(t-1)^{-1}]$ for $t \in [3, \infty) \cap T$. Using the inequality $\ln(1 + \sigma) \leq \sigma$ valid for $\sigma \geq 0$, we see that $\ln[t(t-1)^{-1}] \leq (t-1)^{-1}$ for $t \in [3, \infty) \cap T$, and therefore that v, defined by $v(t) = (Gx - GD_\tau x)(t)$ for $t \in T$, belongs to L. Since here Gx cannot be written as $p + q$ with $p \in P$ and $q \in L$, it follows from the theorem that $v \notin S$.

It is not difficult to verify that the proof given of the theorem can be modified to show that H.1 can be replaced with the following somewhat weaker hypothesis.

H.1′: G:$E(L) \to E(L)$ is a map such that for any $v \in E(L)$, there is an $s \in S$ such that $(GD_\tau v)(t) = (D_\tau Gv)(t) + s(t)$ for $t \in T \cap [\tau, \infty)$.

The simple example: $n = 1$, $(Gv)(t) = v(t) + e^{-t}$ for $t \in T$ and each $v \in E(L)$ is one for which H.1′, but not H.1, is met.

3.2.2 Corollaries (the use of weighting functions)

In this section, and in the Appendix, w denotes any function from T into R^1 such that there is a constant $\beta > 0$ for which $w(t) \geq (1 + \beta t)^2$ when $t \in T$, and such that w is measurable on T and bounded on bounded subsets of T if $T = [0, \infty)$. By wv, where $v \in E(L)$, we mean the element of $E(L)$ defined by $(wv)(t) = w(t)v(t)$ for $t \in T$.

Corollary 1: Suppose that H.1 is met, that $x \in E(L)$, *and that*

$w(Gx - GD_\tau x) \in L$. Then $Gx = p + q$ for some $p \in P$ and some $q \in L$.

Proof: Let $h = w(Gx - GD_\tau x)$, and let s denote $(Gx - GD_\tau x)$. Observe that $s \in L$. For any positive integers J and K with $J > K$,

$$\begin{aligned}\left\| \sum_{k=1}^{J} s(\cdot + k\tau) - \sum_{k=1}^{K} s(\cdot + k\tau) \right\| &= \left\| \sum_{k=K+1}^{J} s(\cdot + k\tau) \right\| \\ &\leq \sum_{k=K+1}^{J} \| s(\cdot + k\tau) \| \\ &\leq \sum_{k=K+1}^{J} (1 + k\beta\tau)^{-2} \| h(\cdot + k\tau) \| \\ &\leq \sum_{k=K+1}^{J} (1 + k\beta\tau)^{-2} \| h \|,\end{aligned}$$

which shows that

$$\left\| \sum_{k=1}^{J} s(\cdot + k\tau) - \sum_{k=1}^{K} s(\cdot + k\tau) \right\| \to 0$$

as J and K approach infinity. By the completeness of L, we have $s \in S$ and the corollary follows.

In Corollary 2, below, $w(\cdot + \tau)[x(\cdot + \tau) - x(\cdot)]$ denotes the element of $E(L)$ whose values are $w(t + \tau)[x(t + \tau) - x(t)]$.

Corollary 2: Assume that H.1 is met, and that there is a positive constant ρ such that

$$\| w(Gu - Gv)_{(\omega)} \| \leq \rho \| w(u - v)_{(\omega)} \| \qquad (5)$$

for u and v in $E(L)$ and $\omega \in T$. If $x \in E(L)$ is such that $w(\cdot + \tau) \cdot [x(\cdot + \tau) - x(\cdot)] \in L$, then $Gx = p + q$ for some $p \in P$ and some $q \in L$.

Proof: We have $\| w(Gx - GD_\tau x)_{(\omega)} \| \leq \rho \| w(x - D_\tau x)_{(\omega)} \|$ for $\omega \in T$ and any $x \in E(L)$. When $w(\cdot + \tau)[x(\cdot + \tau) - x(\cdot)] \in L$, it follows that $\sup_{\omega\in T} \| w(x - D_\tau x)_{(\omega)} \| < \infty$; hence, $w(Gx - GD_\tau x) \in L$. By Corollary 1, $Gx = p + q$ with p and q as indicated.

3.2.3 Comments

The condition that $w(\cdot + \tau)[x(\cdot + \tau) - x(\cdot)] \in L$ is met if $\sup_{t\in T}[w(t + \tau)/w(t)] < \infty$ and $x = p_0 + q_0$ with $p_0 \in P$ and $wq_0 \in L$, and of course $\sup_{t\in T}[w(t + \tau)/w(t)] < \infty$ is satisfied if, for example, $w(t) = e^{\lambda t}$ for $t \in T$ or $w(t) = (1 + \lambda t)^2$ for $t \in T$, with λ a positive constant. Input-output stability theory techniques can frequently be used to show, in specific cases, that eq. (5), with an appropriate w, is met.

Regarding the case in which $T = \{0, 1, 2, \cdots\}$, since τ could have been taken to be unity, Theorem 1 and Corollaries 1 and 2 provide conditions under which r is L-asymptotically *constant* in the sense that $r = c + q$ with $q \in L$ and $c \in C$, where C is the set of constant R^n-valued functions $\{v \in P \mid v(t) = u \text{ for } t \in T \text{ and some } u \in R^n\}$. Corresponding results for $T = [0, \infty)$ are given in the Appendix.

3.3 Time invariance and periodically-varying systems

Hypothesis 1 plays a prominent role in Section 3.2. Here we give definitions which make precise the essentially self-evident proposition that H.1 is met if G is a causal map of $E(L)$ into itself that is either, in the usual sense, time invariant or periodically varying with period τ.

Let H be an arbitrary causal map of $E(L)$ into $E(L)$.

Definition 1: H is *time invariant* if (i) there is an element ν of R^n such that $(H\theta_E)(t) = \nu$ for $t \in T$, and (ii) for any $x \in E(L)$, we have

$$\begin{aligned} (HD_\omega x)(t) &= \nu, & t &\in [0, \omega) \cap T \\ &= (D_\omega Hx)(t), & t &\in [\omega, \infty) \cap T \end{aligned}$$

for each $\omega \in (T - \{0\})$.

Definition 2: H is *periodically varying* with period τ if (i) $H\theta_E = \nu$ for some $\nu \in P$, and (ii) for each $x \in E(L)$ and any positive integer k,

$$\begin{aligned} (HD_{k\tau} x)(t) &= \nu(t), & t &\in [0, k\tau) \cap T \\ &= (D_{k\tau} Hx)(t), & t &\in [k\tau, \infty) \cap T. \end{aligned}$$

Notice that H is "periodically varying" with period τ if H is time invariant. A related definition is the following:

Definition 2′: H is *periodically varying* with period τ if (i) $H\theta_E = \nu$ for some $\nu \in P$, and (ii) for any $x \in E(L)$, we have

$$\begin{aligned} (HD_\tau x)(t) &= \nu(t), & t &\in [0, \tau) \cap T \\ &= (D_\tau Hx)(t), & t &\in [\tau, \infty) \cap T. \end{aligned}$$

To see that Definitions 2 and 2′ are consistent, we observe the following: If H meets the conditions of Definition 2, then obviously H satisfies the conditions of Definition 2′. On the other hand, if H meets the conditions of Definition 2′, and $x \in E(L)$ is given, and if

$$(HD_{k\tau} x)(t) = \nu(t), \quad t \in [0, k\tau) \cap T \tag{6a}$$

$$= (D_{k\tau} Hx)(t), \quad t \in [k\tau, \infty) \cap T \tag{6b}$$

for some k, then, by the conditions of Definition 2′ with x replaced with $D_{k\tau}x$,

$$\begin{aligned} (HD_{(k+1)\tau} x)(t) &= \nu(t), & t &\in [0, \tau) \cap T \\ &= (D_\tau HD_{k\tau} x)(t), & t &\in [\tau, \infty) \cap T. \end{aligned}$$

Since $HD_{k\tau}x$ has the values given by eqs. (6a) and (6b), we see that

$$(HD_{(k+1)\tau}x)(t) = \nu(t), \qquad t \in [0, (k+1)\tau) \cap T$$
$$= (D_{(k+1)\tau}Hx)(t), \qquad t \in [(k+1)\tau, \infty) \cap T,$$

which shows that the conditions of Definition 2 are met.

Notice that our assumption that H is causal is not explicitly used. That assumption restricts the class of operators H so that the definitions given above are appropriate and natural.†

3.4 An example

Let $T = [0, \infty)$, and consider eq. (1) which is repeated below.

$$x(t) = r(t) + \int_0^t k(t - \sigma)\psi[r(\sigma), \sigma]d\sigma, \qquad t \geq 0. \tag{1}$$

Assume the following, in which L_1 denotes the set of functions from $[0, \infty)$ to R^1 that are summable over $[0, \infty)$.

A.1: $x \in E(L)$, k is a measurable real $n \times n$ matrix-valued function defined on $[0, \infty)$ such that each k_{ij} is bounded and belongs to L_1, and ψ is a map from $R^n \times [0, \infty)$ into R^n with the properties that $\psi(\theta, \sigma) = \theta$ for $\sigma \geq 0$, and

(*i*) there is a constant $c > 0$ such that $|\psi(u, t) - \psi(v, t)| \leq c|u - v|$ for all $u, v \in R^n$ and all $t \geq 0$, in which $|\cdot|$ is some norm on R^n, and

(*ii*) $\psi[z(\cdot), \cdot]$ is measurable on $[0, \infty)$ whenever $z \in E(L)$.

Since $x \in E(L)$ and each $k_{ij} \in L_1$, it follows that u defined by

$$u(t) = \int_0^t k(t - \sigma)\psi[x(\sigma), \sigma]d\sigma, \qquad t \geq 0$$

is an element of $E(L)$. Also, since each k_{ij} is bounded, there is a constant c_0 such that $|k(t - \sigma)[\psi(z_1, \sigma) - \psi(z_2, \sigma)]| \leq c_0|z_1 - z_2|$ for all nonnegative t and σ such that $t \geq \sigma$, and for all z_1 and z_2 in R^n. These two observations show that a proof given by Tricomi (see Ref. 8, pp. 42–7) can be modified to prove that $E(L)$ contains a unique solution r of eq. (1).‡

Let G be the map of $E(L)$ into $E(L)$ defined by the condition that for each $x \in E(L)$, $r = Gx$ is the solution in $E(L)$ of eq. (1). Since

† Although the concepts involved are obviously well known, it appears that Definitions 2 and 2′ have not actually been given earlier. Also, Definition 1 is not entirely standard. For example, in Ref. 4, p. 20, time invariance requires that $\nu = \theta$.

‡ The integral on the right side of eq. (1) can easily be shown to be an element of R^n for each t whenever $r \in E(L)$. Since the value of the integral for a given t is unchanged if r is replaced by any element of $E(L)$ that agrees with r almost everywhere, eq. (1) has a solution if there is an element $E(L)$ that satisfies the equation almost everywhere, and, moreover, any solution $r \in E(L)$ is unique and not merely essentially unique.

$\psi(\theta, t) = \theta$ for $t \geq 0$, it is easy to see that H.1 is met when $\psi(z, t + \tau) = \psi(z, t)$ for $t \geq 0$ and $z \in R^n$.

Now consider four additional assumptions.

A.2: $\psi(z, t) = \psi(z, t + \tau)$ for $t \geq 0$ and all $z \in R^n$.

A.3: For any z_a and z_b in $E(L)$, there is a measurable real $n \times n$ matrix-valued function D defined on $[0, \infty)$ such that (i) each D_{ij} is bounded on $[0, \infty)$, (ii) $\psi[z_a(t), t] - \psi[z_b(t), t] = D(t)[z_a(t) - z_b(t)]$ for $t \geq 0$, and (iii) the relation

$$x_0(t) = r_0(t) + \int_0^t k(t - \sigma)D(\sigma)r_0(\sigma)d\sigma, \qquad t \geq 0 \tag{7}$$

implies that we have $r_0 \in L$ whenever $r_0 \in E(L)$ and $x_0 \in L$. (See Ref. 2, pp. 876–8 for conditions under which A.3 holds when ψ has a certain important specific form.)

A.4: For each i and j, $t^p k_{ij} \in L_1$ for $p = 1, 2$.†

A.5: Concerning eq. (1), $x = u_1 + u_2$ with $u_1 \in P$ and $t^p u_2 \in L$ for $p = 0, 1, 2$.

We shall prove the following.

Theorem 2: If A.1 through A.5 hold, then $E(L)$ contains a unique solution r of eq. (1), *and we have $r = p + q$ for some $p \in P$ and some $q \in L$.*

Proof: As indicated earlier, A.1 implies that there is a unique solution of eq. (1) in $E(L)$. Let r and s denote Gx and $GD_\tau x$, respectively, and let D satisfy $\psi[r(t), t] - \psi[s(t), t] = D(t)[r(t) - s(t)]$, $t \geq 0$ with D such that (i) and (iii) of A.3 hold. Then, with $\Delta = r - s$ and $v = x - D_\tau x$,

$$v(t) = \Delta(t) + \int_0^t k(t - \sigma)D(\sigma)\Delta(\sigma)d\sigma, \qquad t \geq 0.$$

Note that $v(t) = x(t)$ for $t \in [0, \tau)$, and $v(t) = u_2(t) - u_2(t - \tau)$ for $t \geq \tau$, from which it easily follows that $(1 + t)^p v \in L$ for $p = 0, 1, 2$.

By A.3, $\Delta \in L$. In addition, observe that we have

$$\begin{aligned}(1 + t)v(t) = {} & (1 + t)\Delta(t) \\ & + \int_0^t k(t - \sigma)D(\sigma)(1 + \sigma)\Delta(\sigma)d\sigma \\ & + \int_0^t (t - \sigma)k(t - \sigma)D(\sigma)\Delta(\sigma)d\sigma, \qquad t \geq 0.\end{aligned}$$

Since $tk_{ij} \in L_1$ for all i and j, I_1 given by

† By $t^p k_{ij}$ we mean, of course, the map from $[0, \infty)$ into R^1 whose value at t is $t^p k_{ij}(t)$.

$$I_1(t) = \int_0^t (t - \sigma)k(t - \sigma)D(\sigma)\Delta(\sigma)d\tau, \qquad t \geq 0$$

belongs to L. Thus, by A.3, $(1 + t)\Delta \in L$. Similarly,

$$(1 + t)^2 v(t) = (1 + t)^2\Delta(t) + \int_0^t k(t - \sigma)D(\sigma)(1 + \sigma)^2\Delta(\sigma)d\sigma$$

$$+ 2\int_0^t (t - \sigma)k(t - \sigma)D(\sigma)(1 + \sigma)\Delta(\sigma)d\sigma$$

$$+ \int_0^t (t - \sigma)^2 k(t - \sigma)D(\sigma)\Delta(\sigma)d\sigma, \qquad t \geq 0,$$

together with the hypothesis that A.3 holds and that $t^2 k_{ij} \in L_1$ for all i and j, shows that $(1 + t)^2\Delta \in L$. By Corollary 1, $r = p + q$ with p and q as indicated.

3.4.1 Comments

Under the conditions of Theorem 2, it can be shown that the integral on the right side of eq. (1) depends continuously on t for $t > 0$. Thus, if u_1 and u_2 of Theorem 2 are continuous, then so is r.

Concerning the standard concept of asymptotic periodicity (see Ref. 6, p. 195 and refer to the footnote in Section I), arguments of the kind used in Ref. 14 show that r is asymptotically τ-periodic whenever x is asymptotically τ-periodic, the conditions of Theorem 2 are met, and k satisfies the additional assumption:

A.6: Each tk_{ij} is bounded on $[0, \infty)$.†

More specifically, let A.6 and the conditions of Theorem 2 be met, and let p and q be as described in Theorem 2. Then, with $\psi[p(\sigma), \sigma]$ defined on $(-\infty, 0)$ by periodically extending $\psi[p(\sigma), \sigma]$ on $[0, \tau)$, the integral

$$\int_{-\infty}^t k(t - \sigma)\psi[p(\sigma), \sigma]d\sigma$$

exists as an element of R^n for each t (see Ref. 14, pp. 2852–3). This integral is periodic in t, and it can be shown to be continuous in t. These facts can be used to verify that when A.1 through A.6 are satisfied,

$$\int_0^t k(t - \sigma)\psi[p(\sigma) + q(\sigma), \sigma]d\sigma,$$

† This hypothesis and A.5 imply that each $(1 + t)k_{ij}$ is square summable over $[0, \infty)$.

which is continuous in t for $t > 0$, can be written as $v_1 + v_2$ with v_1 continuous and τ-periodic and with $v_2(t) \to \theta$ as $t \to \infty$. The rest is obvious.

The proof of Theorem 2 involves the use of a quadratic weighting function w. A similar result can be proved using an exponential weighting function. Specifically, suppose that A.1 holds, that there is an $\alpha > 0$ such that A.3 is met with k replaced with $e^{\alpha t}k$, and that A.5 holds with the integrability conditions on u_2 replaced by the requirement that $e^{\alpha t}u_2 \in L$. Then, since

$$e^{\alpha t}v(t) = e^{\alpha t}\Delta(t) + \int_0^t e^{\alpha(t-\sigma)}k(t-\sigma)D(\sigma)e^{\alpha\sigma}\Delta(\sigma)d\sigma, \qquad t \geq 0$$

we have $e^{\alpha t}\Delta \in L$.†

IV. APPENDIX

Throughout this appendix, δ denotes an arbitrary positive constant, C stands for the subset of $E(L)$ whose elements are constant R^n-valued functions, $T = [0, \infty)$, $P(\omega)$ denotes the set of periodic functions $\{v \in E(L) \,|\, v(t+\omega) = v(t) \text{ for } t \in T\}$ for each $\omega > 0$, and $S(\omega)$ is defined by $S(\omega) = \{v \in L \,|\,$ there is a $v^* \in L$ with the property that

$$\sum_{k=1}^{K} v(\cdot\ + k\omega) \to v^*$$

as $K \to \infty\}$ for any $\omega > 0$.

Consider hypothesis H.2 below.

H.2: $T = [0, \infty)$, and G is a map of $E(L)$ into $E(L)$ with the following property: For each $v \in E(L)$ and each $\omega \in (0, \delta)$, we have $(GD_\omega v)(t) = (D_\omega Gv)(t)$ for $t \geq \omega$.

Theorem 3: Let H.2 hold, and let $x \in E(L)$. Then Gx has the form $c + q$ with $c \in C$ and $q \in L$ if and only if $(Gx - GD_\omega x) \in S(\omega)$ for $\omega \in (0, \delta)$.

Proof: By Theorem 1, $(Gx - GD_{\tau_0}x) \in S(\tau_0)$ for any $\tau_0 \in (0, \delta)$ when Gx has the form indicated.

On the other hand, suppose that $(Gx - GD_\omega x) \in S(\omega)$ for $\omega \in (0, \delta)$, and let $\tau_0 \in (0, \delta)$. By Theorem 1, $Gx = p_{\tau_0} + q_{\tau_0}$ with $p_{\tau_0} \in P(\tau_0)$ and $q_{\tau_0} \in L$. Similarly, for any integer $m > 0$, and with $\tau_1 = \tau_0/m$, we have $Gx = p_{\tau_1} + q_{\tau_1}$ for some $p_{\tau_1} \in P(\tau_1)$ and some $q_{\tau_1} \in L$. Notice that p_{τ_1} and, therefore, $(p_{\tau_0} - p_{\tau_1})$ belong to $P(\tau_0)$, and hence have Fourier series expansions. Since $(p_{\tau_0} - p_{\tau_1})$ also belongs to L, and $m > 0$ is arbitrary, it easily follows that there is a $u \in R^n$ such that $p_{\tau_0}(t) = u$ for almost all $t \geq 0$. This completes the proof.

† Both quadratic and exponential weighting functions have been used earlier for the different purpose of obtaining criteria for the boundedness of solutions of equations (see Refs. 2 and 7, and, for example, Refs. 5 and 6).

Theorem 3 and the material in Section 3.2.2 can be used to immediately obtain the following two results.

Corollary 3: Assume that H.2 is met, that $x \in E(L)$, *and that* $w(Gx - GD_\omega x) \in L$ *for* $\omega \in (0, \delta)$ (w *is defined in Section 3.2.2*). *Then* $Gx = c + q$ *with* $c \in C$ *and* $q \in L$.

Corollary 4: Suppose that H.2 is satisfied, that w (*see Section 3.2.2*) *satisfies* $\sup_{t\geq 0}[w(t + \omega)/w(t)] < \infty$ *for* $\omega \in (0, \delta)$, *that there is a constant* $\rho > 0$ *such that* $\| w(Gu - Gv)_{(\omega)} \| \leq \rho \| w(u - v)_{(\omega)} \|$ *for* u *and* v *in* $E(L)$ *and* $\omega > 0$, *and that* $x = c_0 + q_0$ *with* $c_0 \in C$ *and* $wq_0 \in L$. *Then* $Gx = c + q$ *for some* $c \in C$ *and some* $q \in L$.

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